Tilting exotic sheaves, parity sheaves on affine Grassmannians, and the Mirković–Vilonen conjecture

(joint work with C. Mautner)

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Notation:

$\mathbb{F}$ algebraically closed field of characteristic p.
$\mathbf{G}$ connected reductive algebraic group over $\mathbb{F}$, with simply-connected derived subgroup.
$\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$ maximal torus and Borel subgroup.
$\mathbf{U} \subset \mathbf{G}$ unipotent radical of $\mathbf{B}$.
$\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{u}$ Lie algebras of $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$, $\mathbf{U}$.

Springer resolution:

$$\widetilde{\mathcal{N}} := \{(\xi, g\mathbf{B}) \in \mathfrak{g}^* \times \mathbf{G}/\mathbf{B} \mid \xi_{|g \cdot \mathfrak{b}} = 0\} \cong \mathbb{T}^*(\mathbf{G}/\mathbf{B}).$$

Definition

$$D^{\mathbf{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) := D^{\mathrm{b}} \operatorname{Coh}^{\mathbf{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$$

for the action of $\mathbf{G} \times \mathbb{G}_{\mathrm{m}}$ defined by $(g, z) \cdot (\xi, h\mathbf{B}) = (z^{-2} g \cdot \xi, gh\mathbf{B})$.

Affine Weyl group: $W_{\mathrm{aff}} := W \ltimes X^*(\mathbf{T})$ (where $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is the Weyl group), with its natural length function ℓ.

Definition

The affine braid group $\mathbb{B}_{\mathrm{aff}}$ is the group with

- generators: T_w for $w \in W_{\mathrm{aff}}$;
- relations: $T_v T_w = T_{vw}$ for $v, w \in W_{\mathrm{aff}}$ with $\ell(vw) = \ell(v) + \ell(w)$.

Let $s \in W$ be a simple reflection, and $\mathbf{P}_s$ be the associated minimal standard parabolic subgroup.

$$\rightsquigarrow Z_s := \left\{ (\xi, g\mathbf{B}, h\mathbf{B}) \in \mathfrak{g}^* \times \mathbf{G}/\mathbf{B} \times \mathbf{G}/\mathbf{B} \;\middle|\; \begin{array}{c} g\mathbf{P}_s = h\mathbf{P}_s \\ \xi_{|g\cdot\mathfrak{b}+h\cdot\mathfrak{b}} = 0 \end{array} \right\} \subset \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}.$$

Theorem (Bezrukavnikov–R.)

There exists a unique weak right action of $\mathbb{B}_{\mathrm{aff}}$ on $D^{\mathbf{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ where:

- for a simple reflection $s \in W$, T_s acts by $R(p_s^2)_* \circ L(p_s^1)^* \langle -1 \rangle$, where $p_s^i : Z_s \to \widetilde{\mathcal{N}}$ are the projections;
- for $\lambda \in X^*(\mathbf{T})$ dominant, T_λ acts by tensor product with the line bundle $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)$.

This action "categorifies" the Kazhdan–Lusztig–Ginzburg isomorphism $K^{\mathbf{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}} \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}}) \cong \mathbb{H}_{\mathrm{aff}}$ and the corresponding action on $K^{\mathbf{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$.

Definition (Bezrukavnikov, 2006)

Set

$$D^{\leq 0} := \langle\!\langle (T_w)^{-1} \cdot \mathcal{O}_{\widetilde{\mathcal{N}}}\langle m \rangle [n],\ m \in \mathbb{Z},\ n \in \mathbb{Z}_{\geq 0} \rangle\!\rangle_{\mathrm{ext}},$$
$$D^{\geq 0} := \langle\!\langle T_w \cdot \mathcal{O}_{\widetilde{\mathcal{N}}}\langle m \rangle [n],\ m \in \mathbb{Z},\ n \in \mathbb{Z}_{\leq 0} \rangle\!\rangle_{\mathrm{ext}}.$$

Then $(D^{\leq 0}, D^{\geq 0})$ is a t-structure on $D^{\mathbf{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$, called the *exotic t-structure*.

Theorem (Bezrukavnikov, Mautner–R.)

The heart $\mathcal{E}^{\mathbf{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ of the exotic t-structure has a natural structure of graded highest weight category, with weights $X^*(\mathbf{T})$.

$\rightsquigarrow$ category of tilting objects $\mathrm{Tilt}(\mathcal{E}^{\mathbf{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}))$, with indecomposable objects $\mathcal{T}^\lambda\langle m\rangle$ for $\lambda \in X^*(\mathbf{T})$ and $m \in \mathbb{Z}$.

Proposition (Mautner–R.)

Assume that p is good for **G** and that there exists a **G**-invariant non-degenerate bilinear form on $\mathfrak{g}$. If $\lambda \in X^*(\mathbf{T})$ is dominant, then we have

$$\mathcal{T}^\lambda \cong \mathsf{T}(\lambda) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}$$

where $\mathsf{T}(\lambda)$ is the tilting **G**-module with highest weight λ.

Result closely related to the description of the tilting *perverse coherent sheaves* on the nilpotent cone of **G** (Minn-Thu-Aye).

Notation:

$G^\vee$ connected *complex* reductive group which is Langlands-dual to **G**.
$T^\vee \subset B^\vee$ maximal torus and Borel subgroup (with $X_*(T^\vee) = X^*(\mathbf{T})$).
Iwahori subgroup: $I^\vee := \mathrm{ev}_0^{-1}(B^\vee)$ where $\mathrm{ev}_0 : G^\vee(\mathbb{C}[[x]]) \to G^\vee$ sends x to 0.

Affine Grassmannian:

$$\mathcal{G}\mathrm{r} := G^\vee(\mathbb{C}((x)))/G^\vee(\mathbb{C}[[x]]) = \bigsqcup_{\lambda \in X^*(\mathbf{T})} \mathcal{G}\mathrm{r}_\lambda \qquad \text{where } \mathcal{G}\mathrm{r}_\lambda = I^\vee \cdot L_\lambda.$$

Definition

$D_{(I^\vee)}(\mathcal{G}\mathrm{r}, \mathbb{F})$: derived category of sheaves of $\mathbb{F}$-vector spaces on $\mathcal{G}\mathrm{r}$ which are constant along $I^\vee$-orbits.

Definition (Juteau–Mautner–Williamson)

An object $\mathcal{F}$ in $D_{(I^\vee)}(\mathcal{G}\mathrm{r}, \mathbb{F})$ is *even* if

$$\mathcal{H}^n(\mathcal{F}) = \mathcal{H}^n(\mathbb{D}_{\mathcal{G}\mathrm{r}}(\mathcal{F})) = 0 \quad \text{unless } n \text{ is even.}$$

An object $\mathcal{F}$ is a *parity complex* if $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{G}'[1]$ with $\mathcal{G}$ and $\mathcal{G}'$ even.

$\rightsquigarrow$ additive category $\mathrm{Parity}_{(I^\vee)}(\mathcal{G}\mathrm{r}, \mathbb{F}) \subset D_{(I^\vee)}(\mathcal{G}\mathrm{r}, \mathbb{F})$.

Theorem (Juteau–Mautner–Williamson, 2009)

For any $\lambda \in X^*(\mathbf{T})$, there exists a unique indecomposable parity complex $\mathcal{E}_\lambda$ supported on $\overline{\mathcal{G}\mathrm{r}_\lambda}$ and such that $(\mathcal{E}_\lambda)_{|\mathcal{G}\mathrm{r}_\lambda} \cong \underline{\mathbb{F}}_{\mathcal{G}\mathrm{r}_\lambda}[\dim \mathcal{G}\mathrm{r}_\lambda]$. Moreover, any indecomposable parity complex is isomorphic to $\mathcal{E}_\mu[n]$ for some $\mu \in X^*(\mathbf{T})$ and $n \in \mathbb{Z}$.

Spherical (i.e. $G^\vee(\mathbb{C}[[x]])$-constructible) indecomposable parity complexes: $\mathcal{E}_\lambda[n]$ where λ is *antidominant*.

Geometric Satake equivalence (Lusztig, Ginzburg, Mirković–Vilonen): equivalence of abelian tensor categories

$$\mathbb{S}_\mathbb{F} : \mathsf{Perv}_{(G^\vee(\mathbb{C}[\![x]\!]))}(\mathcal{G}r, \mathbb{F}) \xrightarrow{\sim} \mathsf{Rep}(\mathbf{G}).$$

Proposition (Juteau–Mautner–Williamson)

If, for some $\lambda \in X^*(\mathbf{T})$ antidominant, the object $\mathcal{E}_\lambda$ is perverse, then

$$\mathbb{S}_\mathbb{F}(\mathcal{E}_\lambda) \cong \mathsf{T}(w_0\lambda).$$

Theorem (Juteau–Mautner–Williamson, 2009)

If p is bigger than the following bounds:

A_n	B_n, D_n	C_n	G_2, F_4, E_6	E_7	E_8
1	2	n	3	19	31

then $\mathcal{E}_\lambda$ is perverse for all antidominant λ.

Theorem (Mautner–Riche, 2015)

Assume that

- **G** is a product of groups $\mathrm{GL}_n(\mathbb{F})$ and of quasi-simple simply connected groups;
- p is very good for each quasi-simple factor of **G**.

Then there exists an equivalence of additive categories

$$\Theta : \mathsf{Parity}_{(I^\vee)}(\mathcal{G}r, \mathbb{F}) \xrightarrow{\sim} \mathsf{Tilt}(\mathcal{E}^{\mathbf{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}))$$

such that

- $\Theta \circ [1] \cong \langle -1 \rangle \circ \Theta$;
- $\Theta(\mathcal{E}_\lambda) \cong \mathcal{T}^{-\lambda}$.

Corollary

Assume that p is good for **G**. Then $\mathcal{E}_\lambda$ is perverse for any λ antidominant.

Main ideas of the proof:

- use a "deformation" of the picture;
- replace $\mathbb{F}$ by a finite localization $\mathfrak{R}$ of $\mathbb{Z}$;
- describe both sides in terms of some "Soergel bimodules."

Constructible side (cf. Soergel, Ginzburg):

Proposition (Bezrukavnikov–Finkelberg, 2008)

There exists a canonical graded $\mathfrak{R}$-algebra morphism

$$\mathcal{O}(\mathfrak{t}^*_{\mathfrak{R}} \times_{\mathfrak{t}^*_{\mathfrak{R}}/W} \mathbb{T}(\mathfrak{t}^*_{\mathfrak{R}}/W)) \to \mathsf{H}^\bullet_{I^\vee}(\mathcal{G}r; \mathfrak{R})$$

which is an "isomorphism up to torsion."

$\rightsquigarrow$ fully-faithful functor

$$\mathsf{H}^\bullet_{I^\vee}(\mathcal{G}r, -) : \mathsf{BSParity}_{I^\vee}(\mathcal{G}r, \mathfrak{R}) \to \mathsf{Mod}^{\mathrm{gr}}\big(\mathcal{O}(\mathfrak{t}^*_{\mathfrak{R}} \times_{\mathfrak{t}^*_{\mathfrak{R}}} \mathbb{T}(\mathfrak{t}^*_{\mathfrak{R}}/W))\big)$$

where $\mathsf{BSParity}_{I^\vee}(\mathcal{G}r, \mathfrak{R})$ is a certain category of "Bott–Samelson" $I^\vee$-equivariant parity complexes on $\mathcal{G}r$.

Coherent side (cf. Bezrukavnikov–Finkelberg, Dodd):
One needs to replace $\widetilde{\mathcal{N}}$ by the *Grothendieck resolution*

$$\widetilde{\mathfrak{g}} := \{(\xi, g\mathbf{B}) \in \mathfrak{g}^* \times \mathbf{G}/\mathbf{B} \mid \xi_{|g\cdot\mathfrak{u}} = 0\},$$

or rather its version $\widetilde{\mathfrak{g}}_{\mathfrak{R}}$ over $\mathfrak{R}$.
Under the assumptions of the Main Theorem, one can construct a "Kostant section" $\widetilde{\mathcal{S}} \subset \widetilde{\mathfrak{g}}_{\mathfrak{R}}$. Let $\widetilde{\mathbf{I}}_{\mathcal{S}}$ be the restriction to $\widetilde{\mathcal{S}}$ of the "universal centralizer group scheme" associated with the $\mathbf{G}_{\mathfrak{R}}$-action on $\widetilde{\mathfrak{g}}_{\mathfrak{R}}$.

Proposition

The group scheme $\widetilde{\mathbf{I}}_{\mathcal{S}}$ over $\widetilde{\mathcal{S}}$ is commutative and smooth, and its Lie algebra is isomorphic to $\mathcal{O}(\mathfrak{t}^*_{\mathfrak{R}}) \otimes_{\mathcal{O}(\mathfrak{t}^*_{\mathfrak{R}}/W)} \Omega(\mathfrak{t}^*_{\mathfrak{R}}/W)$.

$\rightsquigarrow$ fully-faithful "Kostant–Whittaker reduction" functor

$$\kappa : \mathrm{BSTilt}(\widetilde{\mathfrak{g}}_{\mathfrak{R}}) \to \mathrm{Mod}^{\mathrm{gr}}\big(\mathcal{O}(\mathfrak{t}^*_{\mathfrak{R}} \times_{\mathfrak{t}^*_{\mathfrak{R}}/W} \mathbb{T}(\mathfrak{t}^*_{\mathfrak{R}}/W))\big)$$

where $\mathrm{BSTilt}(\widetilde{\mathfrak{g}}_{\mathfrak{R}})$ is some category of Bott–Samelson "deformed tilting exotic sheaves" (cf. work of C. Dodd).

For $\lambda \in X^*(\mathbf{T})$ dominant, set $\mathcal{G}r^\lambda := G^\vee(\mathbb{C}[[x]]) \cdot L_\lambda = \bigsqcup_{\mu \in W(\lambda)} \mathcal{G}r_\mu$, and denote by $j^\lambda : \mathcal{G}r^\lambda \hookrightarrow \mathcal{G}r$ the inclusion.

Definition (standard spherical perverse sheaves)

k Noetherian ring of finite global dimension.

$$\mathcal{I}_!(\lambda, k) := {}^p(j^\lambda)_!\big(\underline{k}_{\mathcal{G}r^\lambda}[\dim \mathcal{G}r^\lambda]\big).$$

Proposition (Mirković–Vilonen)

The $\mathbf{G}_k$-module $\mathbb{S}_k\big(\mathcal{I}_!(\lambda, k)\big)$ is the Weyl module with highest weight λ.

Conjecture (Mirković–Vilonen, 2000)

The cohomology modules of the stalks of the perverse sheaf $\mathcal{I}_!(\lambda, \mathbb{Z})$ are free. In other words, for any field $\Bbbk$, for any $n \in \mathbb{Z}$ and $\mu \in X^*(\mathbf{T})$, the dimension

$$\dim_{\Bbbk}\big(\mathsf{H}^n(\mathcal{I}_!(\lambda, \Bbbk)_{|L_\mu})\big)$$

is independent of $\Bbbk$.

Juteau (2008): the stalks of $\mathcal{I}_!(\lambda, \mathbb{Z})$ can have p-torsion if p is bad for $\mathbf{G}$; in particular, the Mirković–Vilonen conjecture is false.

Theorem (Achar–Rider, 2013)

If the indecomposable parity complexes $\mathcal{E}_\lambda$ over $\overline{\mathbb{F}}_p$ are perverse for any λ antidominant, then the stalks of $\mathcal{I}_!(\lambda, \mathbb{Z})$ have no p-torsion.

$\rightsquigarrow$ the stalks of $\mathcal{I}_!(\lambda, \mathbb{Z})$ have no p-torsion if p is bigger than the bounds in the Juteau–Mautner–Williamson theorem:

A_n	B_n, D_n	C_n	G_2, F_4, E_6	E_7	E_8
1	2	n	3	19	31

Corollary

The stalks of $\mathcal{I}_!(\lambda, \mathbb{Z})$ have no p-torsion if p is good for $\mathbf{G}$.

Definition (Achar–R.)

The *modular mixed derived category* is the triangulated category

$$D^{\mathrm{mix}}_{(I^\vee)}(\mathcal{G}r, \mathbb{F}) := K^{\mathrm{b}}\,\mathsf{Parity}_{(I^\vee)}(\mathcal{G}r, \mathbb{F}).$$

It can be endowed with a "Tate twist" autoequivalence $\langle 1 \rangle$ and a "perverse" t-structure whose heart $\mathsf{Perv}^{\mathrm{mix}}_{(I^\vee)}(\mathcal{G}r, \mathbb{F})$ is a graded quasi-hereditary category with weights $X^*(\mathbf{T})$.

Theorem (Achar–Rider 2014, Mautner–R. 2015)

Under the assumptions of the Main Theorem, there exists an equivalence of triangulated categories

$$\Phi : D^{\mathrm{mix}}_{(I^\vee)}(\mathcal{G}r, \mathbb{F}) \xrightarrow{\sim} D^{\mathbf{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$$

which satisfies $\Phi \circ \langle 1 \rangle \cong \langle 1 \rangle[1] \circ \Phi$ and

$$\Phi(\Delta^{\mathrm{mix}}_\lambda) \cong \Delta^{-\lambda}, \quad \Phi(\nabla^{\mathrm{mix}}_\lambda) \cong \nabla^{-\lambda}, \quad \Phi(\mathcal{E}^{\mathrm{mix}}_\lambda) \cong \mathcal{T}^{-\lambda}.$$