BOREL–MOORE HOMOLOGY AND THE SPRINGER CORRESPONDENCE BY RESTRICTION

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1. Introduction

The modular Springer correspondence was first studied by Juteau [Ju], as an analogue of the "ordinay" (i.e. characteristic zero) Springer correspondence due to Springer [Spr]. This theory relates simple objects of the category $\mathsf{Perv}_G(\mathcal{N}, \Bbbk)$ of G-equivariant perverse sheaves on the nilpotent cone $\mathcal{N}$ of a complex reductive algebraic group G (with coefficients in a field $\Bbbk$) and of the category $\mathsf{Rep}(W, \Bbbk)$ of $\Bbbk$-representations of the Weyl group W of G. This correspondence was extended to noetherian rings of finite global dimension in [AHJR].

The main player of this correspondence is the *Springer sheaf* $\underline{\mathsf{Spr}}$, a certain object of $\mathsf{Perv}_G(\mathcal{N}, \Bbbk)$ endowed with an action of W. More precisely, there exist two natural ways of defining this action: one "by restriction", and one via a Fourier transform. The action considered in [Ju, AHJR] was the one defined by Fourier transform. The main goal of this note is to explain that one can also define a Springer correspondence (for general coefficients) using the action by restriction. The main point is to show that this action induces an algebra isomorphism

$$(\star) \qquad \Bbbk[W] \xrightarrow{\sim} \mathrm{End}_{\mathsf{Perv}_G(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}}).$$

(The analogous property for the action by Fourier transform is immediate.) Then, the observation that $\underline{\mathsf{Spr}}$ is a projective object in $\mathsf{Perv}_G(\mathcal{N}, \Bbbk)$ (as proved in [AHR]) readily implies that there exists a Springer correspondence over $\Bbbk$, see §3.2 for a precise statement.

Note that isomorphism ($\star$) was also proved in [AHJR] by comparing the two actions of W on $\underline{\mathsf{Spr}}$. Here we do not use the Fourier action at all. Instead we use an interpretation of the space $\mathrm{End}_{\mathsf{Perv}_G(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}})$ in terms of Borel–Moore homology due to Ginzburg, see e.g. [CG]. In fact, our proof is essentially a reformulation of some considerations in [CG, §3.4]. For this reason, this text will not be published.

2. Reminder on Borel–Moore homology

We begin with some generalities on (equivariant) Borel–Moore homology, from the sheaf-theoretic point of view. Let $\Bbbk$ be a noetherian ring of finite global dimension. All our sheaves will have coefficients in $\Bbbk$. We fix a connected complex algebraic group A. By a "variety" we mean a complex algebraic variety. By an "A-variety" we mean a variety endowed with an algebraic action of A.

2.1. **Notation.** If X is an A-variety, we denote by $\mathscr{D}^{\mathrm{b}}_A(X, \Bbbk)$ the constructible A-equivariant derived category of X with coefficients in $\Bbbk$, as defined in [BL]. We let $\underline{\Bbbk}_X \in \mathscr{D}^{\mathrm{b}}_A(X,\Bbbk)$, respectively $\underline{\mathbb{D}}_X \in \mathscr{D}^{\mathrm{b}}_A(X,\Bbbk)$, denote the constant sheaf, respectively the dualizing sheaf, of X. Then the A-equivariant n-th Borel–Moore homology of X (with coefficients in $\Bbbk$) is defined by

$$\mathsf{H}^A_n(X;\Bbbk) := \mathrm{Hom}^{-n}_{\mathscr{D}^{\mathrm{b}}_A(X,\Bbbk)}(\underline{\Bbbk}_X, \underline{\mathbb{D}}_X).$$

As usual, when $A = \{1\}$ we omit it from the notation.

We will use the same conventions as in [MR2, Appendix A]: if X, Y, Z are A-varieties and $f : X \to Y$, $g : Y \to Z$ are A-equivariant morphisms, then there exist canonical "composition" isomorphisms

$$g_* f_* \cong (g \circ f)_*, \quad g_! f_! \cong (g \circ f)_!, \quad f^* g^* \cong (g \circ f)^*, \quad f^! g^! \cong (g \circ f)^!,$$

which we will all indicate by (Co). Similarly, given a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & \square & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

of A-varieties and A-equivariant morphisms, there exist canonical "base change" isomorphisms

$$f^* g_! \cong g'_! f'^*, \quad f^! g_* \cong g'_* f'^!,$$

which we will indicate by (BC). Finally, if $f : X \to Y$ is an A-equivariant morphism of A-varieties, there exist natural adjunction morphisms

$$\mathrm{id} \to f_* f^*, \quad f^* f_* \to \mathrm{id}, \quad \mathrm{id} \to f^! f_!, \quad f_! f^! \to \mathrm{id},$$

which we will indicate by (Adj).

Below we will use some results of [MR2] which are stated in *loc. cit.* only in the case $\Bbbk = \mathbb{C}$. One can easily check that this assumption is not used in the proof of these results, so that they hold in our level of generality.

2.2. **Proper pushforward.** Let $f : X \to Y$ be an A-equivariant proper morphism of A-varieties. Then the adjunction morphism $f_! f^! \to \mathrm{id}$ applied to $\underline{\mathbb{D}}_Y$ induces, via the canonical isomorphism $f^! \underline{\mathbb{D}}_Y \cong \underline{\mathbb{D}}_X$, a morphism $f_! \underline{\mathbb{D}}_X \to \underline{\mathbb{D}}_Y$. Taking cohomology we deduce a "pushforward" morphism

$$\mathsf{PF}(X,Y) : \mathsf{H}^A_i(X;\Bbbk) \to \mathsf{H}^A_i(Y;\Bbbk).$$

2.3. **Open restriction.** Let X be an A-variety, and let $U \subset X$ be an A-stable open subvariety. Denote by $j : U \hookrightarrow X$ the inclusion. Then the adjunction morphism $\mathrm{id} \to j_* j^*$ applied to $\underline{\mathbb{D}}_X$ induces, via the canonical isomorphism $j^* \underline{\mathbb{D}}_X \cong \underline{\mathbb{D}}_U$, a morphism $\underline{\mathbb{D}}_X \to j_* \underline{\mathbb{D}}_U$. Taking cohomology we deduce a "pullback" morphism

$$\mathsf{PB}(X,U) : \mathsf{H}^A_i(X;\Bbbk) \to \mathsf{H}^A_i(U;\Bbbk).$$

2.4. **Long exact sequence.** Let X be an A-variety, and let $U \subset X$ be an A-stable open subvariety, with complement Y. Consider the inclusions

$$U \xhookrightarrow{j} X \xhookleftarrow{i} Y.$$

Then the canonical triangle of functors $i_! i^! \to \mathrm{id} \to j_* j^* \xrightarrow{+1}$ applied to $\underline{\mathbb{D}}_X$ induces a long exact sequence

$$\cdots \to \mathsf{H}^A_n(Y;\Bbbk) \xrightarrow{\mathsf{PF}(Y,X)} \mathsf{H}^A_n(X;\Bbbk) \xrightarrow{\mathsf{PB}(X,U)} \mathsf{H}^A_n(U;\Bbbk) \to \mathsf{H}^A_{n-1}(Y;\Bbbk) \to \cdots$$

2.5. **Restriction with supports.** Let X be a smooth A-variety of dimension d, and $X' \subset X$ be a smooth A-stable locally closed subvariety of dimension d'. Note that there exist canonical isomorphisms $\underline{\mathbb{D}}_X \cong \underline{\mathbb{C}}_X[2d]$ and $\underline{\mathbb{D}}_{X'} \cong \underline{\mathbb{C}}_{X'}[2d']$, so that we obtain a canonical isomorphism

$$f^*\underline{\mathbb{D}}_X \cong f^*\underline{\mathbb{C}}_X[2d] \cong \underline{\mathbb{C}}_{X'}[2d] \cong \underline{\mathbb{D}}_{X'}[2d-2d']. \tag{2.1}$$

Let $Z \subset X$ be a non necessarily smooth A-stable closed subvariety, and set $Z' := Z \cap X'$. Consider the cartesian diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

The adjunction morphism $\mathrm{id} \to f_*f^*$ induces a morphism of functors $i^! \to i^! f_* f^*$. Applying the base change theorem we obtain a canonical morphism of functors

$$i^! \to f'_* i'^! f^*.$$

Applying this morphism to $\underline{\mathbb{D}}_X$ and using isomorphism (2.1) we obtain a canonical morphism

$$\underline{\mathbb{D}}_Z \to f'_*\underline{\mathbb{D}}_{Z'}[2d-2d']. \tag{2.2}$$

Applying equivariant cohomology we obtain a "restriction with supports" morphism

$$\mathsf{RS}(Z,X,X') : \mathsf{H}^A_n(Z;\Bbbk) \to \mathsf{H}^A_{n+2d'-2d}(Z';\Bbbk).$$

Note that this morphism depends on f (and not only on f').

Later we will need the following easy result. Let V be an A-module and V_1, V_2 be submodules of V. We denote by $g : V_1 \cap V_2 \hookrightarrow V_2$ the inclusion

Lemma 2.1. *If $V_1 + V_2 = V$, the morphism*

$$\underline{\Bbbk}_{V_2}[2\dim(V_2)] \to g_*\underline{\Bbbk}_{V_1\cap V_2}[2\dim(V_2)]$$

obtained from (2.2) *(in the case $X = V$, $X' = V_1$, $Z = V_2$) using the canonical isomorphisms $\underline{\mathbb{D}}_{V_2} \cong \underline{\Bbbk}_{V_2}[2\dim(V_2)]$ and $\underline{\mathbb{D}}_{V_1\cap V_2} \cong \underline{\Bbbk}_{V_1\cap V_2}[2\dim(V_1\cap V_2)]$ coincides with the shift by $2\dim(V_2)$ of the morphism $\underline{\Bbbk}_{V_2} \to g_*\underline{\Bbbk}_{V_1\cap V_2}$ obtained from the adjunction (g^*, g_*).*

Proof. Using a standard reduction (see [BL, Theorem 3.7.3]) we can assume A is reductive, hence V is completely reducible. Then, using the compatibility of our constructions with exterior products, one can assume that either $V_1 = V$ or $V_2 = V$; in both cases the result is obvious. □

2.6. **Compatibility with closed inclusion.** Let us use the same notation as in §2.5. Let also $Z_c \subset Z$ be an A-stable closed subvariety, and set $Z'_c := Z_c \cap X'$. Consider the following diagram, where all squares are cartesian and all triangles are commutative:

$$\begin{array}{ccccc} & & \overset{k'_c}{\frown} & & \\ Z'_c & \xrightarrow{i'_c} & Z' & \xrightarrow{i'} & X' \\ {\scriptstyle f'_c}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z_c & \xrightarrow{i_c} & Z & \xrightarrow{i} & X. \\ & & \underset{k_c}{\smile} & & \end{array}$$

The following lemma is a generalization of [MR2, Lemma A.5.3] (where the case $Z = X$ is considered).

Lemma 2.2. *For any $n \in \mathbb{Z}$ the following diagram is commutative:*

$$\begin{array}{ccc} \mathsf{H}^A_n(Z_c;\Bbbk) & \xrightarrow{\mathsf{PF}(Z_c,Z)} & \mathsf{H}^A_n(Z;\Bbbk) \\ {\scriptstyle \mathsf{RS}(Z_c,X,X')}\downarrow & & \downarrow{\scriptstyle \mathsf{RS}(Z,X,X')} \\ \mathsf{H}^A_{n+2d'-2d}(Z'_c;\Bbbk) & \xrightarrow{\mathsf{PF}(Z'_c,Z')} & \mathsf{H}^A_{n+2d'-2d}(Z';\Bbbk). \end{array}$$

Proof. Consider the following diagram, where unlabelled arrows are induced by the natural identifications $f'_{c*} = f'_{c!}$ and $f'_* = f'_!$:

$$\begin{array}{ccccccccc}
i_{c!}k_c^! & \xrightarrow{(\mathrm{Adj})} & i_{c!}k_c^! f_* f^* & & \xrightarrow[\sim]{(\mathrm{BC})} & & i_{c!} f'_{c*} k'^!_c f^* & \xrightarrow{\sim} & i_{c!} f'_{c!} k'^!_c f^* \\
{\scriptstyle (\mathrm{Co})}\downarrow & & \wr\downarrow{\scriptstyle (\mathrm{Co})} & & (\dagger) & & {\scriptstyle (\mathrm{Co})}\downarrow\wr & & {\scriptstyle (\mathrm{Co})}\downarrow\wr \\
i_{c!}i_c^! i^! & \xrightarrow{(\mathrm{Adj})} & i_{c!}i_c^! i^! f_* f^* & \xrightarrow[\sim]{(\mathrm{BC})} & i_{c!} i_c^! f'_* i'^! f^* & \xrightarrow[\sim]{(\mathrm{BC})} & i_{c!} f'_{c*} i'^!_c i'^! f^* & \longrightarrow & i_{c!} f'_{c!} i'^!_c i'^! f^* \\
 & & & & & \searrow{\scriptstyle (\mathrm{Adj})} & & & \sim\downarrow{\scriptstyle (\mathrm{Co})} \\
{\scriptstyle (\mathrm{Adj})}\downarrow & & {\scriptstyle (\mathrm{Adj})}\downarrow & & & & (\ddagger) & & f'_! i'_{c!} i'^!_c i'^! f^* \\
 & & & & & & & & \downarrow{\scriptstyle (\mathrm{Adj})} \\
i^! & \xrightarrow{(\mathrm{Adj})} & i^! f_* f^* & & \xrightarrow[\sim]{(\mathrm{BC})} & & f'_* i'^! f^* & \xrightarrow{\sim} & f'_! i'^! f^*.
\end{array}$$

The commutativity of part (†) of the diagram follows from [AHR, Lemma B.7(c)]. The commutativity of part (‡) follows from [MR2, Lemma A.4.4]. The commutativity of other parts of the diagram is obvious. Hence the diagram as a whole is commutative. Now we observe that (when applied to $\underline{\mathbb{D}}_X$) this diagram describes the morphisms of the lemma. (In this argument we also use [AHR, Lemma B.4], which implies that one can forget about "(Co)" isomorphisms for $(\cdot)_*$ functors if they are followed by taking cohomology, and similar isomorphisms for $(\cdot)^!$ functors when they are applied to dualizing sheaves.) ☐

2.7. **Compatibility with open inclusion.** Let us use the same notation as in §2.5. Let also $Z_o \subset Z$ be an A-stable open subvariety, and set $Z'_o := Z_o \cap X'$. Consider the following diagram, where all squares are cartesian and all triangles are commutative:

$$\begin{array}{ccccc}
 & & \overset{k'_o}{\frown\!\!\longrightarrow} & & \\
Z'_o & \xrightarrow[j'_o]{} & Z' & \xrightarrow[i']{} & X' \\
{\scriptstyle f'_o}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Z_o & \xrightarrow{j_o} & Z & \xrightarrow{i} & X. \\
 & & \underset{k_o}{\smile\!\!\longrightarrow} & &
\end{array}$$

The proof of the following lemma is easy and left to the reader.

Lemma 2.3. *For any $n \in \mathbb{Z}$ the following diagram is commutative:*

$$\begin{array}{ccc} \mathsf{H}^A_n(Z;\Bbbk) & \xrightarrow{\mathsf{PB}(Z,Z_o)} & \mathsf{H}^A_n(Z_o;\Bbbk) \\ \downarrow{\scriptstyle \mathsf{RS}(Z,X,X')} & & \downarrow{\scriptstyle \mathsf{RS}(Z_o,X,X')} \\ \mathsf{H}^A_{n+2d'-2d}(Z';\Bbbk) & \xrightarrow{\mathsf{PB}(Z',Z'_o)} & \mathsf{H}^A_{n+2d'-2d}(Z'_o;\Bbbk) \end{array}$$

3. Restriction for Springer and Grothendieck sheaves

3.1. **Notation.** In this section we consider a complex connected reductive group G, and we choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We denote by U and U^- the unipotent radical of B and of the opposite Borel subgroup (with respect to T), and by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{u}$ the Lie algebras of G, B, T, U. If α is a root of G, we denote by U_α the corresponding root subgroup (either in U or in U^-).

We let $\mathscr{B} := G/B$ be the flag variety of G. We will consider the following vector bundles over $\mathscr{B}$:

$$\widetilde{\mathcal{N}} := \{(\xi, gB) \in \mathfrak{g}^* \times G/B \mid \xi_{|g\cdot\mathfrak{b}} = 0\}, \quad \widetilde{\mathfrak{g}} := \{(\xi, gB) \in \mathfrak{g}^* \times G/B \mid \xi_{|g\cdot\mathfrak{u}} = 0\},$$

called respectively the Springer resolution and the Grothendieck resolution. We choose a non-degenerate G-invariant bilinear form on $\mathfrak{g}$, and denote by $\mathcal{N} \subset \mathfrak{g}^*$ the image of the nilpotent cone of G under the associated isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. Then there exist natural morphisms

$$\mu : \widetilde{\mathcal{N}} \to \mathcal{N}, \quad \pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}^*$$

induced by projection on the first factor. We set $d := \dim(\mathfrak{g}) = \dim(\widetilde{\mathfrak{g}})$ and $N := \dim(\mathcal{N})/2$, and let r be the rank of G, so that we have $d = 2N + r$. The main players of this section will be the $\Bbbk$-perverse sheaves

$$\underline{\mathsf{Spr}} := \mu_! \underline{\Bbbk}_{\widetilde{\mathcal{N}}}[2N], \quad \underline{\mathsf{Groth}} := \pi_! \underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d].$$

Let $i_{\widetilde{\mathcal{N}}} : \widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathfrak{g}}$ and $i_{\mathcal{N}} : \mathcal{N} \hookrightarrow \mathfrak{g}^*$ be the inclusions; then the following diagram is cartesian:

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \xrightarrow{i_{\widetilde{\mathcal{N}}}} & \widetilde{\mathfrak{g}} \\ \downarrow{\scriptstyle \mu} & \square & \downarrow{\scriptstyle \pi} \\ \mathcal{N} & \xrightarrow{f} & \mathfrak{g}^*. \end{array}$$

Hence using the base change isomorphism $i_{\mathcal{N}}^* \pi_! \cong \mu_! i_{\widetilde{\mathcal{N}}}^*$ we obtain a canonical isomorphism

$$i_{\mathcal{N}}^* \underline{\mathsf{Groth}} \cong \underline{\mathsf{Spr}}[r]. \tag{3.1}$$

3.2. **Statement and the Springer correspondence.** Let A be a connected closed subgroup of $G \times \mathbb{C}^\times$ which satisfies the following condition:

$$\mathsf{H}^i_A(\mathrm{pt};\Bbbk) = 0 \qquad \text{if } i \text{ is odd.} \tag{3.2}$$

Note that this condition is automatically satisfied if the reductive part A_{red} of A is a torus, or more generally if torsion primes for A_{red} are invertible in $\Bbbk$; see [Bo].

The group $G \times \mathbb{C}^\times$ (hence also its subgroup A) acts on $\mathcal{N}$ and on $\mathfrak{g}^*$, where G acts by the coadjoint action and $\mathbb{C}^\times$ acts by $x \cdot \xi = x^{-2}\xi$ for $x \in \mathbb{C}^\times$ and $\xi \in \mathfrak{g}^*$.

Isomorphism (3.1) holds in the equivariant derived category $\mathscr{D}^{\mathrm{b}}_A(\mathcal{N},\Bbbk)$, so that the functor $i_{\mathcal{N}}^*[-r]$ induces a morphism

$$\varphi : \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}},\underline{\mathsf{Groth}}) \to \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}},\underline{\mathsf{Spr}}).$$

The main result of this note is the following.

Theorem 3.1. *Assume that condition* (3.2) *is satisfied. Then the morphism* φ *is an isomorphism.*

Consider in particular the case $A=\{1\}$, and denote by $\mathsf{Perv}_G(\mathcal{N},\Bbbk) \subset \mathscr{D}^{\mathrm{b}}(\mathcal{N},\Bbbk)$ the category of G-equivariant perverse sheaves sheaves on $\mathcal{N}$. Let also W be the Weyl group of (G,T), and denote by $\mathsf{Rep}(W,\Bbbk)$ the category of $\Bbbk$-representations of W. There exists a natural action of W on $\underline{\mathsf{Groth}}$ (induced by the natural W-action on the regular semi-simple part of $\widetilde{\mathfrak{g}}$), which induces a ring isomorphism

$$r : \Bbbk[W] \xrightarrow{\sim} \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}}).$$

see e.g. [AHJR, §3.4]. Composing r with φ (or more precisely its restriction to degree 0 morphisms) we obtain a ring morphism

$$\iota : \Bbbk[W] \to \mathrm{End}_{\mathsf{Perv}_G(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}})$$

(or in other words an action of W on $\underline{\mathsf{Spr}}$ "by restriction"). The following is an immediate corollary of Theorem 3.1. This result is well known (and was first proved in [BM]) in the case $\Bbbk$ is a field of characteristic 0. It was already proved in this level of generality, using completely different methods, in [AHJR].

Corollary 3.2. *The morphism* ι *is an isomorphism.*

Note that the arguments in [AHJR, §5.1] show that Corollary 3.2 (together with the fact that $\underline{\mathsf{Spr}}$ is a projective object of $\mathsf{Perv}_G(\mathcal{N},\Bbbk)$, as proved in [AHR, Proposition 7.10]) implies that there exists a "Springer correspondence over $\Bbbk$", i.e. that the assignment

$$M \mapsto \mathrm{Hom}_{\mathsf{Perv}_G(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}},M)$$

induces a bijection between isomorphism classes of simples objects M of $\mathsf{Perv}_G(\mathcal{N},\Bbbk)$ such that $\mathrm{Hom}_{\mathsf{Perv}_G(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}},M) \neq 0$ and isomorphism classes of simple objects of the category $\mathsf{Rep}(W,\Bbbk)$ of $\Bbbk$-representations of W.

3.3. **Reinterpretation in terms of Borel–Moore homology.** To prove Theorem 3.1 we will re-interpret the morphism φ in terms of Borel–Moore homology. Consider the varieties

$$Z' := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}, \quad Z := \widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}.$$

We observe that we also have $Z' := \widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}}$. In other words the following diagram is cartesian, where all maps are the natural inclusions:

$$\begin{array}{ccc} Z' & \longrightarrow & \widetilde{\mathfrak{g}} \times \widetilde{\mathcal{N}} \\ \downarrow & \square & \downarrow \\ Z & \longrightarrow & \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}. \end{array}$$

Hence the construction of §2.5 provides a "restriction with supports" morphism

$$\mathsf{Res} := \mathsf{RS}(Z, \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}} \times \widetilde{\mathcal{N}}) : \mathsf{H}^A_{\bullet}(Z;\Bbbk) \to \mathsf{H}^A_{\bullet-2r}(Z';\Bbbk).$$

Proposition 3.3. *There exist canonical isomorphisms*

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}},\underline{\mathsf{Groth}}) \cong \mathsf{H}^A_{2d-\bullet}(Z;\Bbbk), \quad \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}},\underline{\mathsf{Spr}}) \cong \mathsf{H}^A_{4N-\bullet}(Z';\Bbbk),$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}},\underline{\mathsf{Groth}}) & \xrightarrow{\sim} & \mathsf{H}^A_{2d-\bullet}(Z;\Bbbk) \\
\varphi \downarrow & & \downarrow \mathsf{Res} \\
\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}},\underline{\mathsf{Spr}}) & \xrightarrow{\sim} & \mathsf{H}^A_{4N-\bullet}(Z';\Bbbk).
\end{array}$$

Proof. By [MR2, Lemma A.4.6], the morphism

$$\underline{\mathsf{Groth}} \xrightarrow{(\mathrm{Adj})} i_{\mathcal{N}*}i^*_{\mathcal{N}}\underline{\mathsf{Groth}} \underset{\sim}{\xrightarrow{(3.1)}} i_{\mathcal{N}*}\underline{\mathsf{Spr}}[r]$$

coincides with the morphism

$$\underline{\mathsf{Groth}} = \pi_!\underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}[d] \xrightarrow{(\mathrm{Adj})} \pi_! i_{\widetilde{\mathcal{N}}*}i^*_{\widetilde{\mathcal{N}}}\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d] \cong \pi_! i_{\widetilde{\mathcal{N}}*}\underline{\Bbbk}_{\widetilde{\mathcal{N}}}[d] \cong i_{\mathcal{N}*}\underline{\mathsf{Spr}}[r]$$

where the final isomorphism is induced by $\pi_! i_{\widetilde{\mathcal{N}}*} = \pi_* i_{\widetilde{\mathcal{N}}*} \underset{\sim}{\xrightarrow{(\mathrm{Co})}} i_{\mathcal{N}*}\mu_* = i_{\mathcal{N}*}\mu_!$.

By adjunction (and isomorphism (3.1)) there exists an isomorphism

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}},\underline{\mathsf{Spr}}) \cong \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}}, i_{\mathcal{N}*}\underline{\mathsf{Spr}}[r])$$

which identifies the morphism φ with the morphism

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}},\underline{\mathsf{Groth}}) \to \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}}, i_{\mathcal{N}*}\underline{\mathsf{Spr}}[r]) \tag{3.3}$$

induced by the morphism $\underline{\mathsf{Groth}} \to i_{\mathcal{N}*}\underline{\mathsf{Spr}}[r]$ considered above.

Now in [MR2, §1.3] we recall (following [CG]) the construction of canonical isomorphisms

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}},\underline{\mathsf{Groth}}) \cong \mathsf{H}^A_{2d-\bullet}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}};\Bbbk),$$
$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}}, i_{\mathcal{N}*}\underline{\mathsf{Spr}}) \cong \mathsf{H}^A_{4N-\bullet}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}};\Bbbk)$$

such that the following diagram commutes (see [MR2, Proposition 2.2.1(1)]):

$$\begin{array}{ccc}
\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}},\underline{\mathsf{Groth}}) & \xrightarrow{(3.3)} & \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\mathrm{b}}_A(\mathfrak{g}^*,\Bbbk)}(\underline{\mathsf{Groth}}, i_{\mathcal{N}*}\underline{\mathsf{Spr}}) \\
\wr\downarrow & & \downarrow\wr \\
\mathsf{H}^A_{2d-\bullet}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}};\Bbbk) & \xrightarrow{\mathsf{RS}(Z,\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}\times\widetilde{\mathcal{N}})} & \mathsf{H}^A_{4N-\bullet}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}};\Bbbk).
\end{array}$$

The proposition follows. □

Using Proposition 3.3 we see that Theorem 3.1 follows from the following result, which will be proved in §3.5.

Theorem 3.4. *Assume that condition* (3.2) *is satisfied. Then the morphism* Res *is an isomorphism.*

Remark 3.5. (1) We have explained in §3.2 the interpretation of Theorem 3.4 in terms of the Springer correspondence. But there are also other cases where $\mathsf{H}^A_{\bullet}(Z';\Bbbk)$ can be described explicitly, at least in the case $\Bbbk = \mathbb{C}$. For instance it is proved in [DR] that $\mathsf{H}_{\bullet}(Z';\mathbb{C})$ is isomorphic to the smash product $\mathbb{C}[W]\#C_{\mathbb{C}}$ where $C_{\mathbb{C}} = \mathrm{S}(\mathfrak{t})/\mathrm{S}(\mathfrak{t})^W_+$ is the coinvariant algebra of

W, and in [Ka, Theorem 3.1] that $\mathsf{H}^G_\bullet(Z';\mathbb{C})$ is isomorphic to the smash product $\mathbb{C}[W]\#\mathrm{S}(\mathfrak{t})$. It is also known that $\mathsf{H}^{G\times\mathbb{C}^\times}_\bullet(Z';\mathbb{C})$ is isomorphic to Lusztig's graded affine Hecke algebra, see [L1, L2].

(2) See [MR1, Lemma 5.2] for a sketch of proof (based on the same ideas) of a similar claim in equivariant K-theory.

3.4. **A preliminary lemma.** Consider the (Bruhat) decomposition of $\mathscr{B}\times\mathscr{B}$ into G-orbits:

$$\mathscr{B}\times\mathscr{B} = \bigsqcup_{w\in W} X_w$$

where $X_w := G\cdot(B/B, wB/B)$. For all $w\in W$ we denote by Z_w (respectively Z'_w) the restriction of Z (respectively Z') to the orbit X_w. Note that Z_w and Z'_w are vector bundles over X_w. The following diagram is cartesian:

$$\begin{array}{ccc} Z'_w & \xrightarrow{i'_w} & \widetilde{\mathfrak{g}}\times\widetilde{\mathcal{N}} \\ {\scriptstyle f_w}\downarrow & \square & \downarrow{\scriptstyle f} \\ Z_w & \xrightarrow{i_w} & \widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}, \end{array}$$

where all morphisms are the natural inclusions. Hence we can consider the corresponding "restriction with supports" morphism

$$\mathsf{Res}_w := \mathsf{RS}(Z_w, \widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}\times\widetilde{\mathcal{N}}) : \mathsf{H}^A_\bullet(Z_w;\Bbbk)\to\mathsf{H}^A_\bullet(Z'_w;\Bbbk).$$

Lemma 3.6. *The morphism* Res_w *is an isomorphism.*

Proof. By construction (see §2.5), the morphism Res_w is obtained by taking equivariant cohomology from a morphism

$$\underline{\mathbb{D}}_{Z_w}\to f_{w*}\underline{\mathbb{D}}_{Z'_w}[2r]$$

in $\mathscr{D}^{\mathrm{b}}_A(Z_w,\Bbbk)$. Now both Z_w and Z'_w are smooth (of respective dimension d and $2N$), so that we have canonical isomorphisms $\underline{\mathbb{D}}_{Z_w}\cong\underline{\Bbbk}_{Z_w}[2d]$, $\underline{\mathbb{D}}_{Z'_w}\cong\underline{\Bbbk}_{Z'_w}[4N]$. It follows that the preceding morphism can also be interpreted as a morphism

$$\underline{\Bbbk}_{Z_w}[2d]\to f_{w*}\underline{\Bbbk}_{Z'_w}[2d] \qquad (3.4)$$

in $\mathscr{D}^{\mathrm{b}}_A(Z_w,\Bbbk)$. We claim that (3.4) is the shift by $2d$ of the morphism $\underline{\Bbbk}_{Z_w}\xrightarrow{(\mathrm{Adj})} f_{w*}\underline{\Bbbk}_{Z'_w}$ induced by adjunction (f_w^*, f_{w*}). This will imply the lemma: indeed the following diagram commutes, where all morphisms are restriction in equivariant cohomology:

$$\begin{array}{ccc} \mathsf{H}^\bullet_A(Z_w;\Bbbk) & \longrightarrow & \mathsf{H}^\bullet_A(Z'_w;\Bbbk) \\ & \searrow \quad \swarrow & \\ & \mathsf{H}^\bullet_A(X_w;\Bbbk). & \end{array}$$

(Here X_w is considered as the zero-section in Z_w and Z'_w.) By our claim the morphism on the upper line identifies in the natural way with Res_w (up to changing the grading), and the other morphisms are isomorphisms since Z_w and Z'_w are vector bundles over X_w; the invertibility of Res_w follows.

By compatibility of all our constructions with forgetful functors (see [AHR, §B.10.1 & Lemma B.11]), it is enough to prove the claim in the case $A = G\times\mathbb{C}^\times$.

Now let

$$\mathscr{Z}' := \{(\xi,(\eta,gB)) \in (\mathfrak{g}/\mathfrak{b})^* \times \widetilde{\mathcal{N}} \mid \xi = \eta\}, \quad \mathscr{Z} := \{(\xi,(\eta,gB)) \in (\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathfrak{g}} \mid \xi = \eta\},$$

so that we have $Z' = G \times^B \mathscr{Z}'$, respectively $Z = G \times^B \mathscr{Z}$, as subvarieties of $G \times^B \big((\mathfrak{g}/\mathfrak{b})^* \times \widetilde{\mathcal{N}}\big) \cong \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$, respectively $G \times^B \big((\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathfrak{g}}\big) \cong \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$. We define $\mathscr{Z}'_w$ and $\mathscr{Z}_w$ as the restrictions of $\mathscr{Z}'$ and $\mathscr{Z}$ to $\mathscr{X}_w := BwB/B \subset \mathscr{B}$, and denote by $g_w : \mathscr{Z}'_w \hookrightarrow \mathscr{Z}_w$ the inclusion. Then we have an "induction equivalence" (see [BL, §2.6.3] or [AHR, §B.17])

$$\mathscr{D}^{\mathrm{b}}_G(Z_w, \Bbbk) \cong \mathscr{D}^{\mathrm{b}}_B(\mathscr{Z}_w, \Bbbk),$$

and the forgetful functor

$$\mathscr{D}^{\mathrm{b}}_B(\mathscr{Z}_w, \Bbbk) \to \mathscr{D}^{\mathrm{b}}_T(\mathscr{Z}_w, \Bbbk)$$

is fully faithful (see [BL, Theorem 3.7.3]). As these functors commute with adjunction morphisms and base change, to prove the claim about morphism (3.4) it is enough to prove that the morphism

$$\underline{\mathbb{D}}_{\mathscr{Z}_w} \to g_{w*}\underline{\mathbb{D}}_{\mathscr{Z}'_w}[2r] \tag{3.5}$$

in $\mathscr{D}^{\mathrm{b}}_T(\mathscr{Z}_w, \Bbbk)$ obtained by the constructions of §2.5 from the cartesian diagram

$$\begin{array}{ccc} \mathscr{Z}'_w & \xrightarrow{k'_w} & (\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathcal{N}} \\ {\scriptstyle g_w}\downarrow & \Box & \downarrow{\scriptstyle g} \\ \mathscr{Z}_w & \xrightarrow{k_w} & (\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathfrak{g}} \end{array}$$

(where all morphisms are natural inclusions) coincides, via the canonical isomorphisms $\underline{\mathbb{D}}_{\mathscr{Z}_w} \cong \underline{\Bbbk}_{\mathscr{Z}_w}[2d-2N]$ and $\underline{\mathbb{D}}_{\mathscr{Z}'_w} \cong \underline{\Bbbk}_{\mathscr{Z}'_w}[2N]$, with a shift of the morphism $\underline{\Bbbk}_{\mathscr{Z}_w} \to g_{w*}\underline{\Bbbk}_{\mathscr{Z}'_w}$ induced by the adjunction (g_w^*, g_{w*}).

Consider the projections

$$p' : (\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}, \quad p : (\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}.$$

Then $p \circ k_w$ and $p' \circ k'_w$ are locally closed inclusions, and the following diagram is cartesian:

$$\begin{array}{ccc} \mathscr{Z}'_w & \xrightarrow{p' \circ k'_w} & \widetilde{\mathcal{N}} \\ {\scriptstyle g_w}\downarrow & \Box & \downarrow{\scriptstyle i_{\widetilde{\mathcal{N}}}} \\ \mathscr{Z}_w & \xrightarrow{p \circ k_w} & \widetilde{\mathfrak{g}}. \end{array} \tag{3.6}$$

Moreover, the morphism

$$\underline{\mathbb{D}}_{(\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathfrak{g}}} \xrightarrow{\text{(Adj)}} g_* g^* \underline{\mathbb{D}}_{(\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathfrak{g}}} \xrightarrow[\sim]{(2.1)} g_* \underline{\mathbb{D}}_{(\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathcal{N}}}[2r]$$

coincides with the composition

$$\underline{\mathbb{D}}_{(\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathfrak{g}}} \cong p^! \underline{\mathbb{D}}_{\widetilde{\mathfrak{g}}} \xrightarrow{\text{(Adj)}} p^! i_{\widetilde{\mathcal{N}}*} i^*_{\widetilde{\mathcal{N}}} \underline{\mathbb{D}}_{\widetilde{\mathfrak{g}}} \xrightarrow[\sim]{\text{(BC)}} g_* p'^! i^*_{\widetilde{\mathcal{N}}} \underline{\mathbb{D}}_{\widetilde{\mathfrak{g}}}$$
$$\xrightarrow[\sim]{(2.1)} g_* p'^! \underline{\mathbb{D}}_{\widetilde{\mathcal{N}}}[2r] \cong g_* \underline{\mathbb{D}}_{(\mathfrak{g}/\mathfrak{u})^* \times \widetilde{\mathcal{N}}}[2r].$$

Using the compatibility of the base change isomorphism with composition (see [AHR, Lemma B.7(c)]) we deduce that our morphism (3.5) coincides with the similar morphism defined using the cartesian square (3.6).

Now consider the open subset $\mathscr{U}_w := wU^- B/B \subset \mathscr{B}$, and let $\widetilde{\mathscr{U}_w}$, respectively $\widetilde{\mathscr{U}'_w}$, be the restriction of $\widetilde{\mathfrak{g}}$, respectively $\widetilde{\mathcal{N}}$, to $\mathscr{U}_w$. Let also

$$U_w^+ := \prod_{\alpha>0, w^{-1}\alpha<0} U_\alpha \subset U, \quad U_w^- := \prod_{\alpha<0, w^{-1}\alpha<0} U_\alpha \subset U^-$$

(where we have fixed an arbitrary order on each set of roots). Then $\mathscr{X}_w \subset \mathscr{U}_w$, and we have an isomorphism $\mathscr{U}_w \cong U_w^- \times U_w^+$ which identifies $\mathscr{X}_w$ with $\{1\} \times U_w^+$. Since $\mathscr{Z}_w$ is included (via $p \circ k_w$) in $\widetilde{\mathscr{U}_w}$, it is easy to check that (3.5) coincides with the similar morphism defined using the the cartesian square

$$\begin{array}{ccc} \mathscr{Z}'_w & \longrightarrow & \widetilde{\mathscr{U}'_w} \\ \downarrow & \square & \downarrow \\ \mathscr{Z}_w & \longrightarrow & \widetilde{\mathscr{U}_w}. \end{array}$$

Identifying the latter diagram with the following one:

$$\begin{array}{ccc} U_w^+ \times (\mathfrak{g}/\mathfrak{u} + w\mathfrak{b})^* & \longrightarrow & U_w^- \times U_w^+ \times (\mathfrak{g}/w\mathfrak{b})^* \\ \downarrow & \square & \downarrow \\ U_w^+ \times (\mathfrak{g}/\mathfrak{u} + w\mathfrak{u})^* & \longrightarrow & U_w^- \times U_w^+ \times (\mathfrak{g}/w\mathfrak{u})^* \end{array}$$

in the natural way, we conclude using Lemma 2.1. □

3.5. **Proof of Theorem 3.4.** For any subset $I \subset W$, we denote by Z_I, respectively Z'_I, the restriction of Z, respectively Z', to $\bigsqcup_{w\in I} X_w$. Then as above the following diagram is cartesian, where all morphisms are natural inclusions:

$$\begin{array}{ccc} Z'_I & \longrightarrow & \widetilde{\mathfrak{g}} \times \widetilde{\mathcal{N}} \\ \downarrow & \square & \downarrow \\ Z_I & \longrightarrow & \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \end{array}$$

so that one can define a "restriction with supports" morphism

$$\mathsf{Res}_I : \mathsf{H}^A_\bullet(Z_I; \Bbbk) \to \mathsf{H}^A_{\bullet-2r}(Z'_I; \Bbbk).$$

We will prove by induction on $\#I$ that Res_I is an isomorphism for any $I \subset W$; the case $I = W$ will prove Theorem 3.4.

The case $\#I = 1$ is proved in Lemma 3.6. Now let $I \subset W$ be a subset of cardinality at least 2, and let $w \in I$ be maximal. Let $J = I \smallsetminus \{w\}$. We claim that the Borel–Moore homology groups $\mathsf{H}^A_\bullet(Z_I; \Bbbk)$ and $\mathsf{H}^A_\bullet(Z'_I; \Bbbk)$ are concentrated in even degrees. Indeed, this claim holds if $A = \{1\}$ since Z_I and Z'_I have affine pavings. The general case follows, using the fact that the natural spectral sequence

$$E_2^{p,q} = \mathsf{H}^p_A(\mathrm{pt}; \Bbbk) \otimes \mathsf{H}_{-q}(Z_I; \Bbbk) \Rightarrow \mathsf{H}^A_{-p-q}(Z_I; \Bbbk)$$

degenerates by a parity vanishing argument using (3.2), and similarly for Z'_I. A similar claim holds also for Z_J, Z'_J, Z_w and Z'_w; it follows that the long exact sequence of §2.4 associated with the decompositions $Z_I = Z_w \sqcup Z_J$ and $Z'_I = Z'_w \sqcup Z'_J$

are in fact collections of short exact sequences. Moreover, by Lemma 2.2 and Lemma 2.3 the following diagram commutes:

$$\begin{array}{ccccc} \mathsf{H}^A_\bullet(Z_J;\Bbbk) & \hookrightarrow & \mathsf{H}^A_\bullet(Z_I;\Bbbk) & \twoheadrightarrow & \mathsf{H}^A_\bullet(Z_w;\Bbbk) \\ \downarrow{\scriptstyle \mathsf{RS}(Z_J,\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}\times\widetilde{\mathcal{N}})} & & \downarrow{\scriptstyle \mathsf{RS}(Z_I,\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}\times\widetilde{\mathcal{N}})} & & \downarrow{\scriptstyle \mathsf{RS}(Z_w,\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}\times\widetilde{\mathcal{N}})} \\ \mathsf{H}^A_\bullet(Z'_J;\Bbbk) & \hookrightarrow & \mathsf{H}^A_\bullet(Z'_I;\Bbbk) & \twoheadrightarrow & \mathsf{H}^A_\bullet(Z'_w;\Bbbk). \end{array}$$

The right and left vertical morphisms are isomorphisms by induction; using the five-lemma it follows that the middle one is also an isomorphism, which finishes the proof.

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