Pramod N. Achar
Simon Riche

CENTRAL SHEAVES ON AFFINE FLAG VARIETIES

Pramod N. Achar

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.

E-mail : pramod.achar@math.lsu.edu

Simon Riche

Université Clermont Auvergne, CNRS, LMBP, F-63000 Clermont-Ferrand, France.

E-mail : simon.riche@uca.fr

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CONTENTS

INTRODUCTION

0.1. Overview

0.1.1. Langlands duality philosophy. — Let G be a complex connected reductive algebraic group. Let $\Bbbk$ be a commutative noetherian ring of finite global dimension, and let $G^\vee_\Bbbk$ be the (split) reductive algebraic group over $\Bbbk$ that is Langlands dual to G. Broadly speaking, the "Langlands duality philosophy" suggests that various algebraic objects related to $G^\vee_\Bbbk$ (representations, coherent sheaves, etc.) should be related to topological objects related to G (especially perverse or constructible sheaves on partial affine flag varieties).(1) These ideas emerged (at the level of combinatorics) in the representation theory of reductive groups over local fields, and their geometric incarnations have a relatively long history in the case where $\Bbbk = \mathbb{C}$. However, recent developments in geometric representation theory (see e.g. [**FM, RW1, AR2, RW2**]) have highlighted the importance of this philosophy for general $\Bbbk$ (or at least in the case when $\Bbbk$ is a field of positive characteristic) as well.

This book, which may be regarded as a sequel to [**BR**], is part of a project to present an exposition of some of the key applications of this philosophy in geometric representation theory.

0.1.2. Geometric Satake equivalence. — The starting point of all of these constructions is the *geometric Satake equivalence*, whose proof was reviewed in detail by P. Baumann and the second author in [**BR**]. (Its main ingredients are recalled in Chapter 1 of the present book.) Consider some group G as in §0.1.1, and let $T \subset G$ be a maximal torus. We consider the loop group $\mathrm{L}G$ and the arc group L^+G, and the affine Grassmannian Gr_G defined as the quotient $\mathrm{L}G/\mathrm{L}^+G$. (See §1.2.1 below for a more formal definition.) Given a commutative noetherian ring $\Bbbk$ of finite global dimension, one can consider the abelian category

$$\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$$

(1)It is expected by some that there should exist more symmetric versions of this duality, in which one does not have to distinguish between an "algebraic side" and a "topological side"; this idea has not been realized concretely so far, however.

of L^+G-equivariant $\Bbbk$-perverse sheaves on Gr_G, namely the heart of the perverse t-structure on the L^+G-equivariant derived category $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. A standard construction (called "convolution") for categories of sheaves on (partial) flag varieties provides a monoidal product $\star^{\mathrm{L}^+G}$ on $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, and a much more specific analysis in this case shows that the bifunctor

$$(\mathscr{A}, \mathscr{B}) \mapsto \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B} := {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B})$$

defines a monoidal structure on $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. Out of these data, the geometric Satake equivalence as considered in [**MV2**] provides a canonical reductive group scheme $G^\vee_\Bbbk$ over $\Bbbk$, an equivalence of monoidal categories

$$(\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star_0^{\mathrm{L}^+G}) \xrightarrow{\sim} (\mathsf{Rep}(G^\vee_\Bbbk), \otimes_\Bbbk) \tag{0.1.1}$$

(where the right-hand side is the category of algebraic representations of $G^\vee_\Bbbk$ on finitely generated $\Bbbk$-modules) and a canonical split maximal torus $T^\vee_\Bbbk \subset G^\vee_\Bbbk$ such that the root datum of $(G^\vee_\Bbbk, T^\vee_\Bbbk)$ is dual to that of (G, T); in rough terms, $G^\vee_\Bbbk$ is Langlands dual to G.

Part of this claim is the assertion that we have a canonical identification between the cocharacter lattice $X_*(T)$ of T and the character lattice $X^*(T^\vee_\Bbbk)$ of $T^\vee_\Bbbk$. The choice of a Borel subgroup $B \subset G$ containing T determines a system of positive roots for G, and hence a subset $X_*^+(T) \subset X_*(T)$ of dominant cocharacters. The L^+G-orbits on Gr_G are naturally parametrized by $X_*^+(T)$, and each orbit is simply connected; in case $\Bbbk$ is a field, we deduce that the simple objects in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ are naturally parametrized by $X_*^+(T)$. On the other hand, under the identification $X_*(T) = X^*(T^\vee_\Bbbk)$, $X_*^+(T)$ is the subset of dominant characters of $T^\vee_\Bbbk$ for an appropriate choice of positive roots; if $\Bbbk$ is a field, this subset therefore also parametrizes the simple objects in $\mathsf{Rep}(G^\vee_\Bbbk)$. In this case, these parametrizations of simple objects on both sides of (0.1.1) match under the geometric Satake equivalence.

This story also has a version for étale sheaves. In this setting, G is defined over an algebraically closed field $\mathbb{F}$; we choose a prime number ℓ invertible in $\mathbb{F}$; and we work with étale $\Bbbk$-sheaves on Gr_G, where $\Bbbk$ may be a finite field of characteristic ℓ, a finite extension of $\mathbb{Q}_\ell$, or the ring of integers of such an extension. (It is also possible to take $\Bbbk$ to be an algebraic closure of either $\mathbb{Q}_\ell$ or $\mathbb{F}_\ell$.) In case $\mathrm{char}(\mathbb{F}) > 0$, since G is in fact defined over the prime subfield of $\mathbb{F}$, one can speak of the "trace of Frobenius" on stalks of various perverse sheaves.

0.1.3. History of the geometric Satake equivalence. — Let us first briefly discuss the history of the geometric Satake equivalence.

0.1.3.1. *Lusztig's work.* — The Satake isomorphism [**Sat**] relates the spherical Hecke algebra of a p-adic group and the representation ring of the Langlands dual group. Since the spherical Hecke algebra is (for local fields of equal characteristic) a space of functions on the points of an ind-scheme, in view of Grothendieck's "faisceaux–fonctions" dictionary, it might seem natural to expect that this statements admits a "geometric" (or "categorical") version relating a certain category of sheaves on this ind-scheme and the category of representations of the Langlands dual group.[(2)] The first clear indication that this isomorphism could have such a geometric version is to be found in work of Lusztig [**Lu1**].[(3)] The main results of this paper are concerned with the Kazhdan–Lusztig basis of the Hecke algebra associated with the affine Weyl group of a complex connected reductive group G, but this algebra is known to be a "categorical trace" of the category of Iwahori-equivariant perverse sheaves on the affine flag variety Fl_G of G, in such a way that the Kazhdan–Lusztig basis corresponds to simple perverse sheaves (with coefficients in a field of characteristic 0). These results can therefore also be interpreted in terms of perverse sheaves on Fl_G or Gr_G. With this translation, the main results of [**Lu1**] take the following form:

- [**Lu1**, Theorem 6.1] the dimension of the stalk of the intersection cohomology complex associated with the L^+G-orbit on Gr_G associated with $\lambda \in X_*^+(T)$ along the orbit associated with $\mu \in X_*^+(T)$ is the multiplicity of μ in the simple representation of the Langlands dual group of highest weight λ;
- [**Lu1**, Corollary 8.7] the convolution of two simple L^+G-equivariant perverse sheaves on Gr_G is perverse; moreover, the multiplicities of simple perverse sheaves in such a convolution product agree with the multiplicities of simple representations in the corresponding tensor product of simple representations for the dual group;
- [**Lu1**, Last line on p. 228] for any dominant coweight λ, the dimension of the total cohomology of the intersection cohomology complex associated with the orbit of a dominant cocharacter λ is the dimension of the simple representation of the dual group with highest weight λ.

This paper also explains how to construct the affine Grassmannian Gr_G in terms of some lattices in $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$ where $\mathfrak{g}$ is the Lie algebra of G; see [**Lu1**, §11]. An immediate interpretation of these results is that they identify the basis corresponding to classes of simple modules under the Satake isomorphism with the basis of characteristic functions of intersection cohomology complexes on Gr_G.

Lusztig remarked that the results of [**Lu1**] suggest an equivalence between the category of spherical perverse sheaves on Gr_G (for coefficients in a field $\Bbbk$ of characteristic 0) and the category of representations of the dual group (over $\Bbbk$), and that the total cohomology functor should correspond to the functor sending a representation to the underlying vector space. However he did not pursue further work on this subject, in the lack of a geometric interpretation of the commutativity of the tensor product of representations on the geometric side. (Note that Lusztig found much later an alternative approach to this problem which replaces geometry with the theory of Soergel bimodules, see [**Lu2**, §18.20].)

0.1.3.2. *Drinfeld's contribution.* — Some years later, and in relation with his work with Beĭlinson [**BD**], Drinfeld also considered the potential for such an equivalence of categories. He proposed defining a commutativity constraint on the category

(2) For a more detailed discussion of the relation between the "classical" Satake isomorphism and the geometric Satake equivalence, see e.g. [**Zh1**, §5.6].

(3) Some of the main results of [**Lu1**] are given a shorter proof in the later paper [**Lu3**].

$\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ based on a new description of the convolution product in terms of the *fusion product*. This construction uses a new family of schemes, now called *Beĭlinson–Drinfeld Grassmannians*, defined as certain "relative" versions of the affine Grassmannian over copies of a curve. Drinfeld explained this construction to a number of people, including V. Ginzburg. It did not appear in written form before the announcement [**MV1**] (see §0.1.3.4 below).

0.1.3.3. *Ginzburg's work.* — In [**Gi**], V. Ginzburg claimed to give the first proof of the geometric Satake equivalence (for coefficients in a field of characteristic 0). This proof relies on Lusztig's results explained in §0.1.3.1, but does *not* use Drinfeld's idea for the construction of the commutativity constraint. It proposes a different construction of this isomorphism, based on the localization theorem in equivariant cohomology. Unfortunately, this construction has a gap: it defines the isomorphism as the unique morphism satisfying an appropriate property (see the proof of [**Gi**, Proposition 2.3.1]), but there is no proof that such a morphism exists. (In later work Zhu [**Zh1**] observed that Ginzburg's condition is *not* the property that the commutativity constraint should satisfy; an appropriate sign must be added.)

The strategy of proof relies on the Tannakian formalism developed by Saavedra-Rivano [**SR**] and Deligne–Milne [**DM**]: one constructs some structures on the category $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ (essentially, in addition to the monoidal structure, a commutativity constraint and a fiber functor), and then uses these structures to apply general results that guarantee that this category must be equivalent to the category of representations of a group scheme. One then proves that this group scheme is in fact the Langlands dual group.

The preprint [**Gi**] introduced many ideas and geometric constructions that were later extremely useful in various applications of the geometric Satake equivalence, but it does not contain a complete proof of this equivalence.

0.1.3.4. *Mirković–Vilonen's work.* — The paper [**MV2**] by Mirković–Vilonen contains the first complete[(4)] proof of the geometric Satake equivalence, and in fact proves such an equivalence for any ring of coefficients which is noetherian and of finite global dimension. (These assumptions are necessary to have a nice theory of constructible derived categories of sheaves.) The proof again uses the ideas of the Tannakian formalism, but requires new ingredients to work over rings since there is no general Tannakian formalism for such coefficients. The construction of the commutativity constraint follows Drinfeld's suggestion. The main new ingredient compared to previous works is the notion of *semi-infinite orbits*, defined as orbits of the loop group of the unipotent radical of B. These orbits appeared earlier in some form in the p-adic group literature and in [**Lu1**] (in the guise of the periodic module); here they are given a geometric structure, and their relation with L^+G-orbits is studied.

The semi-infinite orbits are used in an essential way at (at least) three stages of the argument:

– in the construction of the maximal torus of the dual group;
– in the application of Braden's "hyperbolic localization theorem;"

(4) An erratum correcting some minor inaccuracies later appeared in [**MV3**].

– in the construction of the Tannakian group scheme over rings.

They also provide "canonical" bases of weight spaces of certain representations of the dual group, in terms of certain varieties now called *Mirković–Vilonen cycles*, which have subsequently been extensively studied by various authors.

This proof was first announced in [**MV1**]. However, the proof outlined there had a gap, in the argument used to check that the group scheme constructed by the authors has reduced fibers. This gap was filled in [**MV2**] using a general result on reductive group schemes proved in the meantime by Prasad–Yu [**PY**]. The proof in [**MV2**] is written in a topological setting. The authors claim that a similar proof can be given for étale perverse sheaves; some technical details in fact have to be treated with some care, which is done in particular in [**Zh4**].

0.1.3.5. *Later contributions.* — The geometric Satake equivalence was very influential. Among the numerous contributions that have appeared since [**MV2**], let us mention the following:

– Richarz [**Rc1**] gave a new proof (for étale $\mathbb{Q}_\ell$-sheaves) that also considers the case when G is defined over a not necessarily separably closed field.
– Generalizations to the "ramified" setting—where one starts with a reductive group over $\mathbb{F}((z))$ and a model over $\mathbb{F}[[z]]$, not necessarily obtained by base change from $\mathbb{F}$—were obtained by Zhu [**Zh2**] under a technical assumption, and by Richarz [**Rc2**] in full generality (for étale $\mathbb{Q}_\ell$-sheaves in both cases).
– Versions for "mixed characteristic" affine Grassmannians were obtained by Zhu [**Zh3**] (for étale $\mathbb{Q}_\ell$-sheaves) and Yu [**Yu**] (for integral or modular coefficients).

The papers [**HR1, HR2**] by Haines–Richarz also contain a more careful analysis of the description of affine Grassmannians attached to B and T as attractors and fixed points for an action of $\mathbb{G}_\mathrm{m}$ respectively.

Finally, we would like to cite the new proof of the equivalence (for general coefficients) given recently by Fargues–Scholze in [**FS**, Chap. VI]. This proof is written in the language of spatial diamonds, but it can in fact be "translated" into the world of étale sheaves on schemes using the general theories developed in particular in [**BS**] and [**HS**]. The proof follows closely the strategy of [**MV2**]. The main new ingredient is a way of defining a "Satake category" of sheaves on the Beĭlinson–Drinfeld Grassmannians (via universally locally acyclic (ULA) sheaves and "relative" perverse sheaves), which makes the proof of the compatibility between the various structures of the Satake category more transparent. Note in passing that for integral coefficients the category considered in [**FS**] (see [**FS**, §VI.7.1]) is *not* an immediate counterpart of that considered in [**MV2**]: the definition in [**FS**] imposes a certain flatness condition, which on the representation-theoretic side corresponds to restricting to representations which are flat over $\Bbbk$.

0.1.4. Gaitsgory's central functor. — Let us now discuss the main topics that are studied in this book.

After the Satake isomorphism, another classical fact about the spherical Hecke algebra, due to Bernstein (but whose first appearance in print seems to be in [**Lu1**]), is that it is isomorphic to the center of the affine Hecke algebra. To upgrade this observation to the categorical level, one might seek a functor from the category $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ to the Iwahori-equivariant derived category of the affine flag variety Fl_G of G, which factors through the Drinfeld center of the latter category. Such a functor (known as the "central functor") was first constructed by Gaitsgory [**G1**], following suggestions of Beĭlinson, Haines and Kottwitz. (The context considered in [**G1**] is that of étale $\mathbb{Q}_\ell$-sheaves; however his construction makes sense for general coefficients, and the proofs of its properties also apply in general.) By definition, *central sheaves* are the perverse sheaves on Fl_G in the image of this functor.

Part I of the present book is an account of the theory of central sheaves (mostly following a variant of Gaitsgory's construction suggested by Heinloth [**He**] and explicitly worked out by Zhu [**Zh1**]). The key to defining the central functor is the construction of an ind-scheme $\mathbf{Gr}_G^{\mathrm{Cen}}$ over $\mathbb{A}^1$ that exhibits Fl_G as a degeneration of Gr_G: the central functor Z is then defined in terms of nearby cycles for this family. Certain subschemes of this family (the so-called *global Schubert varieties*) had previously appeared in the number theory literature, in the guise of "local models of Shimura varieties" (see, for instance, [**HN, GH**]). Indeed, central sheaves have recently found new applications in the geometry of Shimura varieties in work of Zhu [**Zh1, Zh2**], Pappas–Zhu [**PZ**] and Haines–Richarz [**HR1, HR2**] in particular.

0.1.5. Arkhipov–Bezrukavnikov equivalence(s). — Part II of the book is devoted to a study of one of the most useful applications of central sheaves in geometric representation theory, again suggested by the Langlands philosophy, and due to Arkhipov–Bezrukavnikov[**AB**]. It consists of a derived equivalence relating equivariant coherent sheaves on the Springer resolution for $G^\vee_\Bbbk$ and Iwahori–Whittaker $\Bbbk$-perverse sheaves on Fl_G. This equivalence can be seen as a categorical upgrade of the fact that both categories' Grothendieck groups are incarnations of the *antispherical module* for the affine Hecke algebra. It has found applications in several problems from representation theory (see e.g. [**Be3, BM**]), and is also the main ingredient of the construction of Bezrukavnikov's equivalence [**Be5**] relating equivariant coherent sheaves of the Steinberg variety of $G^\vee_\Bbbk$ and Iwahori-equivariant perverse sheaves on Fl_G. Besides these applications in the direction of Representation Theory, these constructions have also played a major role in some aspects of the Geometric Langlands Program. We refer to [**Ras2**] for one example (which provides a vast generalization of the equivalence from [**AB**], see [**Ras2**, Remark 1.13.1] for these aspects), and to [**Dh**] for an introduction to this subject which discusses in particular the role of the equivalences of [**AB**] and [**Be5**] in this story. For an alternative proof of the equivalence, and some other extensions, we refer to [**Do**].

The original construction of this equivalence is carried out in the setting of étale $\overline{\mathbb{Q}}_\ell$-sheaves, in order to use some specific features from Deligne's theory of weights for ℓ-adic sheaves together with the easier cohomological behavior of representations of reductive algebraic groups over fields of characteristic 0. In this book, we will see that this proof can be adapted to treat the case when G is a general linear group and $\Bbbk$ is an algebraically closed field of large enough positive characteristic (with an explicit bound depending on the rank of the general linear group under consideration). We also explain the relationship of this construction to the *exotic t-structure* on the derived category of equivariant coherent sheaves on the Springer resolution, and some variants describing the category of representations of the centralizer of a regular nilpotent element, and that of equivariant coherent sheaves on the nilpotent cone. (This first variant is stated in [**AB**, Remark 8]; the second one is due to Bezrukavnikov [**Be4**]).

0.2. Contents

Let us now review in more detail the content of each chapter of this book.

0.2.1. Part I. — Part I provides a detailed account of the construction and main properties of central sheaves.

First, in Chapter 1 we provide a brief overview of the Mirković–Vilonen proof of the geometric Satake equivalence. Detailed expositions of this proof can be found in [**BR, Zh4, Ac3**], and here we only explain the structure of this proof, some technical details involved in crucial constructions, and highlight its main features.

Chapter 2 is concerned with the geometric input for the construction of the central functor and the proofs of its main properties. Following Heinloth [**He**] and Zhu [**Zh1**], we consider a certain (non-constant) smooth affine group scheme $\mathcal{G}$ over $\mathbb{A}^1$ which satisfies the following properties:

- its restriction to $\mathbb{A}^1 \smallsetminus \{0\}$ is the constant group scheme $G \times (\mathbb{A}^1 \smallsetminus \{0\})$.
- its restriction to $\mathrm{Spec}(\mathbb{C}[\![x]\!])$ is the Iwahori group scheme $\mathcal{I}$ determined by Bruhat–Tits theory.

Then, considering a certain moduli problem involving a principal $\mathcal{G}$-bundle with a trivialization we construct the "central affine Grassmannian," an ind-scheme $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ over $\mathbb{A}^1$ which has the following properties:

- its fiber over any point in $\mathbb{A}^1 \smallsetminus \{0\}$ is Gr_G;
- fiber over 0 is Fl_G.

(In other words, $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ can be thought as a degeneration of Gr_G to Fl_G.) Using nearby cycles associated with this family we obtain the functor

$$\mathsf{Z} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$$

from the L^+G-equivariant derived category of $\Bbbk$-sheaves on Gr_G to the I-equivariant derived category of $\Bbbk$-sheaves on Fl_G. (Here, I is the Iwahori subgroup of the arc group L^+G associated with G.)

Gaitsgory's original construction of the functor Z uses a different ind-scheme, whose fiber over a point in $\mathbb{A}^1 \smallsetminus \{0\}$ is the product of Gr_G with the flag variety of G. (This suggests that $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ can be identified with a closed sub-ind-scheme in Gaitsgory's version; this question will not be studied here.) The properties of this ind-scheme are very similar to those of $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$, but for technical reasons it turns out to be easier to work with the latter ind-scheme. (The difference appears in particular when considering the equivariant structure on central sheaves, and the construction of the "centrality" isomorphism; see Remark 3.2.4 for more comments.)

As preparation for the proof of some of this functor's main properties, we also explain in this chapter the construction of "iterated" variants of this ind-scheme, defined in terms of a moduli problem involving *several* principal $\mathcal{G}$-bundles with trivializations. This construction is closely related with Beĭlinson–Drinfeld's factorization Grassmannian, some instances of which already appear in the proof of the geometric Satake equivalence. Some particular cases of this construction are considered in [**Zh1**], but its general treatment is new (to the best of our knowledge). In Section 2.4 we finally consider the nearby cycles functors associated with these ind-schemes (which exhibit the functor Z as a special case), and prove a first general compatibility property with respect to the convolution product.

Chapter 3 is devoted to the proof of the fact that Z is monoidal and factors through the Drinfeld center of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$. This involves the following ingredients:

1. constructing a canonical "centrality" isomorphism
$$\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F} \xrightarrow{\sim} \mathscr{F} \star^I \mathsf{Z}(\mathscr{A})$$
for any $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{F}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$;
2. constructing a "monoidality" isomorphism
$$\mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \xrightarrow{\sim} \mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B})$$
for any $\mathscr{A}, \mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$;
3. and finally proving that these isomorphisms satisfy various compatibility properties with each other and with some structures we have on $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$.

All of these results are due to Gaitsgory: items (1) and (2) were obtained in [**G1**], while (3) is proved in [**G2**]. In particular, one of the properties from (3) involves comparing the isomorphism from (1) in the special case when $\mathscr{F} = \mathsf{Z}(\mathscr{B})$ for some $\mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ with the image under Z of the commutativity constraint in the Satake category; this part of the proof requires the use of a version of "nearby cycles over a 2-dimensional base."

In passing, we remark that most of the constructions of this chapter have obvious variants in the setting where the non-constant group scheme $\mathcal{G}$ is replaced by the constant group scheme $G \times \mathbb{A}^1$; these variants allow one to reinterpret various constructions from the proof of the geometric Satake equivalence in terms of nearby cycles.

Chapter 4 is devoted to the proof of another property of central sheaves, namely that they possess a canonical filtration whose subquotients are "Wakimoto sheaves." This property is due to Arkhipov–Bezrukavnikov, and was obtained in the course of the construction of their equivalence in [**AB**]. Its proof is essentially formal, and based on the observation that, for $\mathscr{A}$ in the Satake category, convolution with the perverse sheaf $\mathsf{Z}(\mathscr{A})$ is exact for the perverse t-structure. (This property holds unconditionally in case $\Bbbk$ is a field, and under the assumption that $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A})$ is $\Bbbk$-flat in general.) We also explain that this filtration can be used to give an alternative description of the canonical maximal torus in $G^\vee_\Bbbk$.

Finally, in Chapter 5 we describe the effect of the central functor at the "combinatorial" level, on Grothendieck groups. For ℓ-adic coefficients, we explain how to compute in terms of the Hecke algebra the composition factors in each piece of the associated graded of the weight filtration on central sheaves.

The existing literature on all of these subjects focuses on the case where $\Bbbk = \mathbb{C}$ (or $\Bbbk = \overline{\mathbb{Q}}_\ell$). It has long been known to experts that most of the arguments go through for general coefficients, but our treatment is the first which systematically allows $\Bbbk$ to be any commutative noetherian ring of finite global dimension (at least, whenever possible). In a few cases, this requires more delicate arguments than those from [**G1, Zh1, AB**].

0.2.2. Part II. — Part II explains the application of central sheaves to the construction of the Arkhipov–Bezrukavnikov equivalence.

First, in Chapter 6 we consider the original setting from [**AB**], when $\Bbbk = \overline{\mathbb{Q}}_\ell$. This equivalence involves the "Iwahori–Whittaker" category $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ of sheaves on Fl_G, whose definition is reminiscent of that of "Whittaker modules" in the representation theory of p-adic groups.(5) This category has a natural perverse t-structure, and its heart is naturally a highest-weight category. An essential ingredient in the proof is the fact that the images of central sheaves in the Iwahori–Whittaker category are *tilting objects* with respect to this highest-weight structure. Under the constructed equivalence

$$D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \cong D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\overline{\mathbb{Q}}_\ell}}(\widetilde{\mathcal{N}}_{\overline{\mathbb{Q}}_\ell}),$$

where $\widetilde{\mathcal{N}}_{\overline{\mathbb{Q}}_\ell}$ is the Springer resolution of $G^\vee_{\overline{\mathbb{Q}}_\ell}$, the central sheaf $\mathsf{Z}(\mathscr{A})$ (were $\mathscr{A}$ is in the Satake category) corresponds to the "free" coherent sheaf $V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}_{\overline{\mathbb{Q}}_\ell}}$, where V is the image of $\mathscr{A}$ under the geometric Satake equivalence, and the images of Wakimoto sheaves correspond to the natural ($G^\vee_{\overline{\mathbb{Q}}_\ell}$-equivariant) line bundles on $\widetilde{\mathcal{N}}_{\overline{\mathbb{Q}}_\ell}$ attached to coweights of G (i.e. weights of $G^\vee_{\overline{\mathbb{Q}}_\ell}$).

In Chapter 7 we discuss some complements to this topic, still in the setting where $\Bbbk = \overline{\mathbb{Q}}_\ell$:

1. we show that the transport of the perverse t-structure on $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ along the equivalence above is the *exotic t-structure* on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\overline{\mathbb{Q}}_\ell}}(\widetilde{\mathcal{N}}_{\overline{\mathbb{Q}}_\ell})$ (whose definition is in terms of a certain exceptional collection of objects);
2. we show that the category of representations of the centralizer of a regular nilpotent element in the Lie algebra of $G^\vee_{\overline{\mathbb{Q}}_\ell}$ can be described in terms of a certain quotient of the category of Iwahori-equivariant perverse sheaves on Fl_G;
3. we explain how to describe the category of equivariant coherent sheaves on the nilpotent cone of $G^\vee_{\overline{\mathbb{Q}}_\ell}$ in terms of the derived category of another quotient of the category of Iwahori-equivariant perverse sheaves on Fl_G.

All of these results are due to Bezrukavnikov (although no proof of (2) was available so far in the literature).

(5) More precisely, the definition resembles a "baby version" of the Whittaker vector condition, involving the pro-unipotent radical of an Iwahori subgroup.

Finally, in Chapter 8 we provide the first example of a modular version of the Arkhipov–Bezrukavnikov equivalence. Namely, we prove an equivalence

$$D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk) \cong D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{N}}_\Bbbk)$$

in the case when $G = \mathrm{GL}(n)$ and $\Bbbk$ is the algebraic closure of a finite field $\mathbb{F}_\ell$ with $2\ell > \binom{n}{\lfloor n/2 \rfloor}$. Adapting the constructions of [**AB**] to the setting of positive-characteristic coefficients presents several rather delicate difficulties, that we are able to overcome in this special case. Obtaining a general proof of this equivalence (under mild assumptions of the characteristic) based on different techniques is an ongoing project of the second author with R. Bezrukavnikov and L. Rider; see [**BRR, BeR3**] for first steps in this direction.

0.2.3. Part III. — This book finishes with two appendices, gathered in Part III.

First, in Chapter 9 we review some aspects of the theory of nearby cycles that are important for our constructions. In particular we explain Beĭlinson's construction of unipotent nearby cycles, and a version of "nearby cycles over a 2-dimensional base" inspired by this construction and due to Gaitsgory.

Then, in Chapter 10 we explain how to extend the well-known construction of Bernstein–Lunts' equivariant derived category to the case of group schemes over a curve. This construction (which is new, to the best of our knowledge) is used in a systematic way in our construction and study of the central functor; this framework lets us obtain stronger results (with less work!) about the equivariant behavior of this functor.

0.3. Some conventions

Given a separated scheme X of finite type over $\mathbb{C}$, and a commutative ring $\Bbbk$, by a $\Bbbk$-sheaf on X we will mean a sheaf of $\Bbbk$-modules on the topological space $X(\mathbb{C})$ of $\mathbb{C}$-points of X, endowed with the "classical" (or "analytic") topology. Of course this notion only depends on the reduced scheme X_{red} associated with X; but to lighten notation it is sometimes useful to allow non-reduced schemes also. The derived category of $\Bbbk$-sheaves on X will be denoted by $D^{\mathrm{b}}(X, \Bbbk)$, and (in case $\Bbbk$ is noetherian) the subcategory of constructible complexes will be denoted $D^{\mathrm{b}}_c(X, \Bbbk)$; see e.g. [**BBDG**, §2.2.1] or [**Ac3**]. We will usually assume that $\Bbbk$ is noetherian and of finite global dimension, so that we have the full "6-functors formalism" for these categories (in particular, the Verdier duality functor).

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PART I

GENERAL THEORY OF CENTRAL SHEAVES

CHAPTER 1

REVIEW OF THE GEOMETRIC SATAKE EQUIVALENCE

The "central sheaves" that are the main object of study in this book are produced starting from certain constructible complexes on the affine Grassmannian Gr_G associated with a connected complex reductive algebraic group G. Perhaps the most important such complexes are the L^+G-equivariant perverse sheaves, which are related to representations of the Langlands dual group via the celebrated *geometric Satake equivalence* (see §0.1.3 for references). A complete exposition of the proof of this theorem can be found in the notes [**BR**] by P. Baumann and the second author; other accounts of this construction can be found in the notes [**Zh4**] by Zhu, or in Chapter 9 of the book [**Ac3**] by the first author.

In this chapter, we review the statement of the geometric Satake equivalence, and we briefly outline the main steps in its proof. On the way to the statement, we will discuss various additional features of L^+G-equivariant perverse sheaves, or of the L^+G-equivariant derived category, including the monoidal structure on these categories. Essentially all the results we will state are taken from [**MV2**], but we will usually give references to [**BR**] for convenience. We take this opportunity to treat in details in §§1.1–1.2 some technical topics that are somewhat overlooked in [**BR**] and [**Ac3**]. We feel it is important to have these details explained somewhere, but they can be skipped at first reading.

Although the geometric Satake equivalence itself is not strictly needed for the construction of central sheaves, the monoidal structure on L^+G-equivariant complexes is essential for the main results of Chapter 3. We will also use the geometric Satake equivalence in a crucial way in Part II for the construction of the Arkhipov–Bezrukavnikov equivalence.

1.1. Some technical preliminaries

Before going into more specific constructions, let us review some very general definitions and constructions that will be essential techniques for our study.

1.1.1. Ind-schemes. — First, let us recall the definition and some basic properties of ind-schemes. Our main reference for this subject will be [**Rc4**].

1.1.1.1. *Definitions.* — We denote by AffSch the category of affine schemes, which identifies with the opposite of the category Rings of unital commutative rings via the Spec construction. Any scheme X defines a functor $\mathsf{AffSch}^{\mathrm{op}} \to \mathsf{Sets}$ via

$$T \mapsto \mathrm{Hom}(T, X),$$

and this assignment defines a fully faithful functor from the category of schemes to the category of functors $\mathsf{AffSch}^{\mathrm{op}} \to \mathsf{Sets}$; we will often identify the category of schemes with its image under this functor. (This follows from the Yoneda lemma and the fact that any scheme admits an affine open cover.) As usual, when $T = \mathrm{Spec}(R)$ for some $R \in \mathsf{Rings}$ we write $X(R) := \mathrm{Hom}(\mathrm{Spec}(R), X)$.

An *ind-scheme* is a functor

$$X : \mathsf{AffSch}^{\mathrm{op}} \to \mathsf{Sets}$$

such that there exists a filtered poset $(I, \leq)$ and an inductive system $(X_i : i \in I)$ of schemes such that[(1)]

$$X \cong \mathrm{colim}_{i \in I} X_i,$$

and moreover each transition morphism $X_i \to X_j$ is a closed immersion (for $i, j \in I$ with $i \leq j$). We denote by IndSch the full subcategory of the category of functors $\mathsf{AffSch}^{\mathrm{op}} \to \mathsf{Sets}$ whose objects are ind-schemes. We will call an isomorphism $X \cong \mathrm{colim}_{i \in I} X_i$ a *presentation* of X; whenever we write an ind-scheme in this way, we implicity assume that the X_i's form an inductive system of schemes with closed immersions as transition morphisms, as above. As explained in [**Rc4**, Lemma 1.10], IndSch is closed under fiber products.[(2)]

Of course, each scheme defines an ind-scheme, and this assignment defines a fully faithful functor from the category of schemes to IndSch. Note that if X is a scheme and $Y = \mathrm{colim}_{i \in I} Y_i$ is an ind-scheme, then the canonical map

$$\mathrm{colim}_{i \in I} \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, Y) \tag{1.1.1}$$

is injective, but not necessarily surjective. It *is* surjective (and hence an isomorphism) if X is quasi-compact, though; see [**Rc4**, Ex. 1.26].[(3)]

We will also consider ind-schemes over a fixed base scheme S. Such a datum consists of an ind-scheme X together with a morphism $X \to S$. We will denote by IndSch_S the category whose objects are ind-schemes over S and whose morphisms are morphisms of ind-schemes compatible with the given morphisms to S. In fact, if we denote by AffSch_S the category of affine schemes T endowed with a morphism $T \to S$,

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(1) Here, recall that colimits of functors can be computed termwise: if $(F_i : i \in I)$ is an inductive system of functors $\mathsf{AffSch}^{\mathrm{op}} \to \mathsf{Sets}$, then $(\mathrm{colim}_i F_i)(T) = \mathrm{colim}_i F_i(T)$ for all $T \in \mathsf{AffSch}$.

(2) Here again, fiber products of functors can be computed termwise, see [**SP**, Tag 0022].

(3) Let us give a sketch of proof of this fact: if $X = \mathrm{Spec}(R)$ is affine, then a morphism $\mathrm{Spec}(R) \to Y$ is the same as an element of $Y(R)$, hence factors through some Y_j by definition. In general, a quasi-compact scheme is a finite union of affine open subschemes; on each such open subscheme the morphism must factor through some Y_j, and then one can use the assumption that I is filtered to see that one can choose j which works for all open subschemes in our covering at the same time.

then the category of schemes over S embeds fully faithfully in the category of functors $\mathsf{AffSch}_S^{\mathrm{op}} \to \mathsf{Sets}$, and IndSch_S identifies with the category of functors $\mathsf{AffSch}_S^{\mathrm{op}} \to \mathsf{Sets}$ isomorphic to $\mathrm{colim}_{i \in I} X_i$ where $(X_i : i \in I)$ is a filtered inductive system of schemes over S such that the transition morphisms $X_i \to X_j$ are closed immersions (over S). Of course, in case $S = \mathrm{Spec}(R)$ for some $R \in \mathsf{Rings}$, then the category AffSch_S identifies with the opposite of the category Alg_R of unital commutative R-algebras.

1.1.1.2. *Immersions.* — If X, Y are ind-schemes and $f : X \to Y$ is a morphism, then we say that f is *representable by schemes*[(4)] if for any scheme Z and any morphism $Z \to Y$ the fiber product $X \times_Y Z$ is a scheme. Similarly, we say that f is *representable by a locally closed, resp. closed, resp. open, immersion* if for any scheme Z and any morphism $Z \to Y$ the fiber product $X \times_Y Z$ is a scheme and the induced morphism $X \times_Y Z \to Z$ is a locally closed, resp. closed, resp. open, immersion of schemes. (In fact, by [**Rc4**, Lemma 1.7] it suffices to check these properties when Z is affine. This turns out to be very useful since (1.1.1) is an isomorphism in this case.)

1.1.1.3. *Underlying topological space and connected components.* — If X is a scheme, we will denote by $|X|$ its underlying topological space; it can be identified as the colimit of the sets $X(K)$ where K runs over fields, with an appropriate topology. If now X is an ind-scheme, we take the latter characterization as a definition: we set $|X| = \mathrm{colim}_K X(K)$: see [**Rc4**, Definition 1.11] for details. Moreover, if $X = \mathrm{colim}_i X_i$ is a presentation, then we have a canonical identification

$$|X| = \mathrm{colim}_i |X_i|$$

where the right-hand side is equipped with the colimit topology.[(5)]

For any scheme X, any connected component of the underlying topological space $|X|$ admits a canonical scheme structure, which is characterized by the property that the corresponding embedding is a flat closed immersion, see [**SP**, Tag 04PX]. If X is an ind-scheme and $X = \mathrm{colim}_i X_i$ is a presentation, the connected components of $|X|$ are increasing unions of connected components of the spaces $|X_i|$. Hence they admit a canonical ind-scheme structure.

It is not clear to us how this structure behaves in a general setting (e.g., if the inclusion of a connected component is representable by a closed immersion), but under appropriate technical conditions that will be satisfied in all the cases we want to consider it is well behaved, as we now explain. Consider an ind-scheme X with a presentation $X = \mathrm{colim}_i X_i$ such that each X_i is noetherian and each transition morphism $X_i \to X_j$ induces an injection on connected components.

Lemma 1.1.1. — *Under the assumptions above, for any connected component Y of X the natural morphism $Y \to X$ is representable by an open and closed immersion.*

Proof. — Our assumptions imply in particular that X_i has a finite number of connected components for any i (see [**SP**, Tag 0052]); in particular, these connected

(4) One sometimes also finds the terminology "f is schematic."

(5) Concretely, this means that a subset of $|X|$ is open, resp. closed, iff its intersection with each $|X_i|$ is open, resp. closed.

components are open and closed. As explained above, if Y is a connected component of X we have a presentation $Y = \mathrm{colim}_i Y_i$ where Y_i is a connected component of X_i for any i, and the closed immersion $X_i \to X_j$ restricts to a closed immersion $Y_i \to Y_j$ for any $i \leq j$. Consider now an affine scheme Z and a morphism $Z \to X$. There exists i such that this morphism factors through X_i, and then we have

$$Y \times_X Z = \mathrm{colim}_{j \geq i} Y_j \times_{X_j} Z.$$

Now, for any $j \geq i$ we have

$$Y_j \times_{X_j} Z = (Y_j \times_{X_j} X_i) \times_{X_i} Z.$$

The fact that the morphism $X_i \to X_j$ induces an injection on connected components means that the underlying topological space of $Y_j \times_{X_j} X_i$ is Y_i; since the natural morphism $Y_j \times_{X_j} X_i \to X_i$ is a flat closed immersion (because so is $Y_j \to X_j$), it follows that $Y_j \times_{X_j} X_i = Y_i$; in particular,

$$Y \times_X Z = Y_i \times_{X_i} Z$$

is a scheme. Since Y_i is open and closed in X_i, we deduce that the morphism $Y \times_X Z \to Z$ is an open and closed immersion, as desired. □

1.1.1.4. *Additional properties.* — An ind-scheme X is said to be *reduced* if it admits a presentation $X = \mathrm{colim}_i X_i$ where each X_i is reduced. By [**Rc4**, Lemma 1.17], given any ind-scheme X, there exists a unique reduced ind-scheme X_{red} together with a monomorphism $X_{\mathrm{red}} \to X$ such that for any reduced affine scheme T the induced map $X_{\mathrm{red}}(T) \to X(T)$ is an equality. In fact, given a presentation $X = \mathrm{colim}_i X_i$, we have a presentation $X_{\mathrm{red}} = \mathrm{colim}_i (X_i)_{\mathrm{red}}$.

If X, Y are ind-schemes and $f : X \to Y$ is a morphism, then f is said to be *ind-affine* if there exist presentations $X = \mathrm{colim}_i X_i$ and $Y = \mathrm{colim}_j Y_j$ such that f is represented by a pro-ind-system of morphisms $f_{i,j} : X_i \to Y_j$ which are affine.

If X is an ind-scheme over $\mathrm{Spec}(k)$ for some base field k, we will say that X is *of ind-finite type* if it admits a presentation $X = \mathrm{colim}_i X_i$ (over k) where each X_i is of finite type over k.

Finally, we say that an ind-scheme X over a scheme S is *separated* if the diagonal morphism $X \to X \times_S X$ is representable by a closed immersion. For this condition to hold, it suffices that X admit a presentation $X = \mathrm{colim}_i X_i$ over S where each morphism $X_i \to S$ is separated, see [**Rc4**, Exercise 1.31]. In addition, if this property holds, given any presentation $X = \mathrm{colim}_i X_i$ over S, each scheme X_i is separated over S.(6)

1.1.2. Attractors and repellers. — Now we explain the theory of attractors, repellers and fixed points for an action of the multiplicative group on a scheme.

(6)To check this, one notes (by consideration of points over each affine scheme) that $(X_i \times_S X_i) \times_{X \times_S X} X = X_i$ where the morphism $X \to X \times_S X$ is the diagonal morphism, and then one uses the definition.

1.1.2.1. *Definitions.* — Let us consider a base scheme S, and a scheme X over S. An *action* of $\mathbb{G}_{\mathrm{m}}$ on X is the datum of a morphism of S-schemes $\mathbb{G}_{\mathrm{m},S} \times_S X \to X$ which satisfies the obvious axioms.[7]

Following [**Dr, Rc3**], we define the functor X^0 of $\mathbb{G}_{\mathrm{m}}$-fixed points in X as sending $T \in \mathsf{AffSch}_S$ to the set of morphisms $\phi : T \to X \times_S T$ over T such that the diagram

$$\begin{tikzcd} \mathbb{G}_{\mathrm{m},T} \arrow[r, "\mathrm{id}_{\mathbb{G}_{\mathrm{m},T}} \times \phi"] \arrow[d] & \mathbb{G}_{\mathrm{m},T} \times_T (X \times_S T) \arrow[d] \\ T \arrow[r, "\phi"] & X \times_S T \end{tikzcd}$$

commutes, where the left vertical arrow is the structure morphism, the right vertical arrow is induced by the action morphism $\mathbb{G}_{\mathrm{m},S} \times_S X \to X$. In other words, given $T \in \mathsf{AffSch}_S$, $X^0(T)$ consists of the T-points of X such that for any affine scheme $T' \to T$ the induced morphism $T' \to X \times_S T'$ commutes with the action of elements in $\mathbb{G}_{\mathrm{m}}(T')$ (where the action on the left-hand side is trivial).

Similarly, we denote by $(\mathbb{A}^1_S)^+$, resp. $(\mathbb{A}^1_S)^-$, the scheme $\mathbb{A}^1_S$ with the natural action of $\mathbb{G}_{\mathrm{m}}$, resp. the opposite of the natural action. Then we define X^+ as the functor sending $T \in \mathsf{AffSch}_S$ to the set of morphisms $\phi : (\mathbb{A}^1_T)^+ \to X \times_S T$ over T such that the diagram

$$\begin{tikzcd} \mathbb{G}_{\mathrm{m},T} \times_T (\mathbb{A}^1_T)^+ \arrow[r, "\mathrm{id}_{\mathbb{G}_{\mathrm{m},T}} \times \phi"] \arrow[d] & \mathbb{G}_{\mathrm{m},T} \times_T (X \times_S T) \arrow[d] \\ (\mathbb{A}^1_T)^+ \arrow[r, "\phi"] & X \times_S T \end{tikzcd}$$

commutes, where the vertical arrows are the action morphisms. In other words, given $T \in \mathsf{AffSch}_S$, $X^+(T)$ consists of the $\mathbb{A}^1_T$-points of X such that for any affine scheme $T' \to T$ the induced morphism $\mathbb{A}^1_{T'} \to X \times_S T'$ commutes with the action of elements in $\mathbb{G}_{\mathrm{m}}(T')$. The functor X^- is defined similarly, replacing $(\mathbb{A}^1_S)^+$ by $(\mathbb{A}^1_S)^-$.

By definition there exists a natural morphism of functors

$$X^0 \to X.$$

On the other hand, using the morphisms[8] $T \to (\mathbb{A}^1_T)^{\pm}$ defined by 0 and 1 we obtain natural morphisms of functors

$$X^0 \leftarrow X^{\pm} \to X.$$

1.1.2.2. *Local linearizability.* — If X is a scheme over S with an action of $\mathbb{G}_{\mathrm{m}}$, this action is said to be *étale (resp. Zariski) locally linearizable* if there exists a $\mathbb{G}_{\mathrm{m}}$-equivariant covering family $(U_i \to X : i \in I)$ where each U_i is affine over S and the maps $U_i \to X$ are étale (resp. open immersions). This condition is interesting in this

(7) Here, $\mathbb{G}_{\mathrm{m},S}$ is the group scheme over S sending $T \in \mathsf{AffSch}_S$ to $\mathscr{O}(T)^{\times}$. In practice, in this book, S will typically be a k-scheme for some algebraically closed field k. The datum of an action of $\mathbb{G}_{\mathrm{m}}$ on X is then equivalent to the datum of an action on X seen as a k-scheme, such that the structure morphism $X \to S$ is $\mathbb{G}_{\mathrm{m}}$-equivariant for the trivial action on S.

(8) Here and below, we use the symbol $\pm$ to mean either $+$ or $-$.

context since, thanks to [**Rc3**, Theorem 1.8, Proposition 1.17], if the $\mathbb{G}_{\mathrm{m}}$-action is étale locally linearizable then:

- the functors X^0, X^+ and X^- are representable by schemes over S;
- the natural morphism $X^0 \to X$ is a closed immersion;
- the natural morphisms $X^{\pm} \to X^0$ are affine, with geometrically connected fibers, and they induce a bijection between sets of connected components.

Moreover, the morphisms $(U_i^0 \to X^0 : i \in I)$, resp. $(U_i^+ \to X^+ : i \in I)$, resp. $(U_i^- \to X^- : i \in I)$, constitute an étale covering of X^0, resp. X^+, resp. X^-.

In practice, all the actions we have to consider in the constructions below will be *Zariski* locally linearizable. In this case the properties considered above have much more elementary proofs. For instance, to see that X^0 is representable and that the morphism $X^0 \to X$ is a closed immersion one can work locally (using e.g. [**GW**, Theorem 8.9] for representability), and thus assume that X is affine over S. In this case, the $\mathbb{G}_{\mathrm{m}}$-action on X defines a $\mathbb{Z}$-grading on the quasi-coherent sheaf of $\mathscr{O}_S$-algebras $f_*\mathscr{O}_X$ (where $f : X \to S$ is the structure morphism), and X^0 is the closed subscheme defined by the quasi-coherent ideal generated by the components of nonzero degree in this sheaf of algebras; see [**Rc3**, §1.3] for details.(9) For the representability of X^+ one first treats the case when $X \to S$ is affine; in this case X^+ is the closed subscheme defined by the quasi-coherent ideal generated by components of negative degree in $f_*\mathscr{O}_X$. One then considers the natural morphism $X^+ \to X^0$. The considerations above show that X^0 is a scheme, and that it admits a Zariski covering by open subschemes of the form U^0 where U runs over the $\mathbb{G}_{\mathrm{m}}$-stable affine open subschemes of X. But for any $\mathbb{G}_{\mathrm{m}}$-stable open subscheme U of X, it is easy to see that the natural morphism

$$U^+ \to X^+ \times_{X^0} U^0$$

is an isomorphism (see [**Dr**, Lemma 1.4.7] for details); this implies that X^+ is a scheme by [**GW**, Theorem 8.9].

Example 1.1.2. — Assume that $S = \mathrm{Spec}(k)$ where k is a field, and V is a k-vector space endowed with a linear action of $\mathbb{G}_{\mathrm{m},k}$. (In other words, V is a representation of $\mathbb{G}_{\mathrm{m},k}$.) Then V decomposes as a sum of its weight spaces

$$V = \bigoplus_{i \in \mathbb{Z}} V_i.$$

Consider the induced action of $\mathbb{G}_{\mathrm{m},k}$ on $\mathbb{P}(V)$. This action is Zariski locally linearisable: in fact the standard affine covering associated with a basis of V consisting of weight vectors is $\mathbb{G}_{\mathrm{m}}$-stable. It is easily seen that we have identifications

$$\mathbb{P}(V)^0 = \bigsqcup_{\substack{i \in \mathbb{Z} \\ V_i \neq 0}} \mathbb{P}(V_i), \qquad \mathbb{P}(V)^+ = \bigsqcup_{\substack{i \in \mathbb{Z} \\ V_i \neq 0}} \mathbb{P}(V_{\geq i}) \smallsetminus \mathbb{P}(V_{\geq i+1})$$

where $V_{\geq j} := \bigoplus_{j' \geq j} V_{j'}$.

(9) The assumption that the base scheme S is connected in [**Rc3**, §1.3] is not necessary.

1.1.2.3. *Compatibility with closed immersions.* — We will need the following facts below.

Lemma 1.1.3. — *Let X be a scheme over S endowed with an étale, resp. Zariski, locally linearizable action of $\mathbb{G}_{\mathrm{m}}$. If $Y \subset X$ is a $\mathbb{G}_{\mathrm{m}}$-stable closed subscheme, then the $\mathbb{G}_{\mathrm{m}}$-action on $Y \to S$ is étale, resp. Zariski, locally linearizable, and the natural morphisms $Y^0 \to X^0$ and $Y^{\pm} \to X^{\pm}$ are closed immersions. More specifically, the canonical morphisms*

$$Y^0 \to Y \times_X X^0, \qquad Y^{\pm} \to Y \times_X X^{\pm}$$

are isomorphisms.

Proof. — If $(U_i \to X : i \in I)$ is an equivariant étale, resp. Zariski, covering for X as above, then of course $(U_i \times_X Y \to Y : i \in I)$ is an equivariant étale, resp. Zariski, covering of Y, and each $U_i \times_X Y$ is affine over S since it is a closed subscheme of the affine scheme U_i. Hence the $\mathbb{G}_{\mathrm{m}}$-action on Y is étale, resp. Zariski, locally linearizable, so that we can consider the schemes Y^0 and $Y^{\pm}$.

We will construct the isomorphism $Y^+ \xrightarrow{\sim} Y \times_X X^+$; the other assertions can be obtained similarly. First, the natural morphisms $Y^+ \to Y$ and $Y^+ \to X^+$ induce a canonical morphism

$$Y^+ \to Y \times_X X^+. \tag{1.1.2}$$

Now, assume that $X \to S$ is affine. Checking that (1.1.2) is an isomorphism can be done Zariski locally over S, so that we can assume that S (hence also X) is affine. In this case for $T \in \mathsf{AffSch}_S$, a T-point of $Y \times_X X^+$ is a certain morphism of T-schemes $\mathbb{A}^1_T \to X$ whose restriction to $\mathbb{G}_{\mathrm{m},T}$ takes values in Y. It is clear that this morphism then factors uniquely through a morphism $\mathbb{A}^1_T \to Y$, which proves that (1.1.2) is an isomorphism in this case.

To treat the general case, consider an equivariant étale covering $(U_i \to X : i \in I)$ where each $U_i \to S$ is affine. Then we have an étale covering $(U_i \times_X Y \to Y : i \in I)$, and hence étale coverings $(U_i^+ \to X^+ : i \in I)$ and $((U_i \times_X Y)^+ \to Y^+ : i \in I)$ by the results recalled in §1.1.2.2, and from the affine case treated above we see that for any i we have a canonical identification

$$(U_i \times_X Y)^+ \xrightarrow{\sim} (U_i \times_X Y) \times_{U_i} (U_i)^+ = Y \times_X U_i^+.$$

This shows that (1.1.2) is an isomorphism étale locally over the target. Hence it is an isomorphism by [**SP**, Tag 02L4]. □

1.1.2.4. *Points over fields.* — Now we assume that $S = \mathrm{Spec}(k)$ for some field k.

Lemma 1.1.4. — *Let X be a proper k-scheme with an étale locally linearizable action of $\mathbb{G}_{\mathrm{m}}$. Then the natural morphism $X^+ \to X$ induces a bijection*

$$X^+(K) \xrightarrow{\sim} X(K)$$

for each field extension $k \to K$. In particular, this morphism induces a bijection

$$|X^+| \xrightarrow{\sim} |X|$$

on the underlying topological spaces.

Proof. — Since X is separated, the morphism $X^+ \to X$ is a monomorphism by [**Rc3**, Remark 1.19(i)] or [**Dr**, §1.3.3(ii)]. In particular, the map $X^+(K) \to X(K)$ is injective for any K. The surjectivity of this map follows from the fact that any morphism $\mathbb{G}_{\mathrm{m},K} \to X \otimes_k K$ can be extended to a morphism $\mathbb{A}^1_K \to X \otimes_k K$ by properness, see [**SP**, Tag 0BXZ]. The final claim follows from the fact that the underlying topological space of a scheme is the colimit of its points over all fields, see [**SP**, Tag 01J9]. □

Remark 1.1.5. — In the setting of Lemma 1.1.4, the map $|X^+| \xrightarrow{\sim} |X|$ is *not* a homeomorphism in general. (This can be seen e.g. in the case considered in Example 1.1.2.)

1.1.3. Attractors and repellers for ind-schemes. — Now we explain how to adapt the constructions of §1.1.2 to the setting of ind-schemes.

1.1.3.1. *Definitions.* — We continue with our base scheme S, and consider an ind-scheme X over S. An action of $\mathbb{G}_{\mathrm{m}}$ on X is the datum of a morphism of S-ind-schemes $\mathbb{G}_{\mathrm{m},S} \times_S X \to X$ which satisfies the obvious axioms. In practice, we will in fact assume that there exists a presentation $X = \mathrm{colim}_i X_i$ by S-schemes such that the action morphism is defined by compatible actions of $\mathbb{G}_{\mathrm{m}}$ on each X_i (in the sense of schemes). As explained in [**RS**, Lemma A.5], this condition is always satisfied if S is noetherian and X is of ind-finite type over S.

Given $X \to S$ as above, we will say that the $\mathbb{G}_{\mathrm{m}}$-action is étale (resp. Zariski) locally linearizable if there exists a presentation $X = \mathrm{colim}_i X_i$ such that the action of $\mathbb{G}_{\mathrm{m}}$ is induced by compatible actions on the X_i's, and the action on $X_i \to S$ is étale (resp. Zariski) locally linearizable for any i. Given such a datum, when writing a presentation $X = \mathrm{colim}_i X_i$ we will always implicitly assume that each X_i is $\mathbb{G}_{\mathrm{m}}$-stable with an étale (resp. Zariski) locally linearizable action.

1.1.3.2. *Representability.* — The following theorem is an easy extension of the first main result of [**Rc3**], treated in [**HR2**, Theorem 2.1].

Theorem 1.1.6. — *Let $X \to S$ be an S-ind-scheme endowed with an étale locally linearizable $\mathbb{G}_{\mathrm{m}}$-action, and write a presentation $X = \mathrm{colim}_i X_i$ as above.*

1. *The functor X^0 is an S-ind-scheme, and we have a presentation $X^0 = \mathrm{colim}_i (X_i)^0$. Moreover, the natural morphism $X^0 \to X$ is representable by a closed immersion.*
2. *The functor $X^\pm$ is an S-ind-scheme, and we have a presentation $X^\pm = \mathrm{colim}_i (X_i)^\pm$. Moreover, the natural morphism $X^\pm \to X$ is representable by schemes.*

Proof. — If $X = \mathrm{colim}_i X_i$ is a presentation such that each X_i has an étale locally linearizable action, then as functors we have $X^0 = \mathrm{colim}_i (X_i)^0$ and $X^\pm = \mathrm{colim}_i (X_i)^\pm$. Hence X^0 and $X^\pm$ are ind-schemes by Lemma 1.1.3. To show that $X^0 \to X$, resp. $X^\pm \to X$, is representable by a closed immersion, resp. representable by schemes, one notices that if Z is an affine scheme and $Z \to X$ is a morphism, then there exists

i such that this morphism is induced by a morphism $Z \to X_i$, and we have

$$Z \times_X X^0 = Z \times_{X_i} (X_i)^0, \quad \text{resp.} \quad Z \times_X X^\pm = Z \times_{X_i} (X_i)^\pm.$$

(For instance, in the case of attractors, we have $Z \times_X X^+ = \mathrm{colim}_{j \geq i} Z \times_{X_j} (X_j)^+$, and for any $j \geq i$ we observe that $Z \times_{X_j} (X_j)^+ = Z \times_{X_i} (X_i \times_{X_j} (X_j)^+) = Z \times_{X_i} (X_i)^+$ by Lemma 1.1.3.) □

1.1.3.3. *Compatibility with immersions.* — We will need the following technical property from [**HR2**, Corollary 2.3]. (In fact, we will only use the "open" case of this statement.)

Proposition 1.1.7. — *Let X and Y be S-ind-schemes equipped with $\mathbb{G}_\mathrm{m}$-actions. Assume that the actions on X and Y are étale locally linearizable, and that Y is separated. Also let $f : X \to Y$ be a $\mathbb{G}_\mathrm{m}$-equivariant morphism. If f is representable by a closed, resp. open, immersion, then so are the morphisms $X^0 \to Y^0$ and $X^\pm \to Y^\pm$.*

Remark 1.1.8. — In [**HR1**, Corollary 2.3] it is assumed that S is affine and connected. However the connectedness is not necessary for the arguments there to apply, and one can reduce to the case where S is affine by considering an affine open cover.

1.2. Affine Grassmannians

In this section we introduce the main geometric object that takes part in the geometric Satake equivalence, namely the affine Grassmannian attached to a complex[(10)] reductive algebraic group.

1.2.1. Definition and representability. —

1.2.1.1. *Loop group, positive loop group, and affine Grassmannian.* — Given a $\mathbb{C}$-algebra R, we denote by $R[[x]]$ the $\mathbb{C}$-algebra of power series in the indeterminate x with coefficients in R, and by $R((x))$ the localization of $R[[x]]$ with respect to x (i.e. the algebra of formal Laurent series in x with coefficients in R). If R is a field, then $R((x))$ is the field of fractions of the integral ring $R[[x]]$. In particular, we will consider $\mathscr{O} := \mathbb{C}[[x]]$ and $\mathscr{K} := \mathbb{C}((x))$.

Recall that if G is a smooth affine group scheme over $\mathbb{C}$, then the associated loop group $\mathrm{L}G$ is the functor $\mathsf{Alg}_\mathbb{C} \to \mathsf{Sets}$ defined by

$$\mathrm{L}G(R) = G(R((x))).$$

The positive loop group (or arc group) L^+G is the subfunctor defined by

$$\mathrm{L}^+G(R) = G(R[[x]]).$$

It is a standard fact that L^+G is represented by an affine group scheme over $\mathbb{C}$, and that $\mathrm{L}G$ is represented by an ind-affine group ind-scheme over $\mathbb{C}$; see [**Zh4**,

(10) We restrict to the case the base field is $\mathbb{C}$ because this is the main case that will be used below, but all the results that appear in this section hold over any algebraically closed field, with identical proofs.

Proposition 1.3.2] or [**Rc4**]. See [**Zh4**, Example 1.3.3] for a concrete example in the case where $G = \mathbb{G}_{\mathrm{a}}$.

The affine Grassmannian Gr_G is the fppf sheaf on the category $\mathsf{Alg}_{\mathbb{C}}$ associated with the functor

$$R \mapsto \mathrm{L}G(R)/\mathrm{L}^+G(R).$$

It is known that Gr_G is represented by a separated ind-scheme of ind-finite type, see [**Zh4**, Theorem 1.2.2 and Proposition 1.3.6] or [**Rc4**, Theorem 3.4 and Proposition 3.18]. The proof of this fact in case G is reductive is reviewed in §1.2.1.4 below; the general case is not very different.

Remark 1.2.1. — Some comments and references on fppf sheaves, and the associated sheafification functor, can be found in §2.1.1 below. (There we will consider sheaves on the category of all schemes rather than affine schemes. The relation between the two versions is explained in Remark 2.1.1.) Other details on the construction of Gr_G, which can be ignored at this stage, will be discussed in §§2.2.1–2.2.2.

The points of Gr_G over separably closed fields admit an explicit description, see [**Rc4**, Corollary 3.22]. In particular, we have

$$\mathrm{Gr}_G(\mathbb{C}) = G(\mathscr{K})/G(\mathscr{O}). \tag{1.2.1}$$

We will denote by $p_{\mathrm{Gr}} : \mathrm{L}G \to \mathrm{Gr}_G$ the quotient morphism.

1.2.1.2. *Big cell.* — We will also consider the functor $\mathrm{L}^-G : \mathsf{Alg}_{\mathbb{C}} \to \mathsf{Sets}$ defined by[11]

$$\mathrm{L}^-G(R) = G(R[x^{-1}]).$$

It is known that L^-G is represented by an ind-affine group ind-scheme of ind-finite type over $\mathbb{C}$; see [**Zh4**, §2.3]. There exists a canonical morphism $\mathrm{L}^-G \to G$ induced by the ring morphisms $R[x^{-1}] \to R$ sending x^{-1} to 0; the kernel of this morphism is denoted $\mathrm{L}^{--}G$. The following statement is somewhat classical; a formal proof can be found in this generality in [**HR2**, Lemma 3.1].

Lemma 1.2.2. — *Let $L_0 \in \mathrm{Gr}_G(\mathbb{C})$ be the base point. Then the orbit morphism*

$$\mathrm{L}^{--}G \to \mathrm{Gr}_G, \quad g \mapsto g \cdot L_0$$

is representable by an open immersion.

A closely related fact is that the multiplication morphism

$$\mathrm{L}^{--}G \times \mathrm{L}^+G \to \mathrm{L}G$$

is representable by an open immersion. Over the image of the morphism of Lemma 1.2.2, the morphism $p_{\mathrm{Gr}} : \mathrm{L}G \to \mathrm{Gr}_G$ restricts to the obvious projection $\mathrm{L}^{--}G \times \mathrm{L}^+G \to \mathrm{L}^{--}G$.

[11] In this formula, x^{-1} is treated as a formal variable; this element is not the inverse of anything.

1.2.1.3. *The case of* $\mathrm{GL}(n)$. — Let R be a $\mathbb{C}$-algebra. An $R[\![x]\!]$-*lattice* in $(R(\!(x)\!))^n$, is, by definition, a locally free finitely generated $R[\![x]\!]$-submodule $\Lambda \subset (R(\!(x)\!))^n$ such that $\Lambda[x^{-1}] = (R(\!(x)\!))^n$. We will now review the description of $\mathrm{Gr}_{\mathrm{GL}(n)}$ in terms of lattices, following [**Rc4**, §2].

Writing $\Lambda_{0,R}$ for the lattice $(R[\![x]\!])^n \subset (R(\!(x)\!))^n$ (for any $R \in \mathsf{Alg}_{\mathbb{C}}$), we have a presentation $\mathrm{Gr}_{\mathrm{GL}(n)} = \mathrm{colim}_{i \geq 0} \mathrm{Gr}_{\mathrm{GL}(n),i}$ where $\mathrm{Gr}_{\mathrm{GL}(n),i}$ is the scheme whose set of R-points is the set of $R[\![x]\!]$-lattices $\Lambda \subset R(\!(x)\!)^n$ with

$$x^i \Lambda_{0,R} \subset \Lambda \subset x^{-i} \Lambda_{0,R}.$$

For any finite-dimensional $\mathbb{C}$-vector space V, it is known that the functor $\mathrm{Grass}(V)$ sending a $\mathbb{C}$-algebra R to the set of R-submodules $M \subset V \otimes_{\mathbb{C}} R$ such that the quotient $(V \otimes_{\mathbb{C}} R)/M$ is locally free is a smooth projective scheme over $\mathbb{C}$ (see [**GW**, §8.4]); in fact it is a disjoint union of the Grassmannians $\mathrm{Grass}_d(V)$ of d-dimensional subspaces in V, and for any d we have a natural closed immersion $\mathrm{Grass}_d(V) \hookrightarrow \mathbb{P}(\bigwedge^d V)$, see [**GW**, §8.10].

Writing $M_i := x^{-i}\Lambda_{0,\mathbb{C}}/x^i\Lambda_{0,\mathbb{C}}$, we then have a closed immersion of schemes

$$\mathrm{Gr}_{\mathrm{GL}(n),i} \hookrightarrow \mathrm{Grass}(M_i)$$

which is defined on R-points by $\Lambda \mapsto \Lambda/x^i\Lambda_{0,R}$, and hence a closed immersion $\mathrm{Gr}_{\mathrm{GL}(n),i} \hookrightarrow \bigsqcup_d \mathbb{P}(\bigwedge^d M_i)$.

For $i \geq 0$, since $\mathrm{Gr}_{\mathrm{GL}(n),i}$ and $\mathrm{Gr}_{\mathrm{GL}(n),i+1}$ are proper (by the previous paragraph), the natural morphism $\mathrm{Gr}_{\mathrm{GL}(n),i} \to \mathrm{Gr}_{\mathrm{GL}(n),i+1}$ is proper as well, see [**SP**, Tag 01W6]. Since this morphism is a monomorphism, it must be a closed immersion by [**SP**, Tag 04XV].

From these considerations we obtain that $\mathrm{Gr}_{\mathrm{GL}(n)}$ is an ind-scheme, and that it admits a presentation in which all schemes that appear are projective (in particular, of finite type) over $\mathbb{C}$.

1.2.1.4. *The case of reductive groups.* — From now on, assume that G is a (connected) reductive algebraic group over $\mathbb{C}$. A choice of a faithful representation of G provides a closed immersion $G \hookrightarrow \mathrm{GL}(n)$ for some n, and the quotient $\mathrm{GL}(n)/G$ is automatically affine by the main result of [**Rs**]. By [**Zh4**, Proposition 1.2.6], it follows that the induced morphism $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}(n)}$ is representable by a closed immersion. In particular, if $\mathrm{Gr}_{\mathrm{GL}(n),i}$ is as above and if we set

$$\mathrm{Gr}_{G,i} := \mathrm{Gr}_G \times_{\mathrm{Gr}_{\mathrm{GL}(n)}} \mathrm{Gr}_{\mathrm{GL}(n),i},$$

then $\mathrm{Gr}_{G,i}$ is a scheme, and the natural morphism $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{\mathrm{GL}(n),i}$ is a closed immersion. It is also easily seen that

$$\mathrm{Gr}_G = \mathrm{colim}_{i \geq 0} \mathrm{Gr}_{G,i},$$

and that for any $i \geq 0$ the natural morphism $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{G,i+1}$ is a closed immersion. In particular, as in the case of $\mathrm{GL}(n)$, Gr_G admits a presentation in which all schemes that appear are projective over $\mathbb{C}$.(12)

(12) One must beware that this property does *not* hold when G is not reductive.

1.2.1.5. *Spherical orbits and Schubert varieties.* — We continue to assume that G is reductive, and choose a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ containing T. Let $B^+ \subset G$ be the Borel subgroup opposite to B with respect to T. We set

$$\mathbf{X} := X^*(T), \qquad \mathbf{X}^\vee := X_*(T).$$

We will denote by $\mathfrak{R} \subset \mathbf{X}$ the root system of (G,T), and by $\mathfrak{R}_+ \subset \mathfrak{R}$ the system of positive roots consisting of the nonzero T-weights in the Lie algebra of B^+. (Thus, B is the "negative" Borel subgroup. Note that this is opposite to the conventions of [**BR, MV2**].) Similarly we have the coroots $\mathfrak{R}^\vee \subset \mathbf{X}^\vee$ and the positive coroots $\mathfrak{R}_+^\vee \subset \mathfrak{R}^\vee$. The positive system $\mathfrak{R}_+$ determines a subset $\mathbf{X}_+^\vee \subset \mathbf{X}^\vee$ of dominant coweights. The Weyl group of (G,T) will be denoted W_{f}. (Here, "f" stands for "finite.")

By [**Rs**] again the quotient G/T is affine, so that the morphism $\mathrm{Gr}_T \to \mathrm{Gr}_G$ is representable by a closed immersion (again by [**Zh4**, Proposition 1.2.6]). Any $\lambda \in \mathbf{X}^\vee$ determines a $\mathbb{C}$-point $x^\lambda \in \mathrm{L}T(\mathbb{C})$, namely the image under the morphism $\mathrm{L}\mathbb{G}_{\mathrm{m}} \to \mathrm{L}T$ induced by λ of $x \in \mathrm{L}\mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathscr{K}^\times$. The image of this point in $\mathrm{Gr}_T(\mathbb{C})$ will be denoted L_λ. We will also denote by x^λ and L_λ the images of these points in $\mathrm{L}G(\mathbb{C})$ and $\mathrm{Gr}_G(\mathbb{C})$ respectively.

Consider a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in §1.2.1.4, so that the L^+G-action on Gr_G is induced by compatible actions on each $\mathrm{Gr}_{G,i}$, and that the action on $\mathrm{Gr}_{G,i}$ factors through a quotient K_i of L^+G which is a smooth group scheme of finite type over $\mathbb{C}$. If $\mu \in \mathbf{X}_+^\vee$, we can choose i such that $L_\mu \in \mathrm{Gr}_{G,i}(\mathbb{C})$. Then it makes sense to consider the K_i-orbit Gr_G^μ of L_μ, which is a reduced locally closed subscheme of $\mathrm{Gr}_{G,i}$. If $\overline{\mathrm{Gr}_G^\mu}$ is the closure of Gr_G^μ, endowed with the reduced closed subscheme structure, then $\overline{\mathrm{Gr}_G^\mu}$ is a projective reduced scheme over $\mathbb{C}$, and the natural morphism $\mathrm{Gr}_G^\mu \to \overline{\mathrm{Gr}_G^\mu}$ is an open immersion, see [**SP**, Tag 03DQ]. It is clear that this construction does not depend on i, nor on the choice of presentation of Gr_G. The varieties Gr_G^μ, resp. $\overline{\mathrm{Gr}_G^\mu}$, are called spherical orbits, resp. Schubert varieties.

The *Cartan decomposition* states that the reduced ind-scheme $(\mathrm{Gr}_G)_{\mathrm{red}}$ associated with Gr_G (see §1.1.1.4) admits a stratification (i.e., a partition into locally closed smooth subschemes satisfying a condition on closures)

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \bigsqcup_{\lambda \in \mathbf{X}_+^\vee} \mathrm{Gr}_G^\lambda.$$

In more detail, by [**Zh4**, Proposition 2.1.5], each Gr_G^λ is a smooth variety, of dimension

$$\dim(\mathrm{Gr}_G^\lambda) = \langle 2\rho, \lambda \rangle,$$

where $\rho \in \frac{1}{2}\mathbf{X}$ is one-half the sum of the positive roots. It is also well known that

$$\overline{\mathrm{Gr}_G^\lambda} = \bigsqcup_{\substack{\mu \in \mathbf{X}_+^\vee \\ \lambda - \mu \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+^\vee}} \mathrm{Gr}_G^\mu. \tag{1.2.2}$$

In particular, L_μ belongs to $\overline{\mathrm{Gr}_G^\lambda}(\mathbb{C})$ iff the unique dominant W_{f}-conjugate μ^+ of μ satisfies $\lambda - \mu^+ \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+^\vee$.

1.2.1.6. *Connected components.* — If G is as in §1.2.1.5, it is a standard fact that the connected components of Gr_G are parametrized by $\mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$, see e.g. [**PR**, Theorem 0.1]. Given a coset $c \in \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$, the subscheme $\overline{\mathrm{Gr}_G^\nu}$ is contained in the component corresponding to c iff $\nu \in c$. We will denote by Gr_G^c the component corresponding to a coset $c \in \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$; it satisfies

$$(\mathrm{Gr}_G^c)_{\mathrm{red}} = \bigsqcup_{\lambda \in c} \mathrm{Gr}_G^\lambda.$$

Since the function $\langle 2\rho, -\rangle$ takes even values on $\mathbb{Z}\mathfrak{R}^\vee$, the parity of the dimension of L^+G-orbits is constant on each connected component of Gr_G. Such a component will be called *even* if these dimensions are even, and *odd* if they are odd.

We will call a coweight λ *minuscule*(13) if $\langle \lambda, \alpha\rangle \in \{0,1\}$ for any positive root α. If we denote by $\mathbf{X}^\vee_{\mathrm{min}} \subset \mathbf{X}^\vee$ the subset of minuscule coweights, then it is well known that the composition

$$\mathbf{X}^\vee_{\mathrm{min}} \hookrightarrow \mathbf{X}^\vee \twoheadrightarrow \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$$

is a bijection. Moreover, for any $c \in \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$, if λ_0 is the unique minuscule coweight in c we have $\lambda_0 \leq \lambda$ for any $\lambda \in c \cap \mathbf{X}^\vee_+$. As a consequence, the orbit $\mathrm{Gr}_G^{\lambda_0}$ is closed, and for any such λ we have $\mathrm{Gr}_G^{\lambda_0} \subset \overline{\mathrm{Gr}_G^\lambda}$. The Schubert varieties attached to minuscule coweights will also be called minuscule; these Schubert varieties are smooth.

This property implies that the assumptions of Lemma 1.1.1 are satisfied in this case: given any presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ such that the L^+G-action on Gr_G is induced by compatible actions on the $\mathrm{Gr}_{G,i}$'s which factor through an action of a smooth group scheme of finite type, the connected components in $\mathrm{Gr}_{G,i}$ are determined by the unique minuscule Schubert variety that they contain (because they are closed and L^+G-stable), so that the morphism $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{G,j}$ indeed induces an injection on sets of connected components if $i \leq j$. In particular, the embedding of any connected component in Gr_G is representable by an open and closed immersion.

Example 1.2.3. — Let us revisit the case of $G = \mathrm{GL}(n)$ from §1.2.1.3. Choose for B the subgroup of lower-triangular matrices, and for T the maximal torus consisting of diagonal matrices. We then have a standard identification

$$\mathbf{X}^\vee = \mathbb{Z}^n,$$

for which the i-th element in the standard basis of $\mathbb{Z}^n$ corresponds to the coweight

$$\varepsilon_i^\vee : t \mapsto \mathrm{diag}(1, \dots, 1, t, 1, \dots, 1)$$

where t appears in i-th position.

We have an identification $\mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee \xrightarrow{\sim} \mathbb{Z}$, induced by $(\lambda_1, \dots, \lambda_n) \mapsto \sum_i \lambda_i$, and the minuscule coweights are those of the form $(k, \dots, k, k-1, \dots, k-1)$ with $k \in \mathbb{Z}$. (Here the number of $k-1$'s can be 0.) In particular, the coweight $\varepsilon_1^\vee$ is minuscule; its Weyl group orbit consists of the coweights $\varepsilon_1^\vee, \dots, \varepsilon_n^\vee$. The orbit $\mathrm{Gr}_{\mathrm{GL}(n)}^{\varepsilon_1^\vee}$ is therefore closed.

(13) Note that our definition is more general than in many other sources, including [**Bou**]. In particular, for us 0 is a minuscule coweight.

In terms of lattices, the point $L_{\varepsilon_1^\vee}$ corresponds to the lattice

$$(x\mathscr{O}) \times \mathscr{O}^{n-1} \subset \mathscr{K}^n.$$

Its orbit under L^+G is

$$\{\Lambda \subset \mathscr{K}^n \text{ lattice} \mid x \cdot \mathscr{O}^n \subset \Lambda \subset \mathscr{O}^n,\ \dim(\mathscr{O}^n/\Lambda) = 1\}.$$

The quotient $\mathscr{O}^n/x \cdot \mathscr{O}^n$ identifies canonically with $\mathbb{C}^n$, and via this identification the variety $\mathrm{Gr}^{\varepsilon_1^\vee}_{\mathrm{GL}(n)}$ identifies with the projective space $\mathbb{P}^{n-1}$ of hyperplanes in $\mathbb{C}^n$.

1.2.2. $\mathbb{G}_\mathrm{m}$-actions via cocharacters. — We continue with our complex reductive algebraic group G, and consider a cocharacter $\chi : \mathbb{G}_\mathrm{m} \to G$. From the action of L^+G on Gr_G, and using the embedding $\mathbb{G}_\mathrm{m} \subset \mathrm{L}^+\mathbb{G}_\mathrm{m}$ (as constant loops) and the morphism $\mathrm{L}^+\mathbb{G}_\mathrm{m} \to \mathrm{L}^+G$ induced by χ we obtain an action of $\mathbb{G}_\mathrm{m}$ on Gr_G.

1.2.2.1. *Local linearizability.* — In order to start considering the formalism of §1.1.2, we need to check that the action under consideration is locally linearizable.

Lemma 1.2.4. — *The $\mathbb{G}_\mathrm{m}$-action on Gr_G is Zariski locally linearizable.*

Proof. — First we consider the case $G = \mathrm{GL}(n)$. Recall the construction of §1.2.1.3; we use the notation introduced there. The cocharacter χ defines a $\mathbb{G}_\mathrm{m}$-action on $\mathbb{A}^n_\mathbb{C}$, and hence on M_i, on $\bigwedge^d M_i$, and finally on $\mathbb{P}(\bigwedge^d M_i)$ (for any i and d), such that the closed embedding $\mathrm{Gr}_{\mathrm{GL}(n),i} \hookrightarrow \bigsqcup_d \mathbb{P}(\bigwedge^d M_i)$ is $\mathbb{G}_\mathrm{m}$-equivariant. Since the $\mathbb{G}_\mathrm{m}$-action on $\mathbb{P}(\bigwedge^d M_i)$ is Zariski locally linearizable (see Example 1.1.2), we deduce that the same holds for the $\mathbb{G}_\mathrm{m}$-action on $\mathrm{Gr}_{\mathrm{GL}(n),i}$, which finishes the proof in this case.

To treat the case of a general reductive group G, we choose a closed embedding $G \hookrightarrow \mathrm{GL}(n)$ for some n as in §1.2.1.4. We then get a presentation $\mathrm{Gr}_G = \mathrm{colim}_{i\geq 0}\mathrm{Gr}_{G,i}$ and closed immersions $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{\mathrm{GL}(n),i}$. The composition of χ with the embedding $G \to \mathrm{GL}(n)$ provides a cocharacter χ' of $\mathrm{GL}(n)$. This cocharacter defines a $\mathbb{G}_\mathrm{m}$-action on each $\mathrm{Gr}_{\mathrm{GL}(n),i}$, such that the closed immersion $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{\mathrm{GL}(n),i}$ is equivariant. Since the action on $\mathrm{Gr}_{\mathrm{GL}(n),i}$ is Zariski locally linearizable by the case treated above, the same is true for the action on $\mathrm{Gr}_{G,i}$ (see Lemma 1.1.3), which finishes the proof. □

In view of Lemma 1.2.4 and Theorem 1.1.6, we can consider the ind-schemes $(\mathrm{Gr}_G)^0$ and $(\mathrm{Gr}_G)^\pm$, and the natural morphisms

$$(\mathrm{Gr}_G)^0 \leftarrow (\mathrm{Gr}_G)^\pm \to \mathrm{Gr}_G. \tag{1.2.3}$$

1.2.2.2. *Description of fixed points, attractors and repellers.* — The cocharacter χ defines via conjugation a $\mathbb{G}_\mathrm{m}$-action on G. If we set

$$M := G^0, \quad P^+ := G^+, \quad P^- := G^-$$

(with respect that this action), then it is known that P^+ and P^- are parabolic subgroups of G, that M is a Levi factor in P^+ and P^-, and that $M = P^+ \cap P^-$, see [**CGP**, §2.1]. The natural maps

$$M \leftarrow P^\pm \to G$$

are the projection to the Levi quotient and the natural embeddings, respectively.

Example 1.2.5. — In case χ is obtained from a cocharacter of T which is moreover regular dominant, we have $P^- = B$, $P^+ = B^+$ and $M = T$.

We can consider the affine Grassmannians Gr_M, $\mathrm{Gr}_{P^\pm}$, and the induced morphisms

$$\mathrm{Gr}_M \leftarrow \mathrm{Gr}_{P^\pm} \rightarrow \mathrm{Gr}_G. \tag{1.2.4}$$

The following result is implicit in many sources; the first explicit proof (to our knowledge) appears in [**HR2**, Proposition 3.4].

Theorem 1.2.6. — *There exist canonical isomorphisms*

$$\mathrm{Gr}_M \xrightarrow{\sim} (\mathrm{Gr}_G)^0, \quad \mathrm{Gr}_{P^\pm} \xrightarrow{\sim} (\mathrm{Gr}_G)^\pm$$

which identify the diagrams (1.2.3) *and* (1.2.4).

For the proof of Theorem 1.2.6 we will need the following preliminary. Recall the ind-affine ind-scheme $\mathrm{L}^{--}G$, see §1.2.1.2. The $\mathbb{G}_\mathrm{m}$-action on G (via conjugation) induces an action on $\mathrm{L}^{--}G$, so that we can consider the (ind-affine) ind-schemes $(\mathrm{L}^{--}G)^0$ and $(\mathrm{L}^{--}G)^\pm$. The closed immersion $M \to G$, resp. $P^\pm \to G$, induces a morphism $\mathrm{L}^{--}M \to \mathrm{L}^{--}G$, resp. $\mathrm{L}^{--}P^\pm \to \mathrm{L}^{--}G$.

Lemma 1.2.7. — *The morphisms above induce isomorphisms*

$$\mathrm{L}^{--}M \xrightarrow{\sim} (\mathrm{L}^{--}G)^0, \qquad \mathrm{L}^{--}P^\pm \xrightarrow{\sim} (\mathrm{L}^{--}G)^\pm.$$

Proof. — [14] It suffices to prove similar claims for L^- instead of L^{--}. We first consider L^-M and $(\mathrm{L}^-G)^0$. By definition, for $R \in \mathsf{Alg}_\mathbb{C}$, $(\mathrm{L}^-G)^0(R)$ consists of the points $g \in G(R[x^{-1}])$ such that for any R-algebra S and any $\lambda \in S^\times$ we have

$$\chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g$$

in $G(S[x^{-1}])$. On the other hand, $(\mathrm{L}^-M)(R) = M(R[x^{-1}])$. Since $M = G^0$ for the $\mathbb{G}_\mathrm{m}$-action on G, the set $M(R[x^{-1}])$ consists of the elements $g \in G(R[x^{-1}])$ such that for any $S' \in \mathsf{Alg}_{R[x^{-1}]}$ and $\lambda \in (S')^\times$ we have

$$\chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g$$

in $G(S')$. We will check that these two subsets of $G(R[x^{-1}])$ coincide.

Given $g \in M(R[x^{-1}])$, for any $S \in \mathsf{Alg}_R$ and $\lambda \in S^\times$ we can consider the $R[x^{-1}]$-algebra $S' := S[x^{-1}]$ and the element $\lambda \in S^\times \subset (S')^\times$. From the description of $M(R[x^{-1}])$ given above we obtain that $\chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g$ in $G(S') = G(S[x^{-1}])$, so that g belongs to $(\mathrm{L}^-G)^0(R)$. In the other direction, consider $g \in (\mathrm{L}^-G)^0(R)$. Then if S' is an $R[x^{-1}]$-algebra, we can consider S' as an R-algebra, and the image of x^{-1} defines an element $s \in S'$. Since g belongs to $(\mathrm{L}^-G)^0(R)$, for any $\lambda \in (S')^\times$ we have

$$\chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g \tag{1.2.5}$$

in $G(S'[x^{-1}])$. We have an S'-algebra morphism $S'[x^{-1}] \twoheadrightarrow S'[x^{-1}]/(s - x^{-1}) \cong S'$, and hence a group homomorphism $G(S'[x^{-1}]) \to G(S')$. Taking the image of the equation (1.2.5) in $G(S')$ we see that g belongs to $(\mathrm{L}^-M)(R)$.

[14] This proof is a corrected version of that appearing in the course of the proof of [**HR2**, Proposition 3.4], which is slightly wrong.

The proof of the isomorphisms involving $P^{\pm}$ is similar. If $R \in \mathsf{Alg}_{\mathbb{C}}$, then $(\mathrm{L}^{-}G)^{\pm}(R)$ is a subset of $(\mathrm{L}^{-}G)(R[t]) = G(R[x^{-1}, t])$ determined by an appropriate equivariance condition, where t is another indeterminate (such that $\mathbb{A}^1_R = \mathrm{Spec}(R[t])$). Similarly we have $(\mathrm{L}^{-}P^{\pm})(R) = P^{\pm}(R[x^{-1}])$, which is a certain subset of $G(R[x^{-1}, t])$ determined by an a priori different equivariance condition. The same considerations as above show that these conditions are in fact equivalent. □

Proof of Theorem 1.2.6. — The action of $\mathbb{G}_{\mathrm{m}}$ is obtained by functoriality from an action on G. Since the embedding $M \hookrightarrow G$ is $\mathbb{G}_{\mathrm{m}}$-equivariant for the trivial action on M, we deduce that the induced morphism $\mathrm{Gr}_M \to \mathrm{Gr}_G$ is also $\mathbb{G}_{\mathrm{m}}$-equivariant for the trivial action on Gr_M, which shows that this embedding factors through a morphism $\mathrm{Gr}_M \to (\mathrm{Gr}_G)^0$. On the other hand, the conjugation action on G stabilizes P^+, and extends to an action of the monoid $\mathbb{A}^1_{\mathbb{C}}$ on this subgroup. It follows that the induced action on Gr_{P^+} also extends to an action of $\mathbb{A}^1_{\mathbb{C}}$.[15] For this action, we therefore have $\mathrm{Gr}_{P^+} = (\mathrm{Gr}_{P^+})^+$. We deduce that the morphism $\mathrm{Gr}_{P^+} \to \mathrm{Gr}_G$ factors through a morphism

$$\mathrm{Gr}_{P^+} = (\mathrm{Gr}_{P^+})^+ \to (\mathrm{Gr}_G)^+.$$

We obtain similarly that the morphism $\mathrm{Gr}_{P^-} \to \mathrm{Gr}_G$ factors through a morphism $\mathrm{Gr}_{P^-} \to (\mathrm{Gr}_G)^-$. (In these considerations we have used the fact that Gr_{P^+}, Gr_{P^-} are separated, so that the morphisms $(\mathrm{Gr}_{P^{\pm}})^{\pm} \to \mathrm{Gr}_{P^{\pm}}$ are monomorphisms, see [**Rc3**, Remark 1.19(i)].)

To conclude, it remains to prove that the morphisms

$$\mathrm{Gr}_M \to (\mathrm{Gr}_G)^0, \quad \mathrm{Gr}_{P^+} \to (\mathrm{Gr}_G)^+, \quad \mathrm{Gr}_{P^-} \to (\mathrm{Gr}_G)^-$$

are isomorphisms. We will treat the case of the morphism $\mathrm{Gr}_{P^+} \to (\mathrm{Gr}_G)^+$; the case of the morphism $\mathrm{Gr}_{P^-} \to (\mathrm{Gr}_G)^-$ follows by applying the previous case to the cocharacter χ^{-1}, and the case of $\mathrm{Gr}_M \to (\mathrm{Gr}_G)^0$ can treated similarly (with some simplifications).

We fix an embedding $G \hookrightarrow \mathrm{GL}(n)$, and consider the presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in the proof of Lemma 1.2.4, so that each $\mathrm{Gr}_{G,i}$ is projective over $\mathbb{C}$ and $\mathbb{G}_{\mathrm{m}}$-stable. For any $g \in \mathrm{L}M(\mathbb{C})$, it follows from Lemma 1.2.2 that the morphism $\mathrm{L}^{--}G \to \mathrm{Gr}_G$ defined by $h \mapsto gh \cdot L_0$ is representable by an open immersion. Moreover these open subschemes form a covering of Gr_G, in the sense that for any i they induce an open covering of $\mathrm{Gr}_{G,i}$. (In fact, since these schemes are of finite type over $\mathbb{C}$, using [**GW**, Corollary 3.36] it suffices to prove that any $\mathbb{C}$-point in some $\mathrm{Gr}_{G,i}$ belongs to such an open subset, which follows e.g. from the Birkhoff decomposition; indeed it suffices to consider elements in $\mathrm{L}A(\mathbb{C})$ where A is a maximal torus contained in M, see [**Fa**, Lemma 4].) By Proposition 1.1.7, for any $g \in \mathrm{L}M(\mathbb{C})$ we deduce a morphism $(\mathrm{L}^{--}G)^+ \to (\mathrm{Gr}_G)^+$ which is representable by an open immersion, and by the results recalled in §1.1.2.2 these open sub-ind-schemes form a covering of $(\mathrm{Gr}_G)^+$.

For fixed $g \in \mathrm{L}M(\mathbb{C})$, by Lemma 1.2.7 our morphism $\mathrm{Gr}_{P^+} \to (\mathrm{Gr}_G)^+$ induces an isomorphism between an open sub-ind-scheme of Gr_{P^+} and the open sub-ind-scheme of $(\mathrm{Gr}_G)^+$ considered above. We can therefore consider the inverse isomorphism, for

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(15) See [**HR2**, p. 153] for an explicit description of this action in terms of a Rees construction.

any $g \in \mathrm{L}M(\mathbb{C})$. Given two elements in $\mathrm{L}M(\mathbb{C})$, these inverse isomorphisms (each defined on the corresponding open sub-ind-scheme of $(\mathrm{Gr}_G)^+$) coincide on the intersection of these sub-ind-schemes; in fact since all the ind-schemes considered here are of ind-finite type (see [**GW**, Example 3.45]), as above it suffices to prove that they coincide on $\mathbb{C}$-points. This follows from the claim that the morphism $\mathrm{Gr}_{P^+}(\mathbb{C}) \to \mathrm{Gr}_G(\mathbb{C})$ is injective, which in turn holds because these sets identify with $P^+(\mathscr{K})/P^+(\mathscr{O})$ and $G(\mathscr{K})/G(\mathscr{O})$ respectively, see (1.2.1). These morphisms therefore glue to define a morphism $(\mathrm{Gr}_G)^+ \to \mathrm{Gr}_{P^+}$, which by construction is an inverse to our given morphism $\mathrm{Gr}_{P^+} \to (\mathrm{Gr}_G)^+$. □

1.2.2.3. *Some geometric consequences.* —

Proposition 1.2.8. — 1. *The natural morphism* $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_M$ *is ind-affine with geometrically connected fibers.*(16) *In particular, it induces a bijection between the sets of connected components of* $\mathrm{Gr}_{P^\pm}$ *and* Gr_M*,*(17) *and for any connected component of* $\mathrm{Gr}_{P^\pm}$ *the embedding in* $\mathrm{Gr}_{P^\pm}$ *is representable by an open and closed immersion.*

2. *The natural morphism* $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_G$ *is bijective and restricts to a morphism representable by a locally closed immersion on each connected component of* $\mathrm{Gr}_{P^\pm}$*.*

Proof. — (1) Consider a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in §1.2.1.4. Then by Theorem 1.2.6 and Theorem 1.1.6 we have

$$\mathrm{Gr}_{P^\pm} = \mathrm{colim}_i (\mathrm{Gr}_{G,i})^\pm, \quad \mathrm{Gr}_M = \mathrm{colim}_i (\mathrm{Gr}_{G,i})^0,$$

and the morphism $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_M$ is induced by the canonical morphisms $(\mathrm{Gr}_{G,i})^\pm \to (\mathrm{Gr}_{G,i})^0$. Each of these morphisms is affine (see §1.1.2.2), proving that our morphism is ind-affine. Regarding fibers, if K is a field, a morphism $\mathrm{Spec}(K) \to \mathrm{Gr}_M$ must factor through a morphism $\mathrm{Spec}(K) \to (\mathrm{Gr}_{G,i})^0$ for some i. Then we have

$$\mathrm{Gr}_{P^\pm} \times_{\mathrm{Gr}_M} \mathrm{Spec}(K) = \mathrm{colim}_{j \geq i} (\mathrm{Gr}_{G,j})^\pm \times_{(\mathrm{Gr}_{G,j})^0} \mathrm{Spec}(K).$$

The underlying topological space of the right-hand side is an increasing union of connected spaces (see again §1.1.2.2), with closed immersions as transition maps, so it is connected. The first part of the second sentence follows, since a continuous map of topological spaces with connected fibers and which admits a section induces a bijection between connected components.

Finally, since each morphism $(\mathrm{Gr}_{G,i})^\pm \to (\mathrm{Gr}_{G,i})^0$ induces a bijection between sets of connected components, and because $(\mathrm{Gr}_{G,i})^0 \to (\mathrm{Gr}_{G,j})^0$ induces an injection between sets of connected components for any $i \leq j$ (by the identification in Theorem 1.2.6 and §1.2.1.6), the same property holds for the morphism $(\mathrm{Gr}_{G,i})^\pm \to (\mathrm{Gr}_{G,j})^\pm$, so that the embedding of each connected component in $\mathrm{Gr}_{P^\pm}$ is representable by an open and closed immersion by Lemma 1.1.1.

(16)By this we mean that for any field K and any morphism $\mathrm{Spec}(K) \to \mathrm{Gr}_M$, the underlying topological space of the ind-scheme $\mathrm{Gr}_{P^\pm} \times_{\mathrm{Gr}_M} \mathrm{Spec}(K)$ is connected.

(17)More specifically, this bijection sends a connected component of $\mathrm{Gr}_{P^\pm}$ to its image in Gr_M, and a connected component of Gr_M to its inverse image in $\mathrm{Gr}_{P^\pm}$.

(2) Fix a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in the proof of Lemma 1.2.4; then we have a presentation $\mathrm{Gr}_{P^{\pm}} = \mathrm{colim}_i (\mathrm{Gr}_{G,i})^{\pm}$. By Lemma 1.1.4, for any i the map

$$|(\mathrm{Gr}_{G,i})^{\pm}| \to |\mathrm{Gr}_{G,i}|$$

is bijective. Passing to colimits we find that $|\mathrm{Gr}_{P^{\pm}}| \to |\mathrm{Gr}_G|$ is bijective, as claimed.

Let us now prove that for any i the morphism

$$(\mathrm{Gr}_{G,i})^{\pm} \to \mathrm{Gr}_{G,i}$$

restricts to a locally closed immersion on each connected component. Let Y be a connected component of $(\mathrm{Gr}_{G,i})^{\pm}$. Using the notation of the proof of Lemma 1.2.4 we have a $\mathbb{G}_{\mathrm{m}}$-equivariant closed immersion $\mathrm{Gr}_{G,i} \hookrightarrow \bigsqcup_d \mathbb{P}(\bigwedge^d M_i)$ and induced closed immersions $Y \hookrightarrow (\mathrm{Gr}_{G,i})^{\pm} \hookrightarrow \bigsqcup_d (\mathbb{P}(\bigwedge^d M_i))^{\pm}$, see Lemma 1.1.3. Let T be the connected component of $(\mathbb{P}(\bigwedge^d M_i))^{\pm}$ containing the image of Y. On the other hand, as seen in Example 1.1.2, the morphism $\mathbb{P}(\bigwedge^d M_i)^{\pm} \to \mathbb{P}(\bigwedge^d M_i)$ restricts to a locally closed immersion on each connected component. That is, $T \to \mathbb{P}(\bigwedge^d M_i)$ factors through an open subscheme $U \subset \mathbb{P}(\bigwedge^d M_i)$. Let $\tilde{U} = U \times_{\mathbb{P}(\bigwedge^d M_i)} \mathrm{Gr}_{G,i}$. These schemes fit into the following commutative diagram.

$$\begin{tikzcd}
Y \arrow[rr, "\text{open \& closed}"] \arrow[dr, "\text{closed}"] \arrow[dd] & & (\mathrm{Gr}_{G,i})^{\pm} \arrow[dr, "\text{closed}"] \arrow[dd] & \\
& T \arrow[rr, "\text{open \& closed}"] \arrow[dd, "\text{closed}"] & & (\mathbb{P}(\bigwedge^d M_i))^{\pm} \arrow[dd] \\
\tilde{U} \arrow[rr, "\text{open}"'] \arrow[dr, "\text{closed}"'] & & \mathrm{Gr}_{G,i} \arrow[dr, "\text{closed}"] & \\
& U \arrow[rr, "\text{open}"'] & & \mathbb{P}(\bigwedge^d M_i)
\end{tikzcd}$$

Since $Y \to U$ is a closed immersion (see [**SP**, Tag 02V0]) and $\tilde{U} \to U$ is a closed immersion, we deduce from [**SP**, Tag 07RK] that $Y \to U$ is as well, and hence that $Y \to \mathrm{Gr}_{G,i}$ is a locally closed immersion, as desired.

Now, consider a connected component Y of $\mathrm{Gr}_{P^{\pm}}$. We can write $Y = \mathrm{colim}_i Y_i$ where Y_i is a connected component of $(\mathrm{Gr}_{G,i})^{\pm}$ for any i. If Z is an affine scheme and $Z \to \mathrm{Gr}_G$ is a morphism, then this morphism factors through $\mathrm{Gr}_{G,i}$ for some i, and we have

$$Y \times_{\mathrm{Gr}_G} Z = \mathrm{colim}_{j \geq i} Y_j \times_{\mathrm{Gr}_{G,j}} Z.$$

Now for any $j \geq i$ we have

$$\begin{aligned}
&Y_j \times_{\mathrm{Gr}_{G,j}} Z = (Y_j \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,i}) \times_{\mathrm{Gr}_{G,i}} Z \\
&= \big(Y_j \times_{(\mathrm{Gr}_{G,j})^{\pm}} ((\mathrm{Gr}_{G,j})^{\pm} \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,i})\big) \times_{\mathrm{Gr}_{G,i}} Z = (Y_j \times_{(\mathrm{Gr}_{G,j})^{\pm}} (\mathrm{Gr}_{G,i})^{\pm}) \times_{\mathrm{Gr}_{G,i}} Z
\end{aligned}$$

by Lemma 1.1.3. Now, as seen in the proof of Lemma 1.1.1 we have $Y_j \times_{(\mathrm{Gr}_{G,j})^{\pm}} (\mathrm{Gr}_{G,i})^{\pm} = Y_i$, so that

$$Y \times_{\mathrm{Gr}_G} Z = Y_i \times_{\mathrm{Gr}_{G,i}} Z.$$

In particular this ind-scheme is a scheme, and the morphism $Y \times_{\mathrm{Gr}_G} Z \to Z$ is a locally closed immersion because so is the morphism $Y_i \to \mathrm{Gr}_{G,i}$. □

1.2.3. Semi-infinite orbits. — Recall that we have fixed subgroups $T \subset B \subset G$ in §1.2.1.5. We consider the setting of §1.2.2, assuming that χ is a cocharacter of T which is regular dominant. In this case we have $G^0 = T$, $G^+ = B^+$, and $G^- = B$, see Example 1.2.5.

1.2.3.1. *Definition.* — Recall from Proposition 1.2.8(1) that the morphism $\mathrm{Gr}_{B^+} \to \mathrm{Gr}_T$ induces a bijection between the connected components of Gr_{B^+} and Gr_T. It is a standard fact that $|\mathrm{Gr}_T|$ is discrete, with

$$|\mathrm{Gr}_T| = \{L_\lambda : \lambda \in \mathbf{X}^\vee\}.$$

Therefore, the map sending $\lambda \in \mathbf{X}^\vee$ to the connected component $\mathrm{Gr}_T^{(\lambda)}$ containing L_λ induces a bijection between $\mathbf{X}^\vee$ and the set of connected components of Gr_T. For any $\lambda \in \mathbf{X}^\vee$ we have $\mathrm{Gr}_T^{(\lambda)}(\mathbb{C}) = \{L_\lambda\} = |\mathrm{Gr}_T^{(\lambda)}|$. By Lemma 1.2.2 and commutativity of $\mathrm{L}T$, the morphism $\mathrm{L}^{--}T \to \mathrm{Gr}_T$ given by $g \mapsto g \cdot L_\lambda$ induces an isomorphism $\mathrm{L}^{--}T \xrightarrow{\sim} \mathrm{Gr}_T^{(\lambda)}$, since it is representable by an open immersion and a bijection on $\mathbb{C}$-points. (A description of $\mathrm{L}^{--}T$ can be derived from [**Rc4**, Example 2.8] or [**PR**, §3.a].)

For any $\lambda \in \mathbf{X}^\vee$ we will denote by

$$S_\lambda$$

the connected component of Gr_{B^+} corresponding to $\mathrm{Gr}_T^{(\lambda)}$ under the bijection considered above; then we have a natural morphism $S_\lambda \to \mathrm{Gr}_G$ which is representable by a locally closed immersion by Proposition 1.2.8(2).

1.2.3.2. *Ind-affineness.* — The setting considered in §1.2.2 can be made slightly more explicit in this case. Namely, choose a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in §1.2.1.4. Then for any i the scheme of finite type $(\mathrm{Gr}_{G,i})^0$ is discrete, and hence the spectrum of a finite-dimensional $\mathbb{C}$-algebra, see [**EGA1**, Chap. I, Prop. 6.4.4]. Moreover this algebra is a finite product of finite-dimensional local algebras (see [**EGA1**, Chap. I, §6.2]), so $(\mathrm{Gr}_{G,i})^0$ is the disjoint union of the spectra of these local algebras, which are the connected components of $(\mathrm{Gr}_{G,i})^0$. If $L_\lambda \in (\mathrm{Gr}_{G,i})^0(\mathbb{C})$, we denote by $(\mathrm{Gr}_{G,i})^{0,(\lambda)}$ the connected component of $(\mathrm{Gr}_{G,i})^0$ containing L_λ; then $(\mathrm{Gr}_{G,i})^{0,(\lambda)}$ is the spectrum of a finite-dimension local $\mathbb{C}$-algebra. The fiber product

$$S_{\lambda,i} := (\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,i})^0} (\mathrm{Gr}_{G,i})^{0,(\lambda)}$$

is an affine connected scheme of finite type over $\mathbb{C}$ by the results of §1.1.2.2, and we have

$$(\mathrm{Gr}_{G,i})^+ = \bigsqcup_{\lambda} S_{\lambda,i} \tag{1.2.6}$$

where λ runs over the (finite) subset of $\mathbf{X}^\vee$ consisting of the elements such that $L_\lambda \in (\mathrm{Gr}_{G,i})^0(\mathbb{C})$.

Fix $\lambda \in \mathbf{X}^\vee$, and choose i such that $L_\lambda \in (\mathrm{Gr}_{G,i})^0(\mathbb{C})$. For $j \geq i$ we have a closed immersion $(\mathrm{Gr}_{G,i})^{0,(\lambda)} \to (\mathrm{Gr}_{G,j})^{0,(\lambda)}$ induced by a surjection of the associated finite-dimensional local $\mathbb{C}$-algebras, and an induced closed immersion

$$(\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,i})^0} (\mathrm{Gr}_{G,i})^{0,(\lambda)} = (\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,i})^{0,(\lambda)} \\ \to (\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,(\lambda)}.$$

Now the natural morphism $(\mathrm{Gr}_{G,i})^+ \to (\mathrm{Gr}_{G,j})^+$ is also a closed immersion (see Lemma 1.1.3), so it induces a closed immersion

$$(\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,(\lambda)} \to (\mathrm{Gr}_{G,j})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,(\lambda)}.$$

Composing these immersions we obtain a closed immersion $S_{\lambda,i} \to S_{\lambda,j}$. Now take colimits on both sides of (1.2.6) to obtain

$$\bigsqcup_{\lambda} S_\lambda = \mathrm{Gr}_{B^+} = \mathrm{colim}_i (\mathrm{Gr}_{G,i})^+ = \bigsqcup_{\lambda} \mathrm{colim}_i S_{\lambda_i}$$

from which we deduce a presentation

$$S_\lambda = \mathrm{colim}_i S_{\lambda,i}.$$

In particular, these considerations show that S_λ is an ind-affine ind-scheme.

Note that this fact can also be seen in a different way, by remarking that the morphism

$$\mathrm{L}^{--}B^+ \to S_\lambda \tag{1.2.7}$$

defined by $g \mapsto x^\lambda g \cdot L_0^{B^+}$ (where $L_0^{B^+}$ is the image of $L_0 \in \mathrm{Gr}_T(\mathbb{C})$ in $\mathrm{Gr}_{B^+}(\mathbb{C})$) is an isomorphism, since it is representable by an open immersion (see Lemma 1.2.2) and gives a bijection on $\mathbb{C}$-points.

Remark 1.2.9. — Let $\mu \in \mathbf{X}_+^\vee$. Using the considerations above, it is not difficult to check that we have

$$\left(\overline{\mathrm{Gr}_G^\mu}\right)^+ = \bigsqcup_{\lambda} S_\lambda \times_{\mathrm{Gr}_G} \overline{\mathrm{Gr}_G^\mu}$$

where λ runs over the (finite) subset of $\mathbf{X}^\vee$ consisting of the elements such that $L_\lambda \in \overline{\mathrm{Gr}_G^\mu}(\mathbb{C})$, and that moreover each $S_\lambda \times_{\mathrm{Gr}_G} \overline{\mathrm{Gr}_G^\mu}$ is an affine scheme of finite type over $\mathbb{C}$.(18) The fact that this fiber product is affine can be used (combined with a general result on the codimension of the complement of an affine open subscheme, see [**SP**, Tag 0BCV]) to give an alternative proof of the crucial dimension estimates [**BR**, Theorem 1.5.2]; see [**FS**, Corollary VI.3.8] for details.

(18) A more complicated proof of this fact is given in [**FS**, Proposition VI.3.7]. The simpler argument was explained to us by T. Richarz.

1.2.3.3. *Relation with the Iwasawa decomposition.* — We have
$$\mathrm{Gr}_{B^+}(\mathbb{C}) = B^+(\mathscr{K})/B^+(\mathscr{O}),$$
see (1.2.1). If we denote by U^+ the unipotent radical of B^+, we have $B^+ \cong T \ltimes U^+$. Denoting by $L_\lambda^{B^+}$ the image of L_λ in $\mathrm{Gr}_{B^+}(\mathbb{C})$ we deduce that
$$\mathrm{Gr}_{B^+}(\mathbb{C}) = \bigsqcup_{\lambda \in \mathbf{X}^\vee} U^+(\mathscr{K}) \cdot L_\lambda^{B^+},$$
and
$$S_\lambda(\mathbb{C}) = U^+(\mathscr{K}) \cdot L_\lambda^{B^+}.$$

It follows from Proposition 1.2.8(2) that for any $\lambda \in \mathbf{X}^\vee$ the morphism $S_\lambda \to \mathrm{Gr}_G$ is representable by a locally closed immersion, and that these morphisms induce a bijection
$$\bigsqcup_{\lambda \in \mathbf{X}^\vee} S_\lambda(\mathbb{C}) \xrightarrow{\sim} \mathrm{Gr}_G(\mathbb{C}).$$
By (1.2.1) once again we have $\mathrm{Gr}_G(\mathbb{C}) = G(\mathscr{K})/G(\mathscr{O})$. These considerations therefore show that
$$G(\mathscr{K}) = \bigsqcup_{\lambda \in X_*(T)} U^+(\mathscr{K}) \cdot x^\lambda \cdot G(\mathscr{O}),$$
which provides a geometric proof of the Iwasawa decomposition in this setting.

Remark 1.2.10. — For any $\lambda \in \mathbf{X}^\vee$, the action of x^λ induces an isomorphism of ind-schemes $S_0 \xrightarrow{\sim} S_\lambda$. This lets us reduce many questions about semi-infinite orbits to the case of S_0.

1.2.3.4. *Opposite version.* — One can of course play the same game with the Borel subgroup B (and its unipotent radical U) instead of B^+. The connected components of Gr_B are again parametrized by $\mathbf{X}^\vee$; the component corresponding to λ is denoted T_λ. All the considerations above have obvious analogues for these variants.

1.3. Main sheaf-theoretic constructions

In this section, we work up to the statement of the geometric Satake equivalence. Along the way, we review a number of other essential definitions and statements. (For more details, see [**BR**] or [**Ac3**, Chapter 9].)

1.3.1. Sheaves and convolution. — Let $\Bbbk$ be a noetherian commutative ring of finite global dimension. This ring will be the coefficient ring for all constructible (complexes of) sheaves.

We denote by
$$D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$$
the constructible L^+G-equivariant derived category of Gr_G in the sense of Bernstein–Lunts, and by
$$\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \subset D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$$
the abelian category of L^+G-equivariant perverse sheaves. (For a discussion of technical issues related to the definition of $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, see [**Ac3**, §9.1] or [**BR**, §1.16.4].

For an overview of equivariant perverse sheaves, see [**Ac3**, §6.2] or [**BR**, §1.4.1 and §1.16.1].)

For $\lambda \in \mathbf{X}_+^\vee$, let $j^\lambda : \mathrm{Gr}_G^\lambda \to \mathrm{Gr}_G$ denote the embedding. We have three L^+G-equivariant perverse sheaves naturally associated with λ:

$$\begin{aligned} \mathscr{J}_*(\lambda, \Bbbk) &:= {}^{\mathrm{p}}\mathscr{H}^0\big((j^\lambda)_* \underline{\Bbbk}_{\mathrm{Gr}_G^\lambda}[\langle \lambda, 2\rho \rangle]\big), \\ \mathscr{J}_!(\lambda, \Bbbk) &:= {}^{\mathrm{p}}\mathscr{H}^0\big((j^\lambda)_! \underline{\Bbbk}_{\mathrm{Gr}_G^\lambda}[\langle \lambda, 2\rho \rangle]\big), \\ \mathscr{J}_{!*}(\lambda, \Bbbk) &:= (j^\lambda)_{!*} \underline{\Bbbk}_{\mathrm{Gr}_G^\lambda}[\langle \lambda, 2\rho \rangle]. \end{aligned}$$

The $\Bbbk$-module $\mathrm{Hom}_{\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)}(\mathscr{J}_!(\lambda, \Bbbk), \mathscr{J}_*(\lambda, \Bbbk))$ is free of rank 1 (with a canonical generator provided by adjunction), and the image of any generator is $\mathscr{J}_{!*}(\lambda, \Bbbk)$. If $\Bbbk$ is a field then $\mathscr{J}_{!*}(\lambda, \Bbbk)$ is a simple perverse sheaf. If $\Bbbk$ is a connected ring then $\mathscr{J}_!(\lambda, \Bbbk)$ and $\mathscr{J}_*(\lambda, \Bbbk)$ are indecomposable.

Example 1.3.1. — 1. In case $\lambda = 0$, the natural morphisms

$$\mathscr{J}_!(0, \Bbbk) \to \mathscr{J}_{!*}(0, \Bbbk) \to \mathscr{J}_*(0, \Bbbk)$$

are clearly isomorphisms. The resulting object will be denoted δ_{Gr}.

2. In the setting considered in Example 1.2.3, we observed that the orbit $\mathrm{Gr}_{\mathrm{GL}(n)}^{\varepsilon_1^\vee}$ is closed. Therefore the natural morphisms

$$\mathscr{J}_!(\varepsilon_1^\vee, \Bbbk) \to \mathscr{J}_{!*}(\varepsilon_1^\vee, \Bbbk) \to \mathscr{J}_*(\varepsilon_1^\vee, \Bbbk)$$

are all isomorphisms, and these perverse sheaves identify with $\underline{\Bbbk}_{\mathrm{Gr}_{\mathrm{GL}(n)}^{\varepsilon_1^\vee}}[n-1]$. Similar comments apply more generally to minuscule coweights (in the sense of §1.2.1.6).

The category $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ is equipped with a bifunctor called the *convolution product*, and denoted by $\star^{\mathrm{L}^+G}$. To define $\star^{\mathrm{L}^+G}$, we introduce the space

$$\mathrm{Gr}_G \mathbin{\widetilde{\times}} \mathrm{Gr}_G := \mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G.$$

(More formally, $\mathrm{Gr}_G \mathbin{\widetilde{\times}} \mathrm{Gr}_G$ is the fppf sheafification of the functor $R \mapsto (\mathrm{L}G(R) \times \mathrm{Gr}_G(R))/\mathrm{L}^+G(R)$, where L^+G acts on $\mathrm{L}G \times \mathrm{Gr}_G$ via $g \cdot (h, x) = (hg^{-1}, g \cdot x)$.) If $\lambda, \mu \in \mathbf{X}_+^\vee$, we also set $\mathrm{Gr}_G^\lambda \mathbin{\widetilde{\times}} \mathrm{Gr}_G^\mu := (p_{\mathrm{Gr}})^{-1}(\mathrm{Gr}_G^\lambda) \times^{\mathrm{L}^+G} \mathrm{Gr}_G^\mu$. Then the collection $(\mathrm{Gr}_G^\lambda \mathbin{\widetilde{\times}} \mathrm{Gr}_G^\mu : \lambda, \mu \in \mathbf{X}_+^\vee)$ provides an algebraic stratification of (the topological space associated with) the ind-scheme $\mathrm{Gr}_G \mathbin{\widetilde{\times}} \mathrm{Gr}_G$. We will denote by

$$m : \mathrm{Gr}_G \mathbin{\widetilde{\times}} \mathrm{Gr}_G \to \mathrm{Gr}_G \tag{1.3.1}$$

the morphism defined by $m([g, h\mathrm{L}^+G]) = gh\mathrm{L}^+G$. For $\mathscr{A}, \mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, we denote by $\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}$ the unique object in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G \mathbin{\widetilde{\times}} \mathrm{Gr}_G, \Bbbk)$ whose pullback to $D^{\mathrm{b}}_{\mathrm{L}^+G \times \mathrm{L}^+G}(\mathrm{L}G \times \mathrm{Gr}_G)$ is $(p_{\mathrm{Gr}})^*(\mathscr{A}) \boxtimes^L_\Bbbk \mathscr{B}$. Then we set

$$\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B} := m_*(\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}). \tag{1.3.2}$$

It is a classical fact that this bifunctor admits a natural associativity constraint, so that the pair $(D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G})$ becomes a monoidal category, with unit object δ_{Gr}. Later in the book we will also consider convolution in the case when the first object is not L^+G-equivariant; indeed, by the same procedure as above, we may define, for any $\mathscr{A}$ in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, the complex

$$\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B} := m_*(\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B})$$

in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$.

We now consider t-exactness properties of $\star^{\mathrm{L}^+G}$. The following claim is due to Mirković–Vilonen, see [**BR**, §1.6.3].

Lemma 1.3.2. — *If $\mathscr{F}$ is a perverse sheaf on $\mathrm{Gr}_G \mathbin{\widetilde{\times}} \mathrm{Gr}_G$ which is constructible with respect to the stratification $(\mathrm{Gr}^\lambda_G \mathbin{\widetilde{\times}} \mathrm{Gr}^\mu_G : \lambda, \mu \in \mathbf{X}^\vee_+)$, then $m_*\mathscr{F}$ is a perverse sheaf on Gr_G, constructible with respect to the stratification $(\mathrm{Gr}^\lambda_G : \lambda \in \mathbf{X}^\vee_+)$.*

In (1.3.2), suppose $\mathscr{A}$ and $\mathscr{B}$ belong to $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. Then one can ask whether $\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}$ is perverse. The answer is yes if at least one of $\mathscr{A}$ or $\mathscr{B}$ has either flat stalks or flat hypercohomology (cf. [**BR**, Lemma 1.10.9]); in particular, this holds if $\Bbbk$ is a field, or, more generally, a semisimple ring. In general, $\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}$ need not be perverse, but it is concentrated in nonpositive perverse degrees. Lemma 1.3.2 implies that $\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}$ again lies in nonpositive perverse degrees, and is perverse if $\Bbbk$ is a semisimple ring. We set

$$\mathscr{A} \star^{\mathrm{L}^+G}_0 \mathscr{B} := {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \cong m_* {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}).$$

Using the above considerations, it is not difficult to check that for $\mathscr{A}, \mathscr{B}, \mathscr{C}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ we have canonical isomorphisms

$$\begin{aligned}(\mathscr{A} \star^{\mathrm{L}^+G}_0 \mathscr{B}) \star^{\mathrm{L}^+G}_0 \mathscr{C} &\cong {}^{\mathrm{p}}\mathscr{H}^0\big((\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \star^{\mathrm{L}^+G} \mathscr{C}\big), \\ \mathscr{A} \star^{\mathrm{L}^+G}_0 (\mathscr{B} \star^{\mathrm{L}^+G}_0 \mathscr{C}) &\cong {}^{\mathrm{p}}\mathscr{H}^0\big(\mathscr{A} \star^{\mathrm{L}^+G} (\mathscr{B} \star^{\mathrm{L}^+G} \mathscr{C})\big).\end{aligned}$$

From this we obtain that the associativity constraint in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ induces one in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, making $(\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G}_0)$ into a monoidal category. Again, the unit object in this monoidal category is δ_{Gr}. Note also that Lemma 1.3.2 implies that for $\mathscr{A}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ the functors $\mathscr{A} \star^{\mathrm{L}^+G}_0 (-)$ and $(-) \star^{\mathrm{L}^+G}_0 \mathscr{A}$ are right exact.

Broadly speaking, the goal of the geometric Satake equivalence is to describe the monoidal category $(\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G}_0)$ in terms of representations of some affine $\Bbbk$-group scheme.

Remark 1.3.3. — Above, we have summarized the t-exactness consequences of Lemma 1.3.2 for $\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}$ in the case where $\mathscr{A}$ and $\mathscr{B}$ both belong to $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. We will see later (see Corollary 3.3.3) that the same conclusions hold if $\mathscr{A}$ is an arbitary object of $\mathsf{Perv}(\mathrm{Gr}_G, \Bbbk)$ (i.e., not assumed to be L^+G-equivariant): $\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}$ is concentrated in nonpositive perverse degrees, and is perverse if $\Bbbk$ is semisimple. However, the proof will require a rather different perspective on convolution.

1.3.2. The fiber functor. — Let $\mathsf{Mof}_{\Bbbk}$ denote the category of finitely generated $\Bbbk$-modules, and consider the functor

$$\mathsf{F} := \mathsf{H}^{\bullet}(\mathrm{Gr}_G, -) : \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to \mathsf{Mof}_{\Bbbk}.$$

(Of course this functor actually yields *graded* $\Bbbk$-modules, but this will mostly be ignored here.) Under the anticipated relation to representations of an affine $\Bbbk$-group scheme, the functor F will correspond to the forgetful functor sending a representation to its underlying $\Bbbk$-module. In order to prove this, we first need to show that F has some basic properties that a forgetful functor obviously has. The first such property is the following: see [**BR**, Theorem 1.10.4].

Proposition 1.3.4. — *The functor* F *is exact and faithful.*

The proof involves equipping F with some additional structure coming from the geometry of the *semi-infinite orbits* encountered in §1.2.3. A crucial step for the proof of the geometric Satake equivalence is the following claim, see [**BR**, Proposition 1.10.1].

Proposition 1.3.5. — *For any* $\lambda \in \mathbf{X}^\vee$ *and* $\mathscr{A}$ *in* $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$*, there exists a canonical (in particular, functorial) isomorphism of* $\Bbbk$*-modules*

$$\mathsf{H}^{\bullet}_c(S_\lambda, \mathscr{A}) \cong \mathsf{H}^{\bullet}_{T_\lambda}(\mathrm{Gr}_G, \mathscr{A}).$$

Moreover, these modules vanish in all degrees different from $\langle \lambda, 2\rho \rangle$*.*

In this statement, the isomorphism between the two $\Bbbk$-modules follows from Braden's theory of *hyperbolic localization* (see [**Bra, DrG, Rc3**]), and applies to a large class of complexes. The vanishing statement, however, only holds for L^+G-equivariant perverse sheaves, and relies on a fine analysis of the intersections $S_\lambda \cap \mathrm{Gr}_G^\mu$ and $T_\lambda \cap \mathrm{Gr}_G^\mu$, see [**BR**, §1.5.2] (see also Remark 1.2.9).

Once this proposition is proved one sets, for $\lambda \in \mathbf{X}^\vee$,

$$\mathsf{F}_\lambda := \mathsf{H}_c^{\langle \lambda, 2\rho \rangle}(S_\lambda, -) \cong \mathsf{H}_{T_\lambda}^{\langle \lambda, 2\rho \rangle}(\mathrm{Gr}_G, -) : \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to \mathsf{Mof}_{\Bbbk}.$$

These functors let us equip F with a new grading indexed by $\mathbf{X}^\vee$, as follows.

Corollary 1.3.6. — *For any* $\mathscr{A}$ *in* $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$*, there exists a canonical (in particular, functorial) isomorphism of* $\Bbbk$*-modules*

$$\mathsf{F}(\mathscr{A}) \cong \bigoplus_{\lambda \in \mathbf{X}^\vee} \mathsf{F}_\lambda(\mathscr{A}).$$

Moreover, this is an isomorphism of $\mathbb{Z}$*-graded* $\Bbbk$*-modules, where the left-hand side carries its natural grading, and the right-hand side is graded by placing* $\mathsf{F}_\lambda(\mathscr{A})$ *in degree* $\langle \lambda, 2\rho \rangle$*.*

Here the corollary follows quite formally from the vanishing statement in Proposition 1.3.5; see [**BR**, Theorem 1.10.4] for details. We will not review its proof in detail, but will only recall that for $\mathscr{A}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, the submodule $\mathsf{F}_\lambda(\mathscr{A}) \subset \mathsf{F}(\mathscr{A})$ is the image of the (injective) map

$$\mathsf{H}_{T_\lambda}^{\langle \lambda, 2\rho \rangle}(\mathrm{Gr}_G, \mathscr{A}) \to \mathsf{H}^{\langle \lambda, 2\rho \rangle}(\mathrm{Gr}_G, \mathscr{A})$$

induced by the $(_!, ^!)$-adjunction for the embedding $\overline{T_\lambda} \hookrightarrow \mathrm{Gr}_G$.

Let us sketch how to deduce Proposition 1.3.4 from Proposition 1.3.5 and Corollary 1.3.6. For exactness of F, it is enough to show that each F_λ is exact, and this follows from the vanishing statement in Proposition 1.3.5 by considering an appropriate long exact sequence in cohomology. Then, given this exactness, the faithfulness of F is equivalent to the property that it does not kill any nonzero object. For this one checks (using a description of $T_\lambda \cap \mathrm{Gr}_G^\mu$ in this case) that $\mathsf{F}_\lambda(\mathscr{A}) \neq 0$ if $\lambda \in \mathbf{X}^\vee_+$ is such that Gr_G^λ is open in the support of $\mathscr{A}$: see [**BR**, Theorem 1.5.9(2)] for details.

Remark 1.3.7. — In order to define the subfunctor $\mathsf{F}_\lambda \subset \mathsf{F}$, we have chosen initially a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. However, the subfunctor $\mathsf{F}_\lambda \subset \mathsf{F}$ does not depend on these choices provided λ is interpreted as a cocharacter of the "universal maximal torus" H of G, as explained in [**BR**, Lemma 1.5.11 and Remark 1.10.5]. (Recall that the universal maximal torus is defined as the quotient of *any* Borel subgroup by its unipotent radical. Different choices of Borel subgroups yield canonically isomorphic tori, but this universal torus is *not* a subgroup of G.)

Let us explain this more concretely in a special case which will be sufficient for our purposes. For any w in the Weyl group $W_{\mathrm{f}} = N_G(T)/T$ one can consider the subgroup $B_w := \dot{w}B\dot{w}^{-1}$ where $\dot{w}$ is any lift of w to $N_G(T)$; its unipotent radical is $U_w := \dot{w}U\dot{w}^{-1}$. Then we have natural isomorphisms

$$\varphi_e : T \xrightarrow{\sim} B/U = H, \quad \varphi_w : T \xrightarrow{\sim} B_w/U_w = H,$$

such that $\varphi_w^{-1} \circ \varphi_e$ is the automorphism of T defined by w. Therefore if $\lambda \in \mathbf{X}^\vee = X_*(T)$ is considered as a cocharacter of H via φ_e, then the corresponding cocharacter of T seen as a maximal torus in B_w is $w(\lambda)$. Therefore, for any $\mathscr{A}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ we have

$$\mathsf{H}^n_{T^w_\mu}(\mathrm{Gr}_G, \mathscr{A}) = 0 \quad \text{unless } n = \langle w^{-1}(\mu), 2\rho \rangle,$$

and $\mathsf{F}_\lambda \subset \mathsf{F}$ is also the image of the canonical morphism

$$\mathsf{H}^{\langle \lambda, 2\rho \rangle}_{T^w_{w(\lambda)}}(\mathrm{Gr}_G, \mathscr{A}) \to \mathsf{H}^{\langle \lambda, 2\rho \rangle}(\mathrm{Gr}_G, \mathscr{A}),$$

where T^w_μ is the connected component of Gr_{B_w} containing the image of L_μ under the embedding $\mathrm{Gr}_T \to \mathrm{Gr}_{B_w}$. (In the vanishing statement in Proposition 1.3.5, the character "2ρ" should be interpreted as the opposite of the character of T on the highest nonzero wedge power of the Lie algebra of B; for the Borel subgroup B_w this character is therefore $w(2\rho)$, which justifies the vanishing statement given here.)

1.3.3. Fusion product and the commutativity constraint. — If the monoidal category $(\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star_0^{\mathrm{L}^+G})$ is expected to be equivalent to the category of representations of some affine $\Bbbk$-group scheme, then the product $\star_0^{\mathrm{L}^+G}$ should be "commutative," i.e. admit a commutativity constraint. In fact, this commutativity constraint is a prerequisite to proving the equivalence. The construction of the commutativity constraint depends on an alternative description of $\star^{\mathrm{L}^+G}$ (called the "fusion product") which we recall now, following [**MV2**]. (The idea of this construction goes back to Drinfeld, see [**BD**].) This construction is phrased in terms of certain moduli problems which will be discussed in more detail in Section 2.2 below.

We set $C = \mathbb{A}^1_{\mathbb{C}}$. For R a $\mathbb{C}$-algebra, and y an R-point of C, we will denote by $\Gamma_y \subset C_R := C \times \mathrm{Spec}(R)$ the graph of y, and by $\widehat{\Gamma}_y$ the completion of C_R along Γ_y. (We regard $\widehat{\Gamma}_y$ as an ordinary (affine) scheme, not as a formal scheme.) We also set $\widehat{\Gamma}_y^\circ := \widehat{\Gamma}_y \smallsetminus \Gamma_y$. Similarly, given two R-points y_1, y_2 of C, we denote by $\widehat{\Gamma_{y_1} \cup \Gamma_{y_2}}$ the completion of C_R along $\Gamma_{y_1} \cup \Gamma_{y_2}$, and set $(\widehat{\Gamma_{y_1} \cup \Gamma_{y_2}})^\circ := \widehat{\Gamma_{y_1} \cup \Gamma_{y_2}} \smallsetminus (\Gamma_{y_1} \cup \Gamma_{y_2})$. Finally, for any $\mathbb{C}$-scheme X we will denote by $\mathcal{F}^0_X$ the trivial principal G-bundle over X. (See [**BR**, §1.7.1] for a very brief survey of principal G-bundles, or §2.1.2 below for more details and references.)

The "fusion space" (or "Beĭlinson–Drinfeld Grassmannian") Fus_G is defined to be the ind-projective ind-scheme over C^2 whose R-points (for R a $\mathbb{C}$-algebra) are given by

$$\left\{ (y_1, y_2, \mathcal{E}, \beta) \;\middle|\; \begin{array}{c} y_1, y_2 \in C(R),\ \mathcal{E} \text{ a principal } G\text{-bundle on } \widehat{\Gamma_{y_1} \cup \Gamma_{y_2}}, \\ \beta : \mathcal{E}_{|(\widehat{\Gamma_{y_1} \cup \Gamma_{y_2}})^\circ} \xrightarrow{\sim} \mathcal{F}^0_{(\widehat{\Gamma_{y_1} \cup \Gamma_{y_2}})^\circ} \text{ a trivialization} \end{array} \right\}.$$

(Here we implicitly consider *isomorphism classes* of quadruples $(y_1, y_2, \mathcal{E}, \beta)$, for the obvious notion of isomorphism.) See [**Zh4**, Theorem 3.1.3] for a discussion of the proof of representability of this functor. This ind-scheme is denoted Gr_{G,C^2} in [**BR**, §1.7.3].

Let ΔC denote the diagonal copy of C inside C^2. It is well known that we have canonical identifications

$$(\mathrm{Fus}_G)_{|C^2 \smallsetminus \Delta C} \cong \mathrm{Gr}_G \times \mathrm{Gr}_G \times (C^2 \smallsetminus \Delta C), \qquad (\mathrm{Fus}_G)_{|\Delta C} \cong \mathrm{Gr}_G \times C, \tag{1.3.3}$$

see [**Zh4**, Proposition 3.1.13]. Denote the corresponding inclusion maps by

$$j : \mathrm{Gr}_G \times \mathrm{Gr}_G \times (C^2 \smallsetminus \Delta C) \hookrightarrow \mathrm{Fus}_G, \qquad i : \mathrm{Gr}_G \times C \hookrightarrow \mathrm{Fus}_G.$$

Remark 1.3.8. — Given a choice of cocharacter $\lambda \in \mathbf{X}^\vee$, as in §1.2.2 we have an induced action of the multiplicative group $\mathbb{G}_{\mathrm{m},C^2}$ on Fus_G. One can prove, essentially as in the case considered in Lemma 1.2.4, that this action is Zariski locally linearisable; see [**HR1**, Lemma 3.16] for details.[(19)] One can thus consider associated schemes $(\mathrm{Fus}_G)^+$, $(\mathrm{Fus}_G)^-$ and $(\mathrm{Fus}_G)^0$. On the other hand, the definition of Fus_G can be extended to more general groups over $\mathbb{C}$, and in particular to the groups P^+, P^- and M of §1.2.2.2. It is proved in [**HR1**, Theorem 3.17] that we have canonical identifications

$$\mathrm{Fus}_M \xrightarrow{\sim} (\mathrm{Fus}_G)^0, \quad \mathrm{Fus}_{P^+} \xrightarrow{\sim} (\mathrm{Fus}_G)^+, \quad \mathrm{Fus}_{P^-} \xrightarrow{\sim} (\mathrm{Fus}_G)^-,$$

and one deduces consequences similar to those of Proposition 1.2.8.

In particular, consider the case when λ is regular dominant, as in §1.2.3. In this case $P^+ = B^+$, $P^- = B$ and $M = T$. The connected components of Fus_T are parametrized by $\mathbf{X}^\vee$. More specifically, under the identifications (1.3.3), the component parametrized by λ identifies with $\mathrm{Gr}_T^{(\lambda)}$ over any point of ΔC, and with

(19) In the more general setting considered in [**HR1**], this action is only *étale* locally linearizable, but the stronger condition of Zariski local linearizability holds in the special case we consider here.

$\bigsqcup_\mu \mathrm{Gr}_T^{(\mu)} \times \mathrm{Gr}_T^{(\lambda-\mu)}$ over any point in $C^2 \smallsetminus \Delta C$. As a consequence, the connected components of Fus_B are also parametrized by $\mathbf{X}^\vee$, and the component T_{λ,C^2} parametrized by λ satisfies

$$(T_{\lambda,C^2})_{|C^2\smallsetminus\Delta C} \cong \bigsqcup_\mu T_\mu \times T_{\lambda-\mu} \times (C^2 \smallsetminus \Delta C), \qquad (T_{\lambda,C^2})_{|\Delta C} \cong T_\lambda \times C.$$

(In [**BR**, §1.8.3], the analogue of T_{λ,C^2} for the group B^+ is denoted $S_\lambda(X^2)$.)

The key point for the construction of the commutativity constraint is that for $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ there exists a canonical isomorphism

$$i^* j_{!*}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_\Bbbk \mathscr{B}) \overset{L}{\boxtimes}_\Bbbk \underline{\Bbbk}_{C^2\smallsetminus\Delta C}[2]) \cong (\mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \overset{L}{\boxtimes}_\Bbbk \underline{\Bbbk}_{\Delta C}[2]. \tag{1.3.4}$$

The functor $(-) \boxtimes^L_\Bbbk \underline{\Bbbk}_{\Delta C}[2]$ is fully faithful on perverse sheaves by [**BBDG**, Proposition 4.2.5], so to specify a natural isomorphism

$$\sigma^{\mathrm{Fus}}_{\mathscr{A},\mathscr{B}} : \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B} \cong \mathscr{B} \star_0^{\mathrm{L}^+G} \mathscr{A}$$

it is enough to give a natural isomorphism

$$i^* j_{!*}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_\Bbbk \mathscr{B}) \overset{L}{\boxtimes}_\Bbbk \underline{\Bbbk}_{C^2\smallsetminus\Delta C}[2]) \cong i^* j_{!*}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{B} \overset{L}{\boxtimes}_\Bbbk \mathscr{A}) \overset{L}{\boxtimes}_\Bbbk \underline{\Bbbk}_{C^2\smallsetminus\Delta C}[2]). \tag{1.3.5}$$

To construct this map, we use the map $\mathrm{swap} : \mathrm{Fus}_G \to \mathrm{Fus}_G$ that swaps y_1 and y_2. The isomorphism (1.3.5) is obtained by combining the natural isomorphisms

$$\mathrm{swap}^*_{|C^2\smallsetminus\Delta C}(\mathscr{A} \overset{L}{\boxtimes}_\Bbbk \mathscr{B} \overset{L}{\boxtimes}_\Bbbk \underline{\Bbbk}_{C^2\smallsetminus\Delta C}[2]) \cong \mathscr{B} \overset{L}{\boxtimes}_\Bbbk \mathscr{A} \overset{L}{\boxtimes}_\Bbbk \underline{\Bbbk}_{C^2\smallsetminus\Delta C}[2]$$

and

$$i^* \circ j_{!*} \circ \mathrm{swap}^*_{|C^2\smallsetminus\Delta C} \cong i^* \circ \mathrm{swap}^* \circ j_{!*} \cong \mathrm{swap}^*_{|\Delta C} \circ i^* \circ j_{!*} \cong i^* \circ j_{!*},$$

where the last isomorphism comes from the fact that swap restricts to the identity map on $(\mathrm{Fus}_G)_{|\Delta C}$.

The isomorphism σ^{Fus} is not quite the commutativity constraint one wants for Tannakian formalism. To define the constraint that will really be used, by additivity one can assume that $\mathscr{A}$ and $\mathscr{B}$ are supported on one connected component of Gr_G. If $\mathscr{A}$ and $\mathscr{B}$ are both supported on odd components one sets

$$\sigma^{\mathrm{Com}}_{\mathscr{A},\mathscr{B}} = -\sigma^{\mathrm{Fus}}_{\mathscr{A},\mathscr{B}},$$

and if at least one of $\mathscr{A}, \mathscr{B}$ is supported on an even connected component one sets

$$\sigma^{\mathrm{Com}}_{\mathscr{A},\mathscr{B}} = \sigma^{\mathrm{Fus}}_{\mathscr{A},\mathscr{B}}.$$

(See §1.2.1.6 for the notions of even and odd components.)

1.3.4. Monoidal structure on total cohomology. — The next task is to construct is a tensor structure on F in the sense of [**DM**, Definition 1.8], i.e. functorial isomorphisms

$$\mathsf{F}(\mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \cong \mathsf{F}(\mathscr{A}) \otimes_{\Bbbk} \mathsf{F}(\mathscr{B}) \tag{1.3.6}$$

for all $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, which are compatible (in the natural sense) with the associativity and commutativity constraints defined above.

This construction again involves the fusion product perspective of §1.3.3. More specifically, using the notation introduced there, and denoting by $f : \mathrm{Fus}_G \to C^2$ the (proper) structure morphism, the main step in the construction is the following claim (see [**BR**, Proposition 1.8.1]).

Lemma 1.3.9. — *For any $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, the complex*

$$\mathscr{C} := f_* j_{!*}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^2 \smallsetminus \Delta C}[2])$$

has the following properties:

1. *for any $x \in \Delta C$ and any $n \in \mathbb{Z}$, the n-th cohomology of the stalk of $\mathscr{C}$ at x is canonically isomorphic to $\mathsf{H}^{n+2}(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B})$;*
2. *for any $x \in C^2 \smallsetminus \Delta C$ and any $n \in \mathbb{Z}$, the n-th cohomology of the stalk of $\mathscr{C}$ at x is canonically isomorphic to $\mathsf{H}^{n+2}(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \boxtimes_{\Bbbk}^L \mathscr{B}))$;*
3. *all cohomology sheaves $\mathscr{H}^n(\mathscr{C})$ are locally constant (and hence constant).*

In this lemma, (1) follows directly from (1.3.4), and (2) is clear from the definition. Property (3) however requires an argument. Once these properties are known, statement (3) allows us to identify the stalks of $\mathscr{H}^n(\mathscr{C})$ at all points in a canonical way. We therefore obtain a canonical isomorphism of graded $\Bbbk$-modules

$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B})).$$

To deduce the isomorphism (1.3.6), one then shows that there exists a canonical isomorphism

$$\mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B})) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\Bbbk} \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{B}), \tag{1.3.7}$$

see [**BR**, Lemma 1.10.10].

Remark 1.3.10. — In case $\Bbbk$ is a field, (1.3.7) follows from the Künneth formula. For the general case, an important step in the proof is the following claim (see [**BR**, Lemma 1.10.9]), which we recall for later use: if $\mathscr{A}, \mathscr{B}$ belong to $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and if $\mathsf{F}(\mathscr{A})$ is flat over $\Bbbk$, then $\mathscr{A} \boxtimes_{\Bbbk}^L \mathscr{B}$ is a perverse sheaf (on $\mathrm{Gr}_G \times \mathrm{Gr}_G$).

Once we have the isomorphism (1.3.6), one needs to prove that it is compatible with the associativity and commutativity constraints for $\star_0^{\mathrm{L}^+G}$, which is not very difficult; see [**BR**, §1.8.2] for details. One also shows that this isomorphism is compatible with the $\mathbf{X}^\vee$-grading provided by Corollary 1.3.6; see [**BR**, Proposition 1.8.3].

1.3.5. Statement. — We are now ready to state the geometric Satake equivalence. Let $G^\vee_{\mathbb{Z}}$ be the unique (up to isomorphism) split reductive group scheme over $\mathbb{Z}$ whose base change to $\mathbb{C}$ has root datum $(\mathbf{X}^\vee, \mathbf{X}, \mathfrak{R}^\vee, \mathfrak{R})$. (The uniqueness of such a group scheme is guaranteed by [**SGA3.3**, Exposé XXIII, Corollaire 5.4]. Also, we insist that the roles of characters/cocharacters and roots/coroots have been switched compared to the root datum of G.) If $\Bbbk$ is a noetherian commutative ring of finite global dimension, we then set

$$G^\vee_{\Bbbk} := \mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\mathbb{Z})} G^\vee_{\mathbb{Z}},$$

and let $\mathsf{Rep}(G^\vee_{\Bbbk})$ denote the symmetric monoidal category of $G^\vee_{\Bbbk}$-modules that are finitely generated over $\Bbbk$. (The monoidal structure is given by the tensor product $\otimes_{\Bbbk}$.) We will also denote by $\mathrm{For}_{G^\vee_{\Bbbk}} : \mathsf{Rep}(G^\vee_{\Bbbk}) \to \mathsf{Mof}_{\Bbbk}$ the natural forgetful functor.

Theorem 1.3.11 **(Mirković–Vilonen [MV2]).** — *Under the assumptions above, there exists an equivalence of abelian symmetric monoidal categories*

$$\mathcal{S} : (\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star_0^{\mathrm{L}^+G}) \to (\mathsf{Rep}(G^\vee_{\Bbbk}), \otimes_{\Bbbk})$$

and an isomorphism of monoidal functors

$$\mathrm{For}_{G^\vee_{\Bbbk}} \circ \mathcal{S} \cong \mathsf{F}.$$

Here, $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ is considered as a symmetric monoidal category with respect to the commutativity constraint σ^{Com}, and $(\mathsf{Rep}(G^\vee_{\Bbbk}), \otimes_{\Bbbk})$ is endowed with the obvious commutativity constraint. The equivalence $\mathcal{S}$ intertwines the associativity and commutativity constraints of $(\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G})$ with those of $(\mathsf{Rep}(G^\vee_{\Bbbk}), \otimes_{\Bbbk})$, and sends the unit object δ_{Gr} to the unit object in $(\mathsf{Rep}(G^\vee_{\Bbbk}), \otimes_{\Bbbk})$, namely the trivial representation $\Bbbk$.

The formulation of Theorem 1.3.11 given here is slightly disappointing, in that the group scheme $G^\vee_{\Bbbk}$ is defined only up to isomorphism, so that it is not possible to say that $\mathcal{S}$ is canonical in any sense. But what is proved in [**MV2**] is slightly better: in fact, while proving this theorem, one actually constructs a *canonical* affine $\Bbbk$-group scheme $\widetilde{G}_{\Bbbk}$ for any $\Bbbk$ as above, then proves that

$$\widetilde{G}_{\Bbbk'} \cong \mathrm{Spec}(\Bbbk') \times_{\mathrm{Spec}(\Bbbk)} \widetilde{G}_{\Bbbk} \tag{1.3.8}$$

canonically for any ring morphism $\Bbbk \to \Bbbk'$, and finally that $\widetilde{G}_{\mathbb{Z}}$ is split reductive and that $\widetilde{G}_{\mathbb{C}}$ has the expected root datum. In the course of the proof, one also constructs a canonical (split) maximal torus $T^\vee_{\Bbbk} \subset \widetilde{G}_{\Bbbk}$ whose weight lattice is identified with $\mathbf{X}^\vee$, along with a (negative) Borel subgroup $B^\vee_{\Bbbk} \subset \widetilde{G}_{\Bbbk}$ containing $T^\vee_{\Bbbk}$. By construction, the direct summand $\mathsf{F}_\lambda(\mathscr{A}) \subset \mathsf{F}(\mathscr{A})$ (see Corollary 1.3.6) is the λ-weight space (for the action of the maximal torus $T^\vee_{\Bbbk}$) of the representation $\mathcal{S}(\mathscr{A})$.

1.4. Overview of the construction of the group scheme

In this section we outline the construction of the group scheme $\widetilde{G}_{\Bbbk}$ and the proof of Theorem 1.3.11. None of these details will be used in the rest of the book.

1.4.1. Tannakian formalism. — The proof of Theorem 1.3.11 is based on "Tannakian formalism," which refers to a family of results on recovering a group scheme from its (tensor) category of representations (see e.g. [**Mi2**, Chapter X]). A typical such result is the following statement, for which we refer to [**DM**, Remark 2.18] or [**BR**, Theorem 1.2.7]. Here we assume that $\Bbbk$ is a field, and denote by $\mathsf{Vect}_{\Bbbk}$ the category of finite-dimensional $\Bbbk$-vector spaces.

Theorem 1.4.1. — *Let $\mathscr{C}$ be an abelian $\Bbbk$-linear category equipped with the following data:*

- *an exact $\Bbbk$-linear faithful functor $\omega : \mathscr{C} \to \mathsf{Vect}_{\Bbbk}$;*
- *a $\Bbbk$-bilinear functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$;*
- *an object $U \in \mathscr{C}$;*
- *an isomorphism $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$, natural in X, Y and Z (the associativity constraint);*
- *isomorphisms $U \otimes X \xrightarrow[\sim]{\lambda_X} X \xleftarrow[\sim]{\rho_X} X \otimes U$, both natural in X (the unit constraints);*
- *an isomorphism $\psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ natural in X and Y (the commutativity constraint).*

We also assume we are given isomorphisms $\upsilon : \Bbbk \xrightarrow{\sim} \omega(U)$ and

$$\tau_{X,Y} : \omega(X) \otimes_{\Bbbk} \omega(Y) \xrightarrow{\sim} \omega(X \otimes Y) \tag{1.4.1}$$

in $\mathsf{Vect}_{\Bbbk}$, with $\tau_{X,Y}$ natural in $X, Y \in \mathscr{C}$. Finally, we assume the following conditions hold:

1. *Taking into account the identifications provided by τ and υ, the isomorphisms $\omega(\phi_{X,Y,Z})$, $\omega(\lambda_X)$, $\omega(\rho_X)$ and $\omega(\psi_{X,Y})$ are the usual associativity, unit and commutativity constraints in $\mathsf{Vect}_{\Bbbk}$.*
2. *If $\dim_{\Bbbk}\big(\omega(X)\big) = 1$, then there exists $X^* \in \mathscr{C}$ such that $X \otimes X^* \cong U$.*

Under these assumptions, there exists an affine $\Bbbk$-group scheme H such that ω admits a canonical factorization

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "\overline{\omega}"] \arrow[dr, "\omega"'] & & \mathsf{Rep}(H) \arrow[dl, "\mathsf{For}_H"] \\ & \mathsf{Vect}_{\Bbbk} & \end{tikzcd}$$

where $\mathsf{Rep}(H)$ is the category of finite-dimensional algebraic representations of H, and where $\overline{\omega}$ is an equivalence of categories that respects the tensor product and the unit in the sense of the compatibility condition (1)*.*

Here, the group scheme H is "reconstructed" from the category $\mathscr{C}$: for any commutative $\Bbbk$-algebra R, an element $\alpha \in H(R)$ is a collection of elements $\alpha_X \in \mathrm{End}_R(\omega(X) \otimes_{\Bbbk} R)$, natural in $X \in \mathscr{C}$, and compatible with $\otimes$ and U via the isomorphisms τ and υ. (There is no need to specifically ask for invertibility: this will automatically follow from condition (2).)

Still assuming that $\Bbbk$ is a field, in the setting where $\mathscr{C} = \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $\omega = \mathsf{F}$, given the structures recalled in Section 1.3, this theorem provides an affine

$\Bbbk$-group scheme $H_\Bbbk$ and an equivalence of symmetric monoidal categories

$$\overline{\omega} : \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \xrightarrow{\sim} \mathsf{Rep}(H_\Bbbk).$$

If we denote by $\mathsf{Vect}_\Bbbk^{\mathbf{X}^\vee}$ the category of $\mathbf{X}^\vee$-graded finite-dimensional $\Bbbk$-vector spaces, our functor F factors through a functor $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to \mathsf{Vect}_\Bbbk^{\mathbf{X}^\vee}$, and all the structures on this functor lift to the category $\mathsf{Vect}_\Bbbk^{\mathbf{X}^\vee}$. If we denote by $T_\Bbbk^\vee$ the unique split $\Bbbk$-torus whose character lattice is $\mathbf{X}^\vee$, then $\mathsf{Vect}_\Bbbk^{\mathbf{X}^\vee}$ is canonically equivalent to $\mathsf{Rep}(T_\Bbbk^\vee)$. In this way one obtains a canonical functor $\mathsf{Rep}(H_\Bbbk) \to \mathsf{Rep}(T_\Bbbk^\vee)$, which by Tannakian formalism (see [**BR**, Proposition 1.2.10]) provides a morphism of $\Bbbk$-group schemes $T_\Bbbk^\vee \to H_\Bbbk$. This morphism can easily be seen to be a closed immersion; we therefore identify $T_\Bbbk^\vee$ with a closed subgroup of $H_\Bbbk$. From this perspective, for any $\lambda \in \mathbf{X}^\vee$ and any $\mathscr{F} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, we have a canonical identification

$$\mathsf{F}_\lambda(\mathscr{F}) \cong \overline{\omega}(\mathscr{F})_\lambda, \tag{1.4.2}$$

where the right-hand side denotes the λ-weight space of the $H_\Bbbk$-representation $\overline{\omega}(\mathscr{F})$.

One can also prove in the context of Tannakian formalism that $H_\Bbbk$ is of finite type and connected (see [**BR**, Lemma 1.14.2] for details). But this is as far as we can go without further restrictions: unless $\mathrm{char}(\Bbbk) = 0$, there is (at present) no known way to say anything more about $H_\Bbbk$ without considering more general coefficient rings. For a general ring $\Bbbk$, there are no general results like Theorem 1.4.1; one will therefore need to construct the group scheme $\widetilde{G}_\Bbbk$ "by hand," exploiting more specific structures we have on the category $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$.

1.4.2. The case of characteristic-0 coefficients. — We explained in §1.4.1 that Theorem 1.4.1 is not sufficient to prove Theorem 1.3.11 for general fields. There is one case where it *is* sufficient, and which is very important (both in its own right, and for the general proof), namely the case when $\mathrm{char}(\Bbbk) = 0$. In fact, under this assumption, there is a simple criterion that can be used to show that a connected affine group scheme of finite type is reductive: this holds if and only if its category of finite-dimensional representations is semisimple (see [**DM**, Proposition 2.23]). For the category $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, semisimplicity is well known in this case, see [**BR**, §1.4] for two slightly different proofs of this fact. Both proofs rely on two key ingredients: the first is a parity-vanishing property for local intersection cohomology (this goes back to [**KL2**] and holds for arbitrary partial flag varieties of Kac–Moody groups), and the second is that the dimensions of L^+G-orbits have constant parity on each connected component of Gr_G (see §1.2.1.6).

Once it is known that $H_\Bbbk$ is reductive, it is not difficult to check that $T_\Bbbk^\vee$ is a maximal torus. One can then consider the root datum of $H_\Bbbk$ with respect to $T_\Bbbk^\vee$. The simple objects in $\mathsf{Rep}(H_\Bbbk)$ are known (because, by the general theory of perverse sheaves, they must be the images of the intersection cohomology complexes associated with the L^+G-orbits in Gr_G), and the root datum can be essentially recovered from this information; see [**BR**, §1.9.2]. This allows one to prove that this root datum is the quadruple $(\mathbf{X}^\vee, \mathbf{X}, \mathfrak{R}^\vee, \mathfrak{R})$ (together with the bijection between roots and coroots of G), or in other words that $H_\Bbbk$ is the split reductive group over $\Bbbk$ which is Langlands dual to G, and therefore finishes the proof of Theorem 1.3.11 in this special case.

1.4.3. Projective generators. — We now drop our special assumptions on $\Bbbk$, and come back to the setting where it is a general noetherian commutative ring of finite global dimension. (The other rings denoted $\Bbbk'$ below will also tacitly be assumed to satisfy these conditions.)

The main ingredient in the construction of the $\Bbbk$-group scheme $\widetilde{G}_{\Bbbk}$ is the representability statement in Lemma 1.4.2 below. Let us define the notation needed for this statement. Let $Z \subset \mathrm{Gr}_G$ be a closed union of finitely many L^+G-orbits, and let $\nu \in \mathbf{X}^\vee$ be such that $Z \cap T_\nu \neq \varnothing$. Then for $n \gg 0$, the action of L^+G on Z factors through an action of the finite-type group scheme $\mathrm{L}^+_n G$ which represents the functor sending a $\mathbb{C}$-algebra R to $G(R[x]/x^n)$. We denote by

$$a, p : \mathrm{L}^+_n G \times Z \to Z$$

the action and projection maps, respectively, and by $i : Z \cap T_\nu \to Z$ the (locally closed) embedding.

Lemma 1.4.2. — *The* L^+G*-equivariant perverse sheaf*

$$P_Z(\nu, \Bbbk) := {}^{\mathrm{p}}\mathscr{H}^0(a_! p^! i_! \underline{\Bbbk}_{Z \cap T_\nu}[-\langle \nu, 2\rho \rangle])$$

represents the restriction of the functor F_ν *to* $\mathsf{Perv}_{\mathrm{L}^+G}(Z, \Bbbk) \subset \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$.

The proof of this lemma is rather straightforward (see [**BR**, Proposition 1.12.1]): by definition the object $i_! \underline{\Bbbk}_{Z \cap T_\nu}[-\langle \nu, 2\rho \rangle]$ "represents" the functor F_ν in the sense that we have $\mathsf{F}_\nu(\mathscr{A}) \cong \mathrm{Hom}_{D^{\mathrm{b}}_c(Z,\Bbbk)}(i_! \underline{\Bbbk}_{Z \cap T_\nu}[-\langle \nu, 2\rho \rangle], \mathscr{A})$ for any $\mathscr{A}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(Z, \Bbbk)$, but it is not an object of $\mathsf{Perv}_{\mathrm{L}^+G}(Z, \Bbbk)$. We therefore "force" it to become L^+G-equivariant by applying $a_! p^!$, and to become perverse by applying ${}^{\mathrm{p}}\mathscr{H}^0$. One then checks that these operations do not alter the fact that our complex represents F_ν.

One can next define the object

$$P_Z(\Bbbk) := \bigoplus_{\substack{\nu \in \mathbf{X}^\vee \\ Z \cap T_\nu \neq \varnothing}} P_Z(\nu, \Bbbk)$$

(where the direct sum is finite). Lemma 1.4.2 and Corollary 1.3.6 imply that $P_Z(\Bbbk)$ represents the restriction of F to $\mathsf{Perv}_{\mathrm{L}^+G}(Z, \Bbbk)$, and from Proposition 1.3.4 we deduce that $P_Z(\Bbbk)$ is a projective generator of the category $\mathsf{Perv}_{\mathrm{L}^+G}(Z, \Bbbk)$.

If Y is another closed finite union of L^+G-orbits such that $Z \subset Y$, then it is not difficult to check that there exists a canonical surjection

$$P_Y(\Bbbk) \twoheadrightarrow {}^{\mathrm{p}}\mathscr{H}^0(P_Y(\Bbbk)_{|Z}) \cong P_Z(\Bbbk), \tag{1.4.3}$$

see [**BR**, Proposition 1.12.2]. Then, a careful study of the objects $\mathscr{J}_!(\lambda, \Bbbk)$ and $\mathscr{J}_*(\lambda, \Bbbk)$ from §1.3.1 (see [**BR**, §1.11]) leads to the following result (see [**BR**, Proposition 1.12.3]).

Proposition 1.4.3. — *Let* $Z \subset \mathrm{Gr}_G$ *be a closed finite union of* L^+G*-orbits.*

1. *The object* $P_Z(\Bbbk)$ *admits a filtration in the abelian category* $\mathsf{Perv}_{\mathrm{L}^+G}(Z, \Bbbk)$ *parametrized by* $\{\lambda \in \mathbf{X}^\vee_+ \mid \mathrm{Gr}^\lambda_G \subset Z\}$ *(endowed with any total order refining the order given by closure inclusions) and with subquotients isomorphic to*

$$\mathsf{F}(\mathscr{J}_*(\lambda, \Bbbk)) \otimes_{\Bbbk} \mathscr{J}_!(\lambda, \Bbbk).$$

2. *For any ring homomorphism* $\Bbbk \to \Bbbk'$ *there exists a canonical isomorphism*
$$P_Z(\Bbbk') \cong \Bbbk' \overset{L}{\otimes}_{\Bbbk} P_Z(\Bbbk).$$
(In particular, the right-hand side is a perverse sheaf.)
3. *The* $\Bbbk$*-module* $\mathsf{F}(P_Z(\Bbbk))$ *is finite free, and for any ring homomorphism* $\Bbbk \to \Bbbk'$ *we have*
$$\mathsf{F}(P_Z(\Bbbk')) = \Bbbk' \otimes_{\Bbbk} \mathsf{F}(P_Z(\Bbbk)).$$

1.4.4. Construction of the group scheme. — For any closed finite union of L^+G-orbits $Z \subset \mathrm{Gr}_G$, we now consider the (unital) $\Bbbk$-algebra
$$A_Z(\Bbbk) := \mathrm{End}_{\mathsf{Perv}_{\mathrm{L}^+G}(Z,\Bbbk)}(P_Z(\Bbbk))^{\mathrm{op}}.$$
Since $P_Z(\Bbbk)$ represents the functor F, as $\Bbbk$-modules we have $A_Z(\Bbbk) \cong \mathsf{F}(P_Z(\Bbbk))$, so that $A_Z(\Bbbk)$ is free of finite rank by Proposition 1.4.3(3). By abstract nonsense (see [**BR**, §1.13.1]), since $P_Z(\Bbbk)$ is a projective generator of $\mathsf{Perv}_{\mathrm{L}^+G}(Z,\Bbbk)$ the functor $\mathrm{Hom}(P_Z(\Bbbk), -)$ induces an equivalence of abelian categories
$$\mathsf{Perv}_{\mathrm{L}^+G}(Z,\Bbbk) \xrightarrow{\sim} \mathsf{Mof}_{A_Z(\Bbbk)}$$
where the right-hand side denotes the category of finitely generated left $A_Z(\Bbbk)$-modules. Moreover, under this equivalence the functor $\mathsf{F} : \mathsf{Perv}_{\mathrm{L}^+G}(Z,\Bbbk) \to \mathsf{Mof}_{\Bbbk}$ corresponds to the natural forgetful functor $\mathsf{Mof}_{A_Z(\Bbbk)} \to \mathsf{Mof}_{\Bbbk}$.

We next set
$$B_Z(\Bbbk) := \mathrm{Hom}_{\Bbbk}(A_Z(\Bbbk), \Bbbk).$$
The (unital) algebra structure on $A_Z(\Bbbk)$ induces a (counital) coalgebra structure on $B_Z(\Bbbk)$, and we have a natural identification
$$\mathsf{Mof}_{A_Z(\Bbbk)} \xrightarrow{\sim} \mathsf{Comof}_{B_Z(\Bbbk)},$$
where the right-hand side denotes the category of finitely generated (over $\Bbbk$) right $B_Z(\Bbbk)$-comodules.

If $Y \subset \mathrm{Gr}_G$ is another closed finite union of L^+G-orbits such that $Z \subset Y$, then from (1.4.3) one gets a surjective algebra map $A_Y(\Bbbk) \twoheadrightarrow A_Z(\Bbbk)$, and then an injective coalgebra morphism $B_Z(\Bbbk) \hookrightarrow B_Y(\Bbbk)$. We can therefore define the coalgebra
$$B(\Bbbk) := \mathrm{colim}_Z B_Z(\Bbbk),$$
where the colimit runs over finite closed unions of L^+G-orbits in Gr_G (ordered by inclusion).

The following properties are easily derived from this construction.

Proposition 1.4.4. — 1. *The coalgebra* $B(\Bbbk)$ *is free (though not of finite rank) as a module over* $\Bbbk$*, and for any ring homomorphism* $\Bbbk \to \Bbbk'$ *we have a canonical identification*
$$B(\Bbbk') = \Bbbk' \otimes_{\Bbbk} B(\Bbbk).$$

2. *There exists a canonical equivalence of categories*

$$\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \xrightarrow{\sim} \mathsf{Comof}_{B(\Bbbk)}$$

under which the functor $\mathsf{F} : \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to \mathsf{Mof}_\Bbbk$ *corresponds to the forgetful functor* $\mathsf{Comof}_{B(\Bbbk)} \to \mathsf{Mof}_\Bbbk$.

The construction so far has not used the convolution product $\star_0^{\mathrm{L}^+G}$ on the category $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. Now is the time when this product enters the game. Namely, via the equivalence of Proposition 1.4.4(2), the associative, commutative monoidal structure on $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ defines an associative, commutative monoidal structure on $\mathsf{Comof}_{B(\Bbbk)}$. By abstract nonsense (see [**SR**, Chap. II, §2.5]) there exists a unique associative, commutative algebra structure on $B(\Bbbk)$ which makes it a bialgebra and such that this monoidal structure corresponds to the natural $B(\Bbbk)$-comodule structure on a tensor product of $B(\Bbbk)$-comodules. In other words, the equivalence of Proposition 1.4.4(2) now becomes an equivalence of monoidal categories, compatible with the commutativity constraints. There exists also a unit for this structure, given by the canonical map $\Bbbk = B_{\mathrm{Gr}_G^0}(\Bbbk) \to B(\Bbbk)$.

If we denote by $\widetilde{G}_\Bbbk$ the spectrum of the commutative $\Bbbk$-algebra $B(\Bbbk)$, then the coassociative coproduct (resp. the counit) on $B(\Bbbk)$ provides an associative product (resp. a unit) on $\widetilde{G}_\Bbbk$ (in the category of affine $\Bbbk$-schemes), which make this scheme a monoid scheme over $\Bbbk$. Moreover, by definition the category $\mathsf{Comof}_{B(\Bbbk)}$ identifies with the category of representations of $\widetilde{G}_\Bbbk$ which are of finite type as $\Bbbk$-modules. Using essentially the same argument as for fields (see §1.4.1) one can check that in fact $\widetilde{G}_\Bbbk$ is a *group* scheme, see [**BR**, Proposition 1.13.4].

We have finally reached the first step towards the proof of Theorem 1.3.11: we have constructed an affine $\Bbbk$-group scheme $\widetilde{G}_\Bbbk$ and an equivalence of monoidal categories

$$\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \xrightarrow{\sim} \mathsf{Rep}(\widetilde{G}_\Bbbk)$$

intertwining the commutativity constraint σ^{Com} (see §1.3.3) with the obvious commutativity constraint on $\mathsf{Rep}(\widetilde{G}_\Bbbk)$, and under which the functor F corresponds to $\mathsf{For}_{\widetilde{G}_\Bbbk}$. Moreover, Proposition 1.4.4(1) implies that we have an identification (1.3.8) for any ring homomorphism $\Bbbk \to \Bbbk'$. Since we know from §1.4.2 that $\widetilde{G}_\mathbb{C}$ is connected reductive with root datum $(\mathbf{X}^\vee, \mathbf{X}, \mathfrak{R}^\vee, \mathfrak{R})$, the remaining step for the proof of Theorem 1.3.11 is proving that $\widetilde{G}_\mathbb{Z}$ is a split reductive group scheme.

Before doing that, we remark that the same considerations as in the characteristic-0 setting allow us to construct a canonical closed embedding $T_\Bbbk^\vee \hookrightarrow \widetilde{G}_\Bbbk$, where $T_\Bbbk^\vee$ is the unique split $\Bbbk$-torus whose character lattice is $\mathbf{X}^\vee$.

1.4.5. Identification of the group scheme. — In order to show that $\widetilde{G}_\mathbb{Z}$ is reductive, we will use two results due to Prasad–Yu. The first one (see [**PY**, Theorem 1.5]) gives a criterion to show that a (flat, affine) $\mathbb{Z}$-group scheme is reductive in terms of the geometric fibers of the group.

Theorem 1.4.5. — *If H is a flat affine group scheme over* $\mathrm{Spec}(\mathbb{Z})$ *such that for any algebraically closed field* $\Bbbk$ *the fiber* $\mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\mathbb{Z})} H$ *is a connected reductive group, whose dimension is independent of* $\Bbbk$*, then H is a reductive group scheme.*

In our setting we already know that the affine $\mathbb{Z}$-group scheme $\widetilde{G}_{\mathbb{Z}}$ is flat (see Proposition 1.4.4(1)). Therefore, given this theorem (and the definition of maximal tori), to conclude it only remains to prove that for any algebraically closed field $\Bbbk$ of characteristic $p > 0$, the $\Bbbk$-group scheme $\widetilde{G}_{\Bbbk}$ (which can be easily checked to coincide with the group denoted $H_{\Bbbk}$ in §1.4.1) is connected reductive, with maximal torus $T^{\vee}_{\Bbbk}$ and root datum $(\mathbf{X}^{\vee}, \mathbf{X}, \mathfrak{R}^{\vee}, \mathfrak{R})$. (Here we use the obvious observation that the root datum determines the dimension.)

So, from now on we fix a prime number $p > 0$, and assume that $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$. The second result of Prasad–Yu that we require (see [**PY**, Theorem 1.2]) is the following.

Theorem 1.4.6. — *Let H be a flat affine group scheme over* $\mathrm{Spec}(\mathbb{Z}_p)$*. Assume that the following two conditions hold:*

1. *the generic fiber* $\mathrm{Spec}(\mathbb{Q}_p) \times_{\mathrm{Spec}(\mathbb{Z}_p)} H$ *is a connected reductive group over* $\mathbb{Q}_p$*;*
2. *the reduced geometric special fiber* $\big(\mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\mathbb{Z}_p)} H\big)_{\mathrm{red}}$ *is of finite type, and its identity component is a reductive group with the same root datum as* $\mathrm{Spec}(\overline{\mathbb{Q}_p}) \times_{\mathrm{Spec}(\mathbb{Z}_p)} H$ *(where* $\overline{\mathbb{Q}_p}$ *is an algebraic closure of* $\mathbb{Q}_p$*).*

Then H is a reductive group scheme. (In particular, $\mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\mathbb{Z}_p)} H$ *is already reduced and connected.)*

Given this theorem, to conclude we now only need to show that $\big(\widetilde{G}_{\Bbbk}\big)_{\mathrm{red}}$ is connected reductive, with maximal torus $T^{\vee}_{\Bbbk}$ and root datum $(\mathbf{X}^{\vee}, \mathbf{X}, \mathfrak{R}^{\vee}, \mathfrak{R})$.

For this, we fix a finite type[(20)] quotient $\widetilde{G}^{*}_{\Bbbk}$ of $\widetilde{G}_{\Bbbk}$ such that all irreducible $\widetilde{G}_{\Bbbk}$-modules factor through $\widetilde{G}^{*}_{\Bbbk}$-modules. (Such a quotient exists because, as can easily be seen by comparing with $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, there exists a finite subset X of the set of isomorphism classes of simple representations such that every simple $\widetilde{G}_{\Bbbk}$-representation appears as a subquotient of a tensor product of modules in X.) Then it is not difficult to check that $\widetilde{G}^{*}_{\Bbbk}$ is connected and of dimension at most $\dim(G)$. Moreover, since $T^{\vee}_{\Bbbk}$ is reduced, the composition $T^{\vee}_{\Bbbk} \hookrightarrow \widetilde{G}_{\Bbbk} \twoheadrightarrow \widetilde{G}^{*}_{\Bbbk}$ factors through a morphism $T^{\vee}_{\Bbbk} \to \big(\widetilde{G}^{*}_{\Bbbk}\big)_{\mathrm{red}}$.

One next denotes by H the reductive quotient of $\big(\widetilde{G}^{*}_{\Bbbk}\big)_{\mathrm{red}}$, and checks (using slightly upgraded versions of the corresponding arguments for characteristic-0 coefficients) that the composition

$$T^{\vee}_{\Bbbk} \to \big(\widetilde{G}^{*}_{\Bbbk}\big)_{\mathrm{red}} \twoheadrightarrow H$$

(20) As explained in §1.4.1, one can in fact prove directly that $\widetilde{G}_{\Bbbk}$ is of finite type, and thereby simplify this step a little bit. However this proof (given in [**BR**, Lemma 1.14.2]) uses results which were not available at the time when [**MV2**] was written. Note also that some arguments involving this finite type quotient $\widetilde{G}^{*}_{\Bbbk}$ were not clear enough in [**MV2**]; details were given later in [**MV3**].

identifies $T^\vee_{\Bbbk}$ with a maximal torus of H, and that the root datum of H with respect to $T^\vee_{\Bbbk}$ is $(\mathbf{X}^\vee, \mathbf{X}, \mathfrak{R}^\vee, \mathfrak{R})$. In particular, this implies that $\dim(H) = \dim(G) \geq \dim((\widetilde{G}^*_{\Bbbk})_{\mathrm{red}})$, so that necessarily $H = \bigl(\widetilde{G}^*_{\Bbbk}\bigr)_{\mathrm{red}}$. Using the fact that the group scheme $\bigl(\widetilde{G}^*_{\Bbbk}\bigr)_{\mathrm{red}}$ does not depend on the choice of $\widetilde{G}^*_{\Bbbk}$, one shows that in fact $\bigl(\widetilde{G}^*_{\Bbbk}\bigr)_{\mathrm{red}} = \bigl(\widetilde{G}_{\Bbbk}\bigr)_{\mathrm{red}}$, so that finally $\bigl(\widetilde{G}_{\Bbbk}\bigr)_{\mathrm{red}}$ is connected reductive with the appropriate root datum.

1.5. Some complements

In Chapters 6 and 8 below we will need detailed information on the images under $\mathcal{S}$ of the perverse sheaves $\mathcal{J}_*(\lambda, \Bbbk)$ and of certain morphisms relating these perverse sheaves, which we review here. (The description of the objects $\mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk))$ is given in [**MV2**], although the argument we give here is slightly different; the description of the maps is stated, in the case of characteristic-0 coefficients, in [**AB**].)

1.5.1. Standard and costandard spherical perverse sheaves. — Recall the perverse sheaves $\mathcal{J}_*(\lambda, \Bbbk)$ and $\mathcal{J}_!(\lambda, \Bbbk)$ defined in §1.3.1. Our goal in this subsection is to describe how the functor $\mathcal{S}$ behaves on these perverse sheaves. Each $\lambda \in \mathbf{X}^\vee$ defines a free rank-1 $T^\vee_{\Bbbk}$-module, which extends uniquely to a $B^\vee_{\Bbbk}$-module; the resulting module will be denoted $\Bbbk_{B^\vee_{\Bbbk}}(\lambda)$. We can then consider the induced (or "co-Weyl") module

$$\mathsf{N}_{\Bbbk}(\lambda) := \mathrm{Ind}_{B^\vee_{\Bbbk}}^{G^\vee_{\Bbbk}}(\Bbbk_{B^\vee_{\Bbbk}}(\lambda)).$$

In more concrete terms, the $\Bbbk$-module underlying $\mathsf{N}_{\Bbbk}(\lambda)$ is the space of functions $f \in \mathscr{O}(G^\vee_{\Bbbk})$ which satisfy $f(gb) = \lambda(b)^{-1} f(g)$ for all $g \in G^\vee_{\Bbbk}$ and $b \in B^\vee_{\Bbbk}$. In these terms, the $G^\vee_{\Bbbk}$-action is given by $(g \cdot f)(h) = f(g^{-1}h)$.

In case $\lambda \in \mathbf{X}^\vee_+$, it is known that $\mathsf{N}_{\Bbbk}(\lambda)$ is free (of finite type) over $\Bbbk$, and moreover that for any ring homomorphism $\Bbbk \to \Bbbk'$ (where $\Bbbk'$ satisfies our running assumptions on rings of coefficients) we have a canonical isomorphism of $G^\vee_{\Bbbk'}$-modules

$$\Bbbk' \otimes_{\Bbbk} \mathsf{N}_{\Bbbk}(\lambda) \xrightarrow{\sim} \mathsf{N}_{\Bbbk'}(\lambda), \tag{1.5.1}$$

see [**J1**, §II.8.8]. Frobenius reciprocity provides a morphism of $B^\vee_{\Bbbk}$-modules

$$\mathbf{f}_\lambda : \mathsf{N}_{\Bbbk}(\lambda) \to \Bbbk_{B^\vee_{\Bbbk}}(\lambda)$$

which identifies with the projection from $\mathsf{N}_{\Bbbk}(\lambda)$ to its weight-λ subspace with respect to its weight space decomposition. (In terms of the "concrete" description of $\mathsf{N}_{\Bbbk}(\lambda)$ given above, we have $\mathbf{f}_\lambda(f) = f(1)$.)

Remark 1.5.1. — In case $\Bbbk$ is a field of characteristic 0, it is well known that $\mathsf{N}_{\Bbbk}(\lambda)$ is simple, see [**J1**, Corollary II.5.6].

In preparation for the next statement, we recall (see [**BR**, Proposition 1.11.1]) that for any $\mu \in \mathbf{X}^\vee$ the weight space $\mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk))_\mu$ is free of finite rank over $\Bbbk$, with a canonical basis parametrized by the irreducible components of the intersection $\mathrm{Gr}^\lambda_G \cap T_\mu$. (Such irreducible components are usually called "Mirković–Vilonen cycles.") In particular, this weight space vanishes unless the dominant W_{f}-translate μ^+ of μ

satisfies $\lambda - \mu^+ \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+^\vee$, and in case $\mu = \lambda$, the intersection $\mathrm{Gr}_G^\lambda \cap T_\lambda$ is $\{L_\lambda\}$, so that this weight space identifies canonically with $\Bbbk$.

Lemma 1.5.2. — *For any $\lambda \in \mathbf{X}_+^\vee$ there exists a canonical isomorphism*

$$\mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk)) \cong \mathsf{N}_\Bbbk(\lambda).$$

Moreover, under this isomorphism the map $\mathbf{f}_\lambda$ sends the canonical basis element in $\mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk))_\lambda$ to 1.

Proof. — As recalled above, we know that λ is a highest weight in $\mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk))$, and that moreover $\mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk))_\lambda$ is free of rank 1, with a canonical basis element. By Frobenius reciprocity we obtain a canonical map of $G_\Bbbk^\vee$-modules

$$\mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk)) \to \mathsf{N}_\Bbbk(\lambda) \tag{1.5.2}$$

whose composition with $\mathbf{f}_\lambda$ sends the canonical basis element to 1. It remains to show that this map is an isomorphism. By compatibility of all our constructions with extensions of scalars (see (1.5.1) and [**BR**, Proposition 1.11.3]), it suffices to do so when $\Bbbk = \mathbb{Z}$. And then, since both sides are free of finite rank it suffices to prove this claim when $\Bbbk$ is a field. In this case, $\mathcal{J}_*(\lambda, \Bbbk)$ has a simple socle (namely, $\mathcal{J}_{!*}(\lambda, \Bbbk)$), and hence so does $\mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk))$. The map of λ-weight spaces $\mathcal{S}(\mathcal{J}_{!*}(\lambda, \Bbbk))_\lambda \to \mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk))_\lambda$ is an isomorphism, as it can be identified (via (1.4.2)) with $\mathsf{F}_\lambda(\mathcal{J}_{!*}(\lambda, \Bbbk)) \to \mathsf{F}_\lambda(\mathcal{J}_*(\lambda, \Bbbk))$. Therefore, the map (1.5.2) remains nonzero when restricted to the socle $\mathcal{S}(\mathcal{J}_{!*}(\lambda, \Bbbk)) \subset \mathcal{S}(\mathcal{J}_*(\lambda, \Bbbk))$.

It follows that (1.5.2) is injective. To conclude, it then suffices to prove that both sides have the same dimension. These dimensions are independent of the field of coefficients, so we may as well assume that $\Bbbk$ has characteristic 0. In this setting, $\mathsf{N}_\Bbbk(\lambda)$ is simple (see Remark 1.5.1) so that (1.5.2) is automatically an isomorphism, proving the desired equality of dimensions. □

Remark 1.5.3. —

1. For $\lambda \in \mathbf{X}_+^\vee$, one can also consider the standard (or "Weyl") module $\mathsf{M}_\Bbbk(\lambda)$, defined as

   $$\mathsf{M}_\Bbbk(\lambda) := \mathrm{Hom}_\Bbbk(\mathsf{N}_\Bbbk(-w_\circ\lambda), \Bbbk),$$

   where $w_\circ$ is the longest element in W_f. Using [**BR**, Proposition 1.11.1] again, we have a canonical morphism of $B_\Bbbk^\vee$-modules $\Bbbk_{B_\Bbbk^\vee}(w_\circ\lambda) \to \mathcal{S}(\mathcal{J}_!(\lambda, \Bbbk))$; using Frobenius reciprocity and duality we deduce a canonical morphism

   $$\mathsf{M}_\Bbbk(\lambda) \to \mathcal{S}(\mathcal{J}_!(\lambda, \Bbbk)).$$

   Arguments similar to those in the proof of Lemma 1.5.2 show that this morphism is an isomorphism as well.
2. See [**MV2**, Proposition 13.1] for a slightly different proof of Lemma 1.5.2.

Example 1.5.4. — Consider the setting of Examples 1.2.3 and 1.3.1(2). It is well known that in this setting the group $G_\Bbbk^\vee$ is a general linear group, and that its standard representation is $\mathsf{N}_\Bbbk(\varepsilon_1^\vee)$. As a $\Bbbk$-module we have

$$\mathcal{S}(\mathcal{J}_*(\varepsilon_1^\vee, \Bbbk)) = \mathsf{H}^\bullet(\mathbb{P}^n; \Bbbk),$$

and therefore we obtain a canonical identification of $G^\vee_\Bbbk$ with the group of $\Bbbk$-automorphisms of this free $\Bbbk$-module.

1.5.2. More on costandard spherical perverse sheaves. — Consider now two dominant coweights $\lambda_1, \lambda_2 \in \mathbf{X}^\vee_+$. By [**BR**, Lemma 1.10.9 and Proposition 1.11.1], the convolution product

$$\mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G} \mathcal{J}_*(\lambda_2, \Bbbk)$$

is in fact a perverse sheaf. It is easily seen from the definitions that this perverse sheaf is supported on $\overline{\mathrm{Gr}^{\lambda_1+\lambda_2}_G}$, and that its restriction to $\mathrm{Gr}^{\lambda_1+\lambda_2}_G$ is canonically isomorphic to $\underline{\Bbbk}_{\mathrm{Gr}^{\lambda_1+\lambda_2}_G}[\dim(\mathrm{Gr}^{\lambda_1+\lambda_2}_G)]$. Therefore, there exists a canonical morphism

$$a_{\lambda_1,\lambda_2} : \mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G}_0 \mathcal{J}_*(\lambda_2, \Bbbk) \to \mathcal{J}_*(\lambda_1+\lambda_2, \Bbbk)$$

in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$.

On the other hand, using Frobenius reciprocity there exists a unique morphism of $G^\vee_\Bbbk$-modules

$$\mathbf{a}_{\lambda_1,\lambda_2} : \mathsf{N}_\Bbbk(\lambda_1) \otimes_\Bbbk \mathsf{N}_\Bbbk(\lambda_2) \to \mathsf{N}_\Bbbk(\lambda_1+\lambda_2)$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{N}_\Bbbk(\lambda_1) \otimes_\Bbbk \mathsf{N}_\Bbbk(\lambda_2) \arrow[r, "\mathbf{a}_{\lambda_1,\lambda_2}"] \arrow[d, "\mathbf{f}_{\lambda_1}\otimes\mathbf{f}_{\lambda_2}"'] & \mathsf{N}_\Bbbk(\lambda_1+\lambda_2) \arrow[d, "\mathbf{f}_{\lambda_1+\lambda_2}"] \\ \Bbbk_{B^\vee_\Bbbk}(\lambda_1) \otimes \Bbbk_{B^\vee_\Bbbk}(\lambda_2) \arrow[r, equal] & \Bbbk_{B^\vee_\Bbbk}(\lambda_1+\lambda_2). \end{tikzcd}$$

Lemma 1.5.5. — *Under the isomorphisms*

$$\mathcal{S}(\mathcal{J}_*(\lambda_i, \Bbbk)) \cong \mathsf{N}_\Bbbk(\lambda_i) \quad \text{and} \quad \mathcal{S}(\mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G}_0 \mathcal{J}_*(\lambda_2, \Bbbk)) \cong \mathsf{N}_\Bbbk(\lambda_1) \otimes_\Bbbk \mathsf{N}_\Bbbk(\lambda_2)$$

provided by Lemma 1.5.2 and the monoidal structure on $\mathcal{S}$, we have

$$\mathcal{S}(a_{\lambda_1,\lambda_2}) = \mathbf{a}_{\lambda_1,\lambda_2}.$$

Proof. — By construction of the morphism $\mathbf{a}_{\lambda_1,\lambda_2}$ (in terms of Frobenius reciprocity), what we have to prove is that

$$\mathbf{f}_{\lambda_1+\lambda_2} \circ \mathcal{S}(a_{\lambda_1,\lambda_2}) = \mathbf{f}_{\lambda_1} \otimes \mathbf{f}_{\lambda_2}.$$

Of course, both maps here vanish on all the $T^\vee_\Bbbk$-weight spaces of weight $\neq \lambda_1+\lambda_2$, so it suffices to consider the restrictions of our maps to the weight space

$$\begin{aligned} \mathcal{S}\big(\mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G}_0 \mathcal{J}_*(\lambda_2, \Bbbk)\big)_{\lambda_1+\lambda_2} &= \mathsf{F}_{\lambda_1+\lambda_2}\big(\mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G}_0 \mathcal{J}_*(\lambda_2, \Bbbk)\big) \\ &= \mathsf{H}^{\langle\lambda_1+\lambda_2,2\rho\rangle}_{T_{\lambda_1+\lambda_2}}\big(\mathrm{Gr}_G, \mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G}_0 \mathcal{J}_*(\lambda_2, \Bbbk)\big). \end{aligned}$$

The perverse sheaf $\mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G}_0 \mathcal{J}_*(\lambda_2, \Bbbk)$ is supported on $\overline{\mathrm{Gr}^{\lambda_1+\lambda_2}_G}$, and the only L^+G-orbit contained in $\overline{\mathrm{Gr}^{\lambda_1+\lambda_2}_G}$ and intersecting $T_{\lambda_1+\lambda_2}$ is the open orbit $\mathrm{Gr}^{\lambda_1+\lambda_2}_G$.

We therefore have

$$\begin{aligned}\mathsf{H}^{\langle\lambda_1+\lambda_2,2\rho\rangle}_{T_{\lambda_1+\lambda_2}}\big(\mathrm{Gr}_G, \mathcal{J}_*(\lambda_1,\Bbbk)\star_0^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_2,\Bbbk)\big)\\ = \mathsf{H}^{\langle\lambda_1+\lambda_2,2\rho\rangle}_{T_{\lambda_1+\lambda_2}\cap\mathrm{Gr}_G^{\lambda_1+\lambda_2}}\big(\mathrm{Gr}_G^{\lambda_1+\lambda_2}, \mathcal{J}_*(\lambda_1,\Bbbk)\star_0^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_2,\Bbbk)_{|\mathrm{Gr}_G^{\lambda_1+\lambda_2}}\big).\end{aligned}$$

In particular, since the restriction of a_{λ_1,λ_2} to $\mathrm{Gr}_G^{\lambda_1+\lambda_2}$ is an isomorphism, this shows that the restriction of $\mathcal{S}(a_{\lambda_1,\lambda_2})$ to the weight space of weight $\lambda_1+\lambda_2$ is an isomorphism. More precisely, we have

$$T_{\lambda_1+\lambda_2}\cap\mathrm{Gr}_G^{\lambda_1+\lambda_2}=\{L_{\lambda_1+\lambda_2}\},$$

and the morphism $\mathbf{f}_{\lambda_1+\lambda_2}\circ\mathcal{S}(a_{\lambda_1,\lambda_2})$ sends the canonical class in

$$\mathsf{H}^{\langle\lambda_1+\lambda_2,2\rho\rangle}_{T_{\lambda_1+\lambda_2}\cap\mathrm{Gr}_G^{\lambda_1+\lambda_2}}\big(\mathrm{Gr}_G^{\lambda_1+\lambda_2}, \underline{\Bbbk}_{\mathrm{Gr}_G^{\lambda_1+\lambda_2}}[\dim\mathrm{Gr}_G^{\lambda_1+\lambda_2}]\big)$$

to $1\in\Bbbk$.

On the other hand, the monoidal structure on F provides a canonical isomorphism

$$\mathsf{F}_{\lambda_1+\lambda_2}\big(\mathcal{J}_*(\lambda_1,\Bbbk)\star_0^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_2,\Bbbk)\big)\cong\mathsf{F}_{\lambda_1}(\mathcal{J}_*(\lambda_1,\Bbbk))\otimes_\Bbbk\mathsf{F}_{\lambda_2}(\mathcal{J}_*(\lambda_2,\Bbbk)),\tag{1.5.3}$$

and by construction the morphism $\mathbf{f}_{\lambda_1}\otimes\mathbf{f}_{\lambda_2}$ sends the tensor product of the canonical classes in

$$\mathsf{F}_{\lambda_i}(\mathcal{J}_*(\lambda_i,\Bbbk))\cong\mathsf{H}^{\langle\lambda_i,2\rho\rangle}_{T_{\lambda_i}\cap\mathrm{Gr}_G^{\lambda_i}}\big(\mathrm{Gr}_G^{\lambda_i},\underline{\Bbbk}_{\mathrm{Gr}_G^{\lambda_i}}[\dim\mathrm{Gr}_G^{\lambda_i}]\big)$$

($i=1,2$) to 1. (Here, as above we have $T_{\lambda_i}\cap\mathrm{Gr}_G^{\lambda_i}=\{L_{\lambda_i}\}$.) Thus, to finish the proof, we have to show that the isomorphism (1.5.3) identifies the canonical class on the left-hand side with the tensor product of the canonical classes on the right-hand side.

To do this, we revisit the construction of the monoidal structure on F from §1.3.4 (see also [**BR**, §1.8.3]). From the perverse sheaves $\mathscr{F}_i=\mathcal{J}_*(\lambda_i,\Bbbk)$ on Gr_G we obtain a perverse sheaf

$$\widetilde{\mathscr{F}}:=j_{!*}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F}_1\overset{L}{\boxtimes}_\Bbbk\mathscr{F}_2)\overset{L}{\boxtimes}_\Bbbk\underline{\Bbbk}_{C^2\smallsetminus\Delta C}[2])$$

on Fus_G whose restriction to the fiber over a point on, resp. away from, the diagonal in C^2 is $\mathscr{F}_1\star_0^{\mathrm{L}^+G}\mathscr{F}_2[2]$, resp. ${}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F}_1\boxtimes^L_\Bbbk\mathscr{F}_2)[2]$. Let

$$f:\mathrm{Fus}_G\to C^2\qquad\text{and}\qquad\tilde{t}_{\lambda_1+\lambda_2}:T_{\lambda_1+\lambda_2,C^2}\hookrightarrow\mathrm{Fus}_G,$$

be the structure map and the inclusion map, respectively. (See Remark 1.3.8 for the locally closed subscheme $T_{\lambda_1+\lambda_2}$.) The isomorphism (1.5.3) is obtained by comparing the stalks of the constant sheaf

$$\mathscr{H}^{\langle\lambda_1+\lambda_2,2\rho\rangle+2}\big((f\tilde{t}_{\lambda_1+\lambda_2})_*(\tilde{t}_{\lambda_1+\lambda_2})^!\widetilde{\mathscr{F}}\big)$$

on and away from the diagonal respectively. In our present setting the intersection of the support of $\widetilde{\mathscr{F}}$ with $T_{\lambda_1+\lambda_2,C^2}$ is a closed subscheme X_{λ_1,λ_2} such that the restriction of $f\tilde{t}_{\lambda_1+\lambda_2}$ to X_{λ_1,λ_2} is an isomorphism to C^2. (In fact, the fiber of X_{λ_1,λ_2} over a point on, resp. away from, the diagonal is $\{L_{\lambda_1+\lambda_2}\}$, resp. $\{(L_{\lambda_1},L_{\lambda_2})\}$.) Moreover, the complex $(\tilde{t}_{\lambda_1+\lambda_2})^!\widetilde{\mathscr{F}}$ is just (canonically) the shifted constant sheaf on this subscheme. Hence (1.5.3) indeed is the obvious isomorphism, which identifies the canonical class in the left-hand side with the tensor product of the canonical classes in the right-hand side. □

Remark 1.5.6. — 1. The claim considered in the proof above about the isomorphism (1.5.3) is in fact a special case of a general result comparing canonical bases in tensor products defined in terms of generalized Mirković–Vilonen cycles (see [**GS**]) and tensor products of canonical basis given by "ordinary" Mirković–Vilonen cycles (as considered in [**BR**, Proposition 1.11.1]); see [**BGL**] for details.

2. One can show that for any $\lambda_1, \lambda_2 \in \mathbf{X}_+^\vee$ the morphisms $\mathbf{a}_{\lambda_1,\lambda_2}$ and a_{λ_1,λ_2} are surjective. (By Lemma 1.5.5, these two surjectivity claims are equivalent.) By a change-of-scalars argument using [**MV2**, Proposition 8.1] or [**BR**, Proposition 1.11.3], our claim can be reduced to the case where $\Bbbk = \mathbb{Z}$, and then to the case where $\Bbbk$ is a field. In the field case, if $\Bbbk$ has characteristic 0 this follows from simplicity of $\mathsf{N}_\Bbbk(\lambda_1 + \lambda_2)$, see [**J1**, Corollary II.5.6]. For general fields, this follows e.g. from the fact that $\mathsf{N}_\Bbbk(\lambda_1) \otimes \mathsf{N}_\Bbbk(\lambda_2)$ has a good filtration (see [**J1**, Proposition II.4.21]) and that $\lambda_1 + \lambda_2$ is maximal among the highest weights of the induced modules appearing in a good filtration of this module, using [**J1**, Remark (4) in §II.4.16].

Corollary 1.5.7. — *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{J}_*(\lambda_1,\Bbbk) \star_0^{\mathrm{L}^+G} \mathcal{J}_*(\lambda_2,\Bbbk) \arrow[rr, "\sigma^{\mathrm{Com}}_{\mathcal{J}_*(\lambda_1,\Bbbk),\mathcal{J}_*(\lambda_2,\Bbbk)}"] \arrow[dr, "a_{\lambda_1,\lambda_2}"'] & & \mathcal{J}_*(\lambda_2,\Bbbk) \star_0^{\mathrm{L}^+G} \mathcal{J}_*(\lambda_1,\Bbbk) \arrow[dl, "a_{\lambda_2,\lambda_1}"] \\
& \mathcal{J}_*(\lambda_1+\lambda_2,\Bbbk). &
\end{tikzcd}$$

Proof. — It suffices to prove the commutativity after application of $\mathcal{S}$. Then, in view of Lemma 1.5.5, the claim transforms into the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{N}_\Bbbk(\lambda_1) \otimes_\Bbbk \mathsf{N}_\Bbbk(\lambda_2) \arrow[rr, "\sigma_{\lambda_1,\lambda_2}"] \arrow[dr, "\mathbf{a}_{\lambda_1,\lambda_2}"'] & & \mathsf{N}_\Bbbk(\lambda_2) \otimes_\Bbbk \mathsf{N}_\Bbbk(\lambda_1) \arrow[dl, "\mathbf{a}_{\lambda_2,\lambda_1}"] \\
& \mathsf{N}_\Bbbk(\lambda_1+\lambda_2), &
\end{tikzcd}$$

where the upper horizontal arrow is the obvious commutativity isomorphism. By Frobenius reciprocity, to prove the commutativity of this diagram it suffices to prove that

$$\mathbf{f}_{\lambda_1+\lambda_2} \circ \mathbf{a}_{\lambda_1,\lambda_2} = \mathbf{f}_{\lambda_1+\lambda_2} \circ \mathbf{a}_{\lambda_2,\lambda_1} \circ \sigma_{\lambda_1,\lambda_2}.$$

And for this it suffices to consider the restrictions of these maps to the $T_\Bbbk^\vee$-weight spaces of weight $\lambda_1 + \lambda_2$. Then the equality is clear, since $\mathbf{a}_{\lambda_i,\lambda_j}$ sends the tensor product of the canonical vectors in $\mathsf{N}_\Bbbk(\lambda_i)$ and $\mathsf{N}_\Bbbk(\lambda_j)$ to 1 (for $(i,j) \in \{(1,2),(2,1)\}$) and $\sigma_{\lambda_1,\lambda_2}$ intertwines these tensor products of canonical vectors. □

CHAPTER 2

CENTRAL SHEAVES

In Chapter 1, we have introduced the affine Grassmannian Gr_G and studied L^+G-equivariant objects on it. In this chapter, we will consider another closely related space, the *affine flag variety*, denoted by Fl_G, along with an action of a subgroup $I \subset \mathrm{L}^+G$, called an *Iwahori subgroup*. One goal of this chapter is to define a certain t-exact (for the perverse t-structures) functor

$$\mathsf{Z} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk).$$

A *central sheaf* is defined to be a perverse sheaf on Fl_G of the form $\mathsf{Z}(\mathscr{F})$, where $\mathscr{F} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. (The use of the term "central" will be explained in Chapter 3.)

The definition of Z will involve a family of ind-schemes over $C \cong \mathbb{A}^1_{\mathbb{C}}$ that exhibits Fl_G as a degeneration of Gr_G. The definition of this family (which will be denoted by $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$), and then of Z, could easily have been given in a couple pages. Unfortunately, the ind-scheme $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ alone is inadequate for the study of the fundamental properties of Z in Chapter 3 and beyond.

Anticipating the needs of later chapters, we devote the bulk of this chapter to developing a framework for constructing many different families of ind-schemes over C, whose fibers may be ordinary or twisted products of copies of Fl_G or Gr_G in various configurations. The spaces arising in this framework are called *iterated global affine Grassmannians*, and include $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ as a special case. Along the way, we define several different families of (ind-)affine group ind-schemes over C that act on iterated global affine Grassmannians, and we study principal bundles for these group schemes.

2.1. Preliminaries on torsors and quotients

This section contains background material on notions such as torsors and principal bundles, including a review of some "basic" results on these objects which are sometimes difficult to absorb for non-experts (such as the authors, for instance). A reader used to working with such concepts might want to skip this section.

2.1.1. Schemes as sheaves. — Consider a base scheme S, and let Sch_S be the category of S-schemes, i.e. pairs (X, f) consisting of a scheme X and a morphism $f : X \to S$. (As usual, the morphism f will usually be omitted from the notation.) We begin by reviewing the basics on some "standard" Grothendieck topologies on the category Sch_S.

Recall that the *fppf topology*, resp. *étale topology*, resp. *Zariski topology* on this category is the Grothendieck topology for which coverings of an object U are collections $(U_i \to U)_{i\in I}$ where each morphism $U_i \to U$ is flat and locally of finite presentation, resp. étale, resp. an open immersion, and the map

$$\bigsqcup_{i\in I} U_i \to U$$

is surjective (see [**Ol**, Examples 2.1.6, 2.1.13 & 2.1.14] or [**SP**, Tag 021M, Tag 0215 & Tag 020O]). Similarly, the *fpqc topology* is the Grothendieck topology for which coverings of an object U are collections $(U_i \to U)_{i\in I}$ where each morphism $U_i \to U$ is flat and for any affine open subscheme $V \subset U$ there exists a map $a : \{1, \dots, n\} \to I$ and affine open subschemes $V_j \subset U_{a(j)}$ such that the map

$$\bigsqcup_{j=1}^{n} V_j \to V$$

is surjective, see [**SP**, Tag 022B]. (Of course, this condition implies that the morphism $\bigsqcup_{i\in I} U_i \to U$ is surjective, as for the other topologies considered above.)

For each such topology, a *sheaf* on Sch_S is a presheaf F (i.e. a functor from $(\mathsf{Sch}_S)^{\mathrm{op}}$ to the category Sets) such that for any U in Sch_S and any covering $(U_i \to U)_{i\in I}$, the natural map

$$F(U) \to \prod_{i\in I} F(U_i)$$

identifies the left-hand side with the equalizer of the two natural maps

$$\prod_{i\in I} F(U_i) \to \prod_{i,j\in I} F(U_i \times_U U_j)$$

(see [**Ol**, Definition 2.2.2]). It is clear that a Zariski covering is also an étale covering (see [**SP**, Tag 0216]), and that an étale covering is also an fppf covering (see [**SP**, Tag 021N]). It is also true that an fppf covering is also an fpqc covering (see [**SP**, Tag 022C]). Therefore:

– a presheaf which is a sheaf for the fpqc topology is also a sheaf for the fppf topology;
– a presheaf which is a sheaf for the fppf topology is also a sheaf for the étale topology;
– a presheaf which is a sheaf for the étale topology is also a sheaf for the Zariski topology.

For any of the Grothendieck topologies on Sch_S considered above (and in fact, for any site), the functor of inclusion of the category of sheaves into the category of presheaves admits a left adjoint, called the *sheafification* functor, see [**SP**, Tag

00WH] or [**Ol**, Theorem 2.2.4]. This construction requires some care in dealing with set-theoretic issues; for the Zariski, étale and fppf topologies these subtleties can be safely ignored (see [**SP**, Tag 03YY] for one possible precise formulation of this idea), but things are more subtle for the fpqc topology (see [**Wa**]); for this reason it is preferable to avoid considering sheafification for this topology.

In particular, any object X in Sch_S defines a presheaf on this category via the Yoneda functor; more precisely, to X we associate the presheaf h_X sending a scheme Y over S to the set $\mathrm{Hom}_{\mathsf{Sch}_S}(Y, X)$ of morphisms of S-schemes from Y to X. Such a presheaf will be called representable. It is known that a representable presheaf is a sheaf for the fpqc topology (and hence also for the fppf, étale and Zariski topologies), see [**SP**, Tag 023Q] or [**Ol**, Theorem 4.1.2 and Remark 4.1.4]. The Yoneda lemma ensures that the assignment $X \mapsto h_X$ is fully faithful; we can (and will) therefore identify Sch_S with a full subcategory of the category of sheaves on Sch_S (for one of the topologies considered above), and will sometimes write X for h_X. More generally, the Yoneda lemma says that

$$\mathrm{Hom}(h_X, F) = F(X) \tag{2.1.1}$$

for any presheaf F and any scheme X. Note also that one can form products of sheaves in the obvious way, and that for $X, Y \in \mathrm{Sch}_S$ we have

$$h_X \times h_Y = h_{X \times_S Y}.$$

Remark 2.1.1. — Any sheaf for the Zariski topology is determined by its values on affine schemes; see [**SP**, Tag 020W] or [**GW**, Exercise 8.1] for precise formulations of this idea. For this reason, as mentioned in Remark 1.2.1, one can study schemes (or more general sheaves) as functors on the category of affine schemes over S rather than on all schemes. (See [**SP**, Tag 021E], resp. [**SP**, Tag 021V], for the special case of étale, resp. fppf, sheaves.) This is convenient when considering ind-schemes as in §1.1.1, but less interesting in the present discussion of torsors.

2.1.2. Torsors and principal bundles. — As in §2.1.1, we consider a base scheme S, the category Sch_S of S-schemes, and one of the "standard" Grothendieck topologies on this category. Let $\mathscr{G}$ be a sheaf of groups for this topology. Then a *$\mathscr{G}$-torsor* is a sheaf $\mathscr{P}$ for the same topology endowed with a right action of $\mathscr{G}$ (i.e. a morphism $\mathscr{P} \times \mathscr{G} \to \mathscr{P}$ satisfying the obvious axioms) and which satisfies the following conditions:

1. for any X in Sch_S there exists a covering $(X_i \to X)_{i \in I}$ such that $\mathscr{P}(X_i) \neq \varnothing$ for all $i \in I$;
2. the map
$$\mathscr{P} \times \mathscr{G} \to \mathscr{P} \times \mathscr{P}$$
defined by $(x, g) \mapsto (x, xg)$ is an isomorphism of sheaves;

see [**Ol**, §4.5.1]. Here, the second condition can be restated as saying that whenever $\mathscr{P}(X) \neq \varnothing$, the action of $\mathscr{G}(X)$ on $\mathscr{P}(X)$ is simply transitive. In particular, a torsor $\mathscr{P}$ satisfies $\mathscr{P}(S) \neq \varnothing$ iff it is isomorphic to $\mathscr{G}$ with its standard right action. Such a torsor is called trivial.

In practice, the sheaves of groups we will consider will always be representable. In other words, we start with an S-group scheme G, and set $\mathscr{G} = h_G$. In this setting, a *principal G-bundle* is a scheme X endowed with a right action of G (over S) such that

1. the morphism of schemes

$$X \times_S G \to X \times_S X$$

   defined by $(x, g) \mapsto (x, xg)$ is an isomorphism;
2. there exists an fpqc covering $(S_i \to S)_{i \in I}$ such that each $X \times_S S_i \to S_i$ is G-equivariantly isomorphic to $G \times_S S_i \to S_i$ (i.e. admits a section),

see [**SP**, Tag 049A]. We will say that this principal G-bundle is fppf locally trivial, resp. étale locally trivial, resp. Zariski locally trivial, if the covering in condition (2) can be chosen to be an fppf, resp. étale, resp. Zariski, covering. We will say that X is trivial if there it is G-equivariantly isomorphic to G (over S), or in other words if the covering can be chosen to be $S \xrightarrow{\mathrm{id}_S} S$.

Remark 2.1.2. — 1. Suppose we are given a scheme X endowed with a right action of G (over S) such that the morphism of schemes

$$X \times_S G \to X \times_S X$$

   defined by $(x, g) \mapsto (x, xg)$ is an isomorphism. If the morphism $X \to S$ is surjective, flat and locally finitely presented (in other words, an fppf covering) then X is automatically an fppf locally trivial principal G-bundle.
2. The property of being locally finitely presented is fpqc on the base (see [**SP**, Tag 02KY]), and so is the property of being flat (see [**SP**, Tag 02L2]), quasi-compact (see [**SP**, Tag 02KQ]), affine (see [**SP**, Tag 02L5]), of finite type (see [**SP**, Tag 02KZ]), or smooth (see [**SP**, Tag 02VL]). Thus, if G is locally finitely presented, flat, quasi-compact, affine, of finite type, or smooth, then so is any principal G-bundle.
3. Combining (1) and (2) (and the fact that surjectivity is also fpqc local on the base, see [**SP**, Tag 02KV]), we obtain that if G is flat and locally finitely presented, then a principal G-bundle is automatically fppf locally trivial.
4. Similarly, in case G is smooth, then any principal G-bundle is smooth over S. Since surjective smooth morphisms admit sections étale locally (see [**EGA4.4**, Corollaire 17.16.3(ii)] or [**SP**, Tag 055U] or [**Ol**, Corollary 1.3.10]), it follows that any principal G-bundle is étale locally trivial.
5. In case $G = \mathrm{GL}(n)_S$, it is known that any principal G-bundle is in fact Zariski locally trivial. (This is one possible formulation of Hilbert's Theorem 90.)

If X is a scheme, then by full faithfulness of the assignment $X \mapsto h_X$, the datum of a right h_G-action on h_X is equivalent to the datum of a right G-action on X. Moreover, given such data, X is a principal G-bundle, resp. an fppf locally trivial principal G-bundle, resp. an étale locally trivial principal G-bundle, resp. a Zariski locally trivial principal G-bundle, iff h_X is an h_G-torsor in the fpqc topology, resp. in the fppf topology, resp. in the étale topology, resp. in the Zariski topology, see [**SP**, Tag 049B].

It is in general a delicate question to determine if all h_G-torsors are representable, i.e. are the sheaves associated with principal G-bundles. However, in case G is affine, any h_G-torsor in the fpqc, fppf or étale topology is representable by a principal G-bundle, see [**Ol**, Proposition 4.5.6] or [**Mi1**, Chap. III, Theorem 4.3]. Here the representing scheme is constructed by (affine) descent. Namely, if $\mathscr{P}$ is an h_G-bundle (in one of the topologies above), and if $(S_i \to S)_{i \in I}$ is a covering (for the corresponding topology) over which $\mathscr{P}$ is trivial, then the restriction of $\mathscr{P}$ to each S_i is representable by a scheme P_i (non canonically isomorphic to $G \times_S S_i$). Each P_i is then affine over S_i by our assumption on G, and these schemes are naturally endowed with a descent datum relative to the covering $(S_i \to S)_{i \in I}$ in the sense of [**SP**, Tag 023W]. By affine descent (see [**SP**, Tag 0245]) this datum is automatically effective, which provides the scheme X representing $\mathscr{P}$.

In conclusion we obtain that if G is a smooth affine group scheme over S, the following notions all coincide:

– h_G-torsors for the fpqc topology;
– h_G-torsors for the fppf topology;
– h_G-torsors for the étale topology;
– principal G-bundles;
– fppf locally trivial principal G-bundles;
– étale locally trivial principal G-bundles.

In a minor abuse of language, if G is an S-group scheme and X is an S-scheme, we will say *principal G-bundle over X* to mean a principal $G \times_S X$-bundle (in Sch_X).

Remark 2.1.3. — Assume that G is flat and quasi-compact over S. Let $\bar{X}$ and Y be S-schemes, let X be a principal G-bundle over $\bar{X}$, and let $f : X \to Y$ be a morphism of S-schemes that is G-invariant (i.e. whose compositions with the projection and the action maps $X \times_S G \to X$ coincide). Then f factors through a unique morphism of S-schemes $\bar{X} \to Y$: this follows from the fact that h_Y is an fpqc sheaf, applied to the fpqc cover $X \to \bar{X}$ (see Remark 2.1.2(2)).

2.1.3. Trivializations over certain base schemes. — Fix a ring R, and let $I \subset R[x]$ be an ideal such that $R[x]/I$ is finitely generated as an R-module, i.e., such that $\mathrm{Spec}(R[x]/I)$ is finite over $\mathrm{Spec}(R)$. For any integer $k \geq 1$, let $S_k = \mathrm{Spec}(R[x]/I^k)$, and let

$$S = \varinjlim_k S_k = \mathrm{Spec}(\widehat{R[x]}_I),$$

where $\widehat{R[x]}_I = \varprojlim_k R[x]/I^k$ is the completion of $R[x]$ with respect to I. (Note, however, that S is considered as an ordinary scheme, not a formal scheme.) Some typical examples of this situation are listed below, using the notation of §1.3.3:

I	S_k	S
(x)	$\mathrm{Spec}(R[x]/(x^k))$	$\widehat{\Gamma}_0 = \mathrm{Spec}(R[\![x]\!])$
$(x-y),\ y \in R$	$\mathrm{Spec}(R[x]/((x-y)^k))$	$\widehat{\Gamma}_y = \mathrm{Spec}([\![x-y]\!])$
$(x) \cap (x-y),\ y \in R$	$\mathrm{Spec}(R[x]/((x) \cap (x-y))^k)$	$\widehat{\Gamma_0 \cup \Gamma_y}$

Now let G be a smooth, affine group scheme over S. We have explained in the preceding subsection that any principal G-bundle $X \to S$ is étale locally trivial, i.e., becomes trivial over some étale cover $(V_j \to S)_{j \in J}$. The following proposition gives a refinement of this fact: it says that $X \to S$ becomes trivial over an étale cover of S obtained by base change of an étale cover of $\mathrm{Spec}(R)$.

Proposition 2.1.4. — *Let R and S be as above. Let G be a smooth, affine group scheme over S, and let $X \to S$ be a principal G-bundle. Then $\mathrm{Spec}(R)$ admits an étale cover $(U_i \to \mathrm{Spec}(R))_{i \in I}$ such that for all $i \in I$,*

$$X \times_{\mathrm{Spec}(R)} U_i \to S \times_{\mathrm{Spec}(R)} U_i$$

is a trivial principal $G \times_{\mathrm{Spec}(R)} U_i$-bundle.

Of course, the étale maps $U_i \to \mathrm{Spec}(R)$ may be assumed to be affine, so the proposition can be rephrased as follows: for each point $\mathfrak{p} \in \mathrm{Spec}(R)$, there is an étale ring map $R \to R'$ such that $\mathfrak{p}$ lies in the image of $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$, and such that $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \to S \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$ admits a section.

Proof. — Let $\mathfrak{p}$ be a point in $\mathrm{Spec}(R)$. Let $R_{\mathfrak{p}}$ be the localization of R at $\mathfrak{p}$, and choose a separable closure κ of the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Let $R_{\mathfrak{p}}^{\mathrm{sh}}$ be the strict Henselization of $R_{\mathfrak{p}}$ whose residue field is κ. Recall (say from [**SP**, Tag 04GP]) that this can written as a limit

$$R_{\mathfrak{p}}^{\mathrm{sh}} = \varinjlim_{\substack{\text{factorizations } R \to R' \to \kappa \\ \text{of } R \to \kappa, \text{ with } R \to R' \text{ étale}}} R'. \tag{2.1.2}$$

Let $I_{\mathfrak{p}}^{\mathrm{sh}} = I \cdot R_{\mathfrak{p}}^{\mathrm{sh}}[x]$. Since $R[x]/I$ is a finite R-algebra, the ring

$$R_{\mathfrak{p}}^{\mathrm{sh}}[x]/I_{\mathfrak{p}}^{\mathrm{sh}} \cong (R[x]/I) \otimes_R R_{\mathfrak{p}}^{\mathrm{sh}} \cong \varinjlim R'[x]/IR'[x]$$

(where the indexing set of the direct limit is as above) is a finite $R_{\mathfrak{p}}^{\mathrm{sh}}$-algebra, and therefore a finite direct product of Henselian local rings, see [**SP**, Tag 04GH]. Moreover, the residue fields of these local rings are finite extensions of κ, and are hence also separably closed. In other words, $R_{\mathfrak{p}}^{\mathrm{sh}}[x]/I_{\mathfrak{p}}^{\mathrm{sh}}$ is in fact a finite product of strictly Henselian local rings.

Let $S_{\mathfrak{p}}^{\mathrm{sh}} = \mathrm{Spec}(R_{\mathfrak{p}}^{\mathrm{sh}}[x]/I_{\mathfrak{p}}^{\mathrm{sh}})$, and let $X_{\mathfrak{p}}^{\mathrm{sh}} = X \times_S S_{\mathfrak{p}}^{\mathrm{sh}}$. The map $X_{\mathfrak{p}}^{\mathrm{sh}} \to S_{\mathfrak{p}}^{\mathrm{sh}}$ is smooth and affine, as it is obtained by base change from $X \to S$, which is smooth and affine by Remark 2.1.2. Any smooth morphism over a strictly Henselian local ring admits a section (see [**BLR**, Corollary 13 on p. 42, Proposition 14 on p. 43, and Proposition 4 on p. 46]), so there is a section s as shown below:

$$\begin{tikzcd} X_{\mathfrak{p}}^{\mathrm{sh}} \arrow[r] \arrow[d] & X \times_S S_1 \arrow[d] \\ S_{\mathfrak{p}}^{\mathrm{sh}} \arrow[r] \arrow[u, dashed, bend left, "s"] & S_1 \end{tikzcd}$$

or

$$\begin{tikzcd} \mathscr{O}(X)/I\mathscr{O}(X) \otimes_{R[x]} \varinjlim R'[x] \arrow[d, bend right, "s"'] & \mathscr{O}(X)/I\mathscr{O}(X) \arrow[l] \\ \varinjlim R'[x]/IR'[x] \arrow[u] & R[x]/I \arrow[l] \arrow[u] \end{tikzcd}$$

Now, $\mathscr{O}(X)/I\mathscr{O}(X)$ is finitely generated over $R[x]/I$ (because $\mathscr{O}(X)$ is finitely generated over $R[x]$), so the composition

$$\mathscr{O}(X)/I\mathscr{O}(X) \to \mathscr{O}(X)/I\mathscr{O}(X) \otimes_{R[x]} \varinjlim R'[x] \to \varinjlim R'[x]/IR'[x]$$

must factor through some term $R'[x]/IR'[x]$ in the limit. Let $\bar{s} : \mathscr{O}(X)/I\mathscr{O}(X) \otimes_{R[x]} R'[x] \to R'[x]/IR'[x]$ be the induced map.

Let $S' = S \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$, $X' = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$, and $S'_1 = S_1 \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$. The map $\bar{s}$ defined above is a section of the principal $G \times_S S'_1$-bundle $X' \times_{S'} S'_1 \to S'_1$. By Lemma 2.1.5 below (applied with R' and S' in place of R and S), the $G \times_S S'$-bundle $X' \to S'$ also admits a section, as desired. □

Lemma 2.1.5. — *In the setting of Proposition 2.1.4, we have that $X \to S$ is a trivial G-bundle if and only if for some $k \geq 1$, $X \times_S S_k \to S_k$ is a trivial $(G \times_S S_k)$-bundle.*

Proof. — The "only if" direction is obvious; we need only prove the "if" direction. Recall from Remark 2.1.2 that $X \to S$ is affine and smooth, and hence also formally smooth (see [**SP**, Tag 00TN]). If for some $n \geq 1$ there is a section $s_n : S_n \to X \times_S S_n$, then by formal smoothness, we can find a map t making the following diagram commute:

$$\begin{tikzcd}
\mathscr{O}(X) \arrow[r] \arrow[drr, dashed, "t"] & \mathscr{O}(X)/I^n\mathscr{O}(X) \arrow[r, "s_n"] & R[x]/(I^n) \\
\widehat{R[x]}_I \arrow[u] \arrow[rr] & & R[x]/(I^{n+1}). \arrow[u]
\end{tikzcd}$$

The map t clearly factors uniquely through a map $s_{n+1} : \mathscr{O}(X)/I^{n+1}\mathscr{O}(X) \to R[x]/(I^{n+1})$. In other words, if we have a section $s_n : S_n \to X \times_S S_n$, it can be extended to a section $s_{n+1} : S_{n+1} \to X \times_S S_{n+1}$.

Now let $s_k : S_k \to X \times_S S_k$ be a section. By induction, the previous paragraph gives us sections $s_n : S_n \to X \times_S S_n$ for all $n \geq k$. Passing to the limit, we obtain the desired section $S \to X$. □

We conclude this subsection with a lemma about the case $I = (x) \cap (x - y) \subset R[x]$, where y is some element of R. This lemma will make use of the following explicit description of the completion $\widehat{R[x]}_{(x)\cap(x-y)}$. It is easy to check that

$$((x) \cap (x - y))^2 \subset (x^2 - xy) \subset (x) \cap (x - y),$$

so $\widehat{R[x]}_{(x)\cap(x-y)}$ is canonically identified with $\widehat{R[x]}_{(x^2-xy)}$. Next, $R[x]$ is isomorphic to $R[x, u]/(x^2 - xy - u)$, where u is another indeterminate, and under this isomorphism, the ideal $(x^2 - xy) \subset R[x]$ is identified with $(u) \subset R[x, u]/(x^2 - xy - u)$. We conclude that

$$\widehat{R[x]}_{(x)\cap(x-y)} \cong R[x][\![u]\!]/(x^2 - xy - u) \cong R[\![u]\!][x]/(x^2 - xy - u). \tag{2.1.3}$$

(Here, the second isomorphism can be checked by observing that both sides are free $R[\![u]\!]$-modules of rank 2, with basis $\{1, x\}$.)

Lemma 2.1.6. — *Suppose that* 2 *is invertible in* R *and that* $y \in R$ *is invertible. Let* $S = \operatorname{Spec}(\widehat{R[x]}_{(x)\cap(x-y)})$*, and let* G *be a smooth, affine group scheme over* S*.*

1. *There exists a canonical isomorphism* $S \cong \operatorname{Spec}(R[\![x]\!]) \sqcup \operatorname{Spec}(R[\![x-y]\!])$ *as schemes over* $\operatorname{Spec}(R[x])$*.*
2. *The datum of a principal* G*-bundle* $X \to S$ *is equivalent to the datum of a pair of principal* G*-bundles* $X' \to \operatorname{Spec}(R[\![x]\!])$*,* $X'' \to \operatorname{Spec}(R[\![x-y]\!])$*.*

Proof. — Both parts of the lemma follow from the observation that there is a canonical (continuous) isomorphism of $R[x]$-algebras

$$\widehat{R[x]}_{(x)\cap(x-y)} \cong R[\![x]\!] \times R[\![x-y]\!]. \tag{2.1.4}$$

To see this, we first note that $1+4u$ has a unique square root $\phi(u)$ in $\mathbb{Z}[\![u]\!]$ whose constant term is 1, namely

$$\phi(u) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(1-2n)(n!)^2} u^n = 1 + 2u - 2u^2 + 4u^3 - \cdots .$$

Since y is invertible, it makes sense to consider $\phi(y^{-2}u) \in R[\![u]\!]$. Then we have

$$x^2 - xy - u = \left(x - \tfrac{1}{2}y(1+\phi(y^{-2}u))\right)\left(x - \tfrac{1}{2}y(1-\phi(y^{-2}u))\right).$$

It follows easily that there is an isomorphism

$$R[\![u]\!][x]/(x^2 - xy - u) \cong R[\![u]\!] \times R[\![u]\!]$$

given by $f \mapsto (f_{|x=\frac{1}{2}y(1-\phi(y^{-2}u))}, f_{|x=\frac{1}{2}y(1+\phi(y^{-2}u))})$. In view of (2.1.3), this yields (2.1.4). □

2.1.4. Descent and associated bundles. —

2.1.4.1. *Descent for schemes.* — Let G be a group scheme over a base scheme S. Let Z be an S-scheme, and let $f : X \to Z$ be a principal G-bundle over Z. Finally, let Y be another S-scheme with a left G-action. Informally, the idea of an "associated bundle" is to replace the fibers of $f : X \to Z$ (which are copies of G) with copies of Y.

Here is a more precise definition. Equip $X \times_S Y$ with a right G-action by $(x,y)\cdot g = (xg, g^{-1}y)$. Suppose there exists an S-scheme P along with a map $\tilde{f} : X \times_S Y \to P$ that makes $X \times_S Y$ into a principal G-bundle over P. Such a scheme is unique up to unique isomorphism if it exists. We typically denote it by

$$X \times_S^G Y,$$

and call it the *bundle associated to* X *with fiber* Y. If this scheme exists, it fits into a cartesian square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\tilde{f}"] \arrow[d, "\mathrm{pr}_1"'] & X \times_S^G Y \arrow[d] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

where the top horizontal arrow is a projection map, and the right vertical arrow is a "locally trivial fibration with fibers isomorphic to Y."

If $X \to Z$ is Zariski locally trivial,(1) then it is easy to construct $X \times_S^G Y$, but the existence of $X \times_S^G Y$ in general is a delicate problem. The following proposition gives a criterion for this.

Proposition 2.1.7. — *Let G be a group scheme over S; let $X \to Z$ be a principal G-bundle over Z; and let Y be another S-scheme with a left G-action. Assume the following conditions hold:*

1. *The group scheme G is flat and quasi-compact (e.g. affine) over S.*
2. *The scheme Y admits a G-equivariant line bundle $\mathscr{L}$ that is relatively ample for the structure map $Y \to S$.*

Then the associated bundle $X \times_S^G Y$ exists and admits a line bundle $\widetilde{\mathscr{L}}$ that is relatively ample for $X \times_S^G Y \to Z$.

The existence of $\mathscr{L}$ implies (by definition, see [**SP**, Tag 01VH]) that $Y \to S$ is quasi-compact. A statement very close to this, but with all schemes assumed to be of finite type over S, can be found in [**MFK**, Proposition 7.1]. However, the proof goes through under the weaker assumptions above: the following argument is copied almost verbatim from [**MFK**].

Proof. — Our assumptions imply that $X \to Z$ is faithfully flat and quasi-compact. The main idea is to construct the desired scheme (which will be denoted by P in this proof) by fpqc descent along $X \to Z$. First, we must equip $X \times_S Y$ with a descent datum, i.e., an isomorphism

$$\varphi : (X \times_S Y) \times_Z X \xrightarrow{\sim} X \times_Z (X \times_S Y)$$

satisfying a certain cocycle condition. Since $X \to Z$ is a principal G-bundle, we have an identification $X \times_S G \xrightarrow{\sim} X \times_Z X$ given by $(x, g) \mapsto (x, xg)$. Writing S-points of $X \times_Z X$ in the form (x, xg), we define φ by $\varphi(x, y, xg) = (x, xg, g^{-1}y)$.

Next, our assumption on $\mathscr{L}$ implies that the line bundle $\mathscr{O}_X \boxtimes_S \mathscr{L}$ is relatively ample for the projection map $X \times_S Y \to X$ (see [**SP**, Tag 0893]). Moreover, $\mathscr{O}_X \boxtimes_S \mathscr{L}$ is G-equivariant; and as explained in the proof of [**MFK**, Proposition 7.1], the G-equivariant structure is the same as a descent datum for $\mathscr{O}_X \boxtimes_S \mathscr{L}$ with respect to $X \to Z$.

According to [**SGA1**, Exp. VIII, Proposition 7.8], the preceding paragraph implies that φ is effective, i.e., that there exists an S-scheme P together with maps making the square

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & P \arrow[d] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

cartesian, as well as a line bundle $\widetilde{\mathscr{L}}$ on P induced by $\mathscr{O}_X \boxtimes_S \mathscr{L}$ that is relatively ample for $P \to Z$.

(1)In the larger setting of *algebraic spaces*, the same comment applies to étale locally trivially bundles [**KM**]. In this book, however, we stay in the world of schemes.

It remains to check that the morphism $X \times_S Y \to P$ above is a principal G-bundle over P. This map is certainly faithfully flat and quasi-compact (since $X \to Z$ is), and hence an fpqc covering. From the isomorphism $X \times_S G \xrightarrow{\sim} X \times_Z X$ we obtain that the analogous map $(X \times_S Y) \times_S G \xrightarrow{\sim} (X \times_S Y) \times_P (X \times_S Y)$ is an isomorphism. As in Remark 2.1.2(1), this implies the desired claim. □

Remark 2.1.8. — 1. In the setting of Proposition 2.1.7, if the morphism $Y \to S$ is quasi-affine then $\mathscr{O}_Y$ is relatively ample for this morphism, see [**SP**, Tag 0891]. Since this line bundle is obviously G-equivariant, the proposition applies in this case.

2. Assume that $S = \mathrm{Spec}(\Bbbk)$ with $\Bbbk$ an algebraically closed field, and that G is a $\Bbbk$-algebraic group (i.e. G is affine and of finite type). If X is a reduced quasi-projective irreducible normal $\Bbbk$-scheme endowed with an action of G, then the existence of a G-equivariant ample line bundle is automatic. Indeed, by definition of quasi-projectivity X admits an ample line bundle. Then a power of this line bundle admits a G-equivariant structure by [**Bri2**, Theorem 5.2.1].

2.1.4.2. *The case of ind-schemes.* — In practice, we will need a version of Proposition 2.1.7 that applies when Z and Y are ind-schemes, rather than just ordinary schemes. Suppose we have an ind-scheme Y with a presentation

$$Y = \mathrm{colim}_{i \in I} Y_i, \tag{2.1.5}$$

where each Y_i is an ordinary S-scheme. Denote by $\phi_{ij} : Y_i \to Y_j$ the transition maps in this limit. (These morphisms are closed immersions.) Recall that a *line bundle on* Y is a collection $\mathscr{L} = ((\mathscr{L}_i)_{i \in I}, (\theta_{ij})_{ij})$, where each $\mathscr{L}_i$ is a line bundle on Y_i, and the θ_{ij}'s are isomorphisms

$$\theta_{ij} : \phi_{ij}^* \mathscr{L}_j \xrightarrow{\sim} \mathscr{L}_i$$

satisfying the obvious compatibility condition $\theta_{ik} = \theta_{ij} \circ \phi_{ij}^* \theta_{jk}$ whenever i, j, and k are such that this equation makes sense. We say that $\mathscr{L}$ is *relatively ample* with respect to $Y \to S$ if each $\mathscr{L}_i$ is relatively ample with respect to $Y_i \to S$.

Suppose now that Y is equipped with a left G-action that is compatible with the limit (2.1.5): that is, each Y_i has a left G-action, and the transition maps ϕ_{ij} are G-equivariant. A *G-equivariant line bundle on* Y is a collection $\mathscr{L} = ((\mathscr{L}_i)_{i \in I}, (\theta_{ij})_{ij})$ as above in which each $\mathscr{L}_i$ is G-equivariant, and the θ_{ij}'s are isomorphisms of G-equivariant line bundles.

Finally, if Z is an ind-scheme over S, we define a principal G-bundle over Z to be an ind-scheme X equipped with a right action of G (over S) and a morphism $X \to Z$ such that for any scheme Z' and any morphism $Z' \to Z$ the fiber product $X \times_Z Z'$ is a principal G-bundle over Z' (in particular, is a scheme). In this case, if $Z = \mathrm{colim}_{i \in I} Z_i$ is a presentation of Z, then we have a presentation of X given by

$$X = \mathrm{colim}_{i \in I} X \times_Z Z_i,$$

and each $X \times_Z Z_i$ is a principal G-bundle over Z_i. We will say that this principal G-bundle is étale, resp. fppf, locally trivial if for any Z' as above the principal G-bundle $X \times_Z Z'$ is étale, resp. fppf, locally trivial.

Proposition 2.1.9. — *Let G be a group scheme over S. Let Z and Y be ind-schemes over S obtained as limits over the same indexing set: say*

$$Z = \mathrm{colim}_{i\in I} Z_i \qquad \textit{and} \qquad Y = \mathrm{colim}_{i\in I} Y_i. \tag{2.1.6}$$

Let X be a principal G-bundle over Z, and suppose Y has a left G-action that is compatible with the limit above. Assume also that the following conditions hold:

1. *The group scheme G is flat and quasi-compact over S.*
2. *The ind-scheme Y admits a G-equivariant line bundle $\mathscr{L}$ that is relatively ample for the structure map $Y \to S$.*

Then there is an ind-scheme

$$X \times_S^G Y = \mathrm{colim}_{i\in I}(X \times_Z Z_i) \times_S^G Y_i \tag{2.1.7}$$

along with a map $X \times_S Y \to X \times_S^G Y$ that is a principal G-bundle.

Proof. — Let $X_i = X \times_Z Z_i$. Then $X_i \to Z_i$ is a principal G-bundle, and Proposition 2.1.7 gives us the existence of the scheme $X_i \times_S^G Y_i$. Let us now show that each transition map $X_i \times_S^G Y_i \to X_j \times_S^G Y_j$ is a closed immersion. For that, we write this map as the bottom row of the following diagram:

$$\begin{tikzcd}
X_i \times_S Y_i \arrow[r] \arrow[d] & X_j \times_S Y_i \arrow[r] \arrow[d] & X_j \times_S Y_j \arrow[d] \\
X_i \times_S^G Y_i \arrow[r] & X_j \times_S^G Y_i \arrow[r] & X_j \times_S^G Y_j.
\end{tikzcd}$$

Here, each vertical map is flat, quasi-compact, and surjective (and hence an fpqc covering); the squares are cartesian; and the top horizontal arrows are closed immersions. Since the property of being a closed immersion is fpqc local [**SP**, Tag 02L6], the bottom arrows are closed immersions as well.

Once this property is established, passing to the colimit, we obtain the principal G-bundle $X \times_S Y \to X \times_S^G Y$. □

Remark 2.1.10. —

1. From the proof of Proposition 2.1.9 we see that this proposition still holds if one replaces assumption (2) by the following slightly weaker variant: for any $i \in I$ there is a G-equivariant line bundle $\mathscr{L}_i$ on Y_i which is relatively ample for the structure map $Y_i \to S$. (In other words, we do not need any compatibility property between these line bundles.)
2. Continuing the theme of the preceding proof, the properties of being separated, of finite type, or proper are all fpqc local on the base: see [**SP**, Tag 02KU, Tag 02KZ, Tag 02L1]. As a consequence, if $Y \to S$ (and hence $X \times_S Y \to X$) is separated, or of (ind-)finite type, or (ind-)proper, then $X \times_S^G Y \to Z$ has that property as well. (Note that, by contrast, the property of being projective is not fpqc local on the base, see [**SP**, Lemma 08J1].)

2.1.4.3. *Compatibility with products.* — One nice feature of this construction is its compatibility with fiber products. Namely, consider a cartesian square of G-schemes over S

$$\begin{array}{ccc} Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z, \end{array}$$

and assume that all schemes in this diagram admit relatively (to S) ample G-equivariant line bundles. Then the induced diagram

$$\begin{array}{ccc} X \times_S^G Y' & \longrightarrow & X \times_S^G Z' \\ \downarrow & & \downarrow \\ X \times_S^G Y & \longrightarrow & X \times_S^G Z \end{array}$$

is also cartesian. In fact, by [**We**, Lemma 2.5], for this it suffices to check that the diagram becomes cartesian after pullback along the fpqc cover $X \times_S Z \to X \times_S^G Z$. (This is an application of the fact that representable presheaves are fpqc sheaves.) However, our diagram then becomes the product of X with our original cartesian diagram, so it is indeed cartesian.

2.2. Global affine Grassmannians

In this section we fix a complex connected reductive group G, with a fixed choice of Borel subgroup B.

2.2.1. The affine Grassmannian and the affine flag variety. — Recall from §1.2.1.1 that the affine Grassmannian Gr_G associated with G is the (ind-scheme representing the) fppf sheafification of the functor sending a $\mathbb{C}$-algebra R to the quotient $\mathrm{L}G(R)/\mathrm{L}^+G(R)$. We then have a canonical morphism $\mathrm{L}G \to \mathrm{Gr}_G$, which is known to be a Zariski locally trivial principal L^+G-bundle. (See e.g. [**Rc4**, §§2.1–2.2] for an explicit study of this property in the case of general linear groups.)

There is an obvious morphism of $\mathbb{C}$-group schemes $\mathrm{L}^+G \to G$: in terms of R-points for a $\mathbb{C}$-algebra R, it is the map $G(R[\![x]\!]) \to G(R)$ induced by the quotient map $R[\![x]\!] \to R[\![x]\!]/(x) = R$. The *Iwahori subgroup* $I \subset \mathrm{L}^+G$ associated with B is the subgroup scheme defined as the preimage of B under this map $\mathrm{L}^+G \to G$. One can then define the *affine flag variety*

$$\mathrm{Fl}_G$$

as the fppf sheafification of the functor sending a $\mathbb{C}$-algebra R to $\mathrm{L}G(R)/I(R)$. Again, this sheaf is represented by an ind-projective ind-scheme of ind-finite type. There is an obvious natural map

$$\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G, \tag{2.2.1}$$

which is a locally trivial fibration (for the Zariski topology) with fiber G/B.

Example 2.2.1. — Consider the setting of Example 1.2.3. In this case, the subgroup $I \subset \mathrm{L}^+G$ consists of the matrices $g \in \mathrm{GL}(n, \mathscr{O})$ whose image in $\mathrm{GL}(n, \mathbb{C})$ stabilizes the standard flag

$$\mathbb{C}^n \supset \{0\} \times \mathbb{C}^{n-1} \supset \{0\}^2 \times \mathbb{C}^{n-2} \supset \cdots \supset \{0\}^{n-1} \times \mathbb{C} \supset \{0\}.$$

Therefore, $I \subset \mathrm{GL}(n, \mathscr{K})$ is the stabilizer of the collection of lattices

$$\mathscr{O}^n \supset (x\mathscr{O}) \times \mathscr{O}^{n-1} \supset (x\mathscr{O})^2 \times \mathscr{O}^{n-2} \supset \cdots \supset (x\mathscr{O})^{n-1} \times \mathscr{O} \supset x \cdot \mathscr{O}^n.$$

Concretely, $I(\mathbb{C})$ consists of matrices of the form

$$\begin{pmatrix} a_{11} & xa_{12} & xa_{13} & \cdots & xa_{1n} \\ a_{21} & a_{22} & xa_{23} & \cdots & xa_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & xa_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

with $a_{ij} \in \mathscr{O}$. The space Fl_G can be identified with the set of collections $(\Lambda_i : i = 1, \ldots, n)$ of $\mathscr{O}$-lattices in $\mathscr{K}^n$ such that

$$\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset x\Lambda_1$$

and $\dim(\Lambda_i/\Lambda_{i+1}) = 1$ for $i = 1, \ldots, n-1$. In this interpretation (and that of §1.2.1.3), the map $\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G$ sends $(\Lambda_1, \ldots, \Lambda_n)$ to Λ_1.

For $\Bbbk$ a noetherian commutative ring of finite global dimension, we will denote by $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ the constructible I-equivariant derived category of $\Bbbk$-sheaves on Fl_G in the sense of Bernstein–Lunts [**BL**]. (As for $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, some technical difficulties arise when defining this category; see §1.3.1 for a discussion and references.) A construction similar to that explained in §1.3.1 using the natural morphism

$$m' : \mathrm{L}G \times^I \mathrm{Fl}_G \to \mathrm{Fl}_G \tag{2.2.2}$$

allows us to define a convolution product $\star^I$ on $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$.

2.2.2. Moduli interpretation. —

2.2.2.1. *The case of* Gr_G. — We will now recall the description of the affine Grassmannian as a certain moduli space of G-bundles on a formal disc. Explicitly, let $\overline{\mathrm{Gr}}_G$ be the functor from $\mathbb{C}$-algebras to sets given by

$$\overline{\mathrm{Gr}}_G(R) = \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{c} \mathcal{E} \text{ a principal } G\text{-bundle over } \mathrm{Spec}(R[\![x]\!]), \\ \beta : \mathcal{E}_{|\mathrm{Spec}(R(\!(x)\!))} \xrightarrow{\sim} G \times \mathrm{Spec}(R(\!(x)\!)) \text{ a trivialization} \end{array} \right\}. \tag{2.2.3}$$

(Here and below, we implicitly consider *equivalence classes* of geometric data, for the obvious equivalence relation.)

Proposition 2.2.2. — *There exists a canonical isomorphism* $\mathrm{Gr}_G \xrightarrow{\sim} \overline{\mathrm{Gr}}_G$.

In view of this proposition, below we will not make any distinction between Gr_G and $\overline{\mathrm{Gr}}_G$. For a discussion of the proof of Proposition 2.2.2, see [**BR**, §1.7.1]; see also [**Zh4**, Proposition 1.3.6] or [**Rc4**, Proposition 3.18] for more details. One important point in the proof is showing that if R is a $\mathbb{C}$-algebra and $\mathcal{E}$ is a principal G-bundle over $\mathrm{Spec}(R[\![x]\!])$, there exists an étale cover $R \to R'$ such that the pullback $\mathrm{Spec}(R'[\![x]\!]) \times_{\mathrm{Spec}(R[\![x]\!])} \mathcal{E}$ is trivial, which follows from Proposition 2.1.4 (applied to the constant group scheme $G \times \mathrm{Spec}(R[\![x]\!])$ over $\mathrm{Spec}(R[\![x]\!])$).

Remark 2.2.3. — In [**Rc4**, Theorem 3.4] it is proved directly that the functor $\overline{\mathrm{Gr}}_G$ is representable (in a much more general setting, in fact), and later it is deduced (again in a general setting, though additional conditions have to be imposed) that this scheme represents the étale sheafification of the functor sending R to $\mathrm{L}G(R)/\mathrm{L}^+G(R)$. In this way one sees that the latter étale sheafification is already an fppf (in fact, even fpqc) sheaf, and therefore that it coincides with the fppf sheafification of this functor, which was our original definition for Gr_G.

In the course of the proof of Proposition 2.2.2 one also establishes the following moduli interpretations of the groups $\mathrm{L}G$ and L^+G.

Lemma 2.2.4. — *The group ind-scheme* $\mathrm{L}G$ *represents the functor*

$$R \mapsto \left\{ (\mathcal{E},\beta,\gamma) \;\middle|\; \begin{array}{c} \mathcal{E} \textit{ a principal } G\textit{-bundle over } \mathrm{Spec}(R[\![x]\!]), \\ \beta : \mathcal{E}_{|\,\mathrm{Spec}(R((x)))} \xrightarrow{\sim} G \times \mathrm{Spec}(R((x))) \textit{ a trivialization,} \\ \textit{and } \gamma : \mathcal{E} \xrightarrow{\sim} G \times \mathrm{Spec}(R[\![x]\!]) \textit{ a trivialization} \end{array} \right\},$$

and the group scheme L^+G *represents the functor*

$$R \mapsto \left\{ (\mathcal{E},\beta,\gamma) \;\middle|\; \begin{array}{c} \mathcal{E} \textit{ a principal } G\textit{-bundle over } \mathrm{Spec}(R[\![x]\!]), \\ \beta : \mathcal{E} \xrightarrow{\sim} G \times \mathrm{Spec}(R[\![x]\!]) \textit{ a trivialization,} \\ \textit{and } \gamma : \mathcal{E} \xrightarrow{\sim} G \times \mathrm{Spec}(R[\![x]\!]) \textit{ a trivialization} \end{array} \right\}.$$

In this language, the group operations for $\mathrm{L}G$ and L^+G are given by
(2.2.4)

$$(\mathcal{E},\beta,\gamma)\cdot(\mathcal{E}',\beta',\gamma') = \begin{cases} (\mathcal{E}', \beta \circ \gamma^{-1}_{|\,\mathrm{Spec}(R((x)))} \circ \beta', \gamma') & \text{in } \mathrm{L}G, \\ (\mathcal{E}', \beta\circ\gamma^{-1}\circ\beta', \gamma') = (\mathcal{E},\beta,\gamma'\circ(\beta')^{-1}\circ\gamma) & \text{in } \mathrm{L}^+G, \end{cases}$$

while the inverse of an element is given by

$$(\mathcal{E},\beta,\gamma)^{-1} = \begin{cases} (G\times\mathrm{Spec}(R[\![x]\!]), \gamma_{|\,\mathrm{Spec}(R((x)))}\circ\beta^{-1}, \mathrm{id}) & \text{in } \mathrm{L}G, \\ (G\times\mathrm{Spec}(R[\![x]\!]), \gamma\circ\beta^{-1}, \mathrm{id}) & \text{in } \mathrm{L}^+G. \end{cases}$$

2.2.2.2. *The case of* Fl_G. — To give an analogous description of the affine flag variety, one needs to introduce the *Iwahori group scheme* $\mathcal{I}$, a smooth (nonconstant!) affine group scheme over $\mathrm{Spec}(\mathscr{O})$ whose restriction to $\mathrm{Spec}(\mathscr{K})$ is $G\times\mathrm{Spec}(\mathscr{K})$ and whose $\mathscr{O}$-points are $I(\mathbb{C})$ as defined in §2.2.1. (In fact, as explained in [**PR**, §2.a.1], these properties characterize $\mathcal{I}$ uniquely.) The construction of this group scheme uses Bruhat–Tits theory; see [**Zh4**, Example 1.2.9(2)] and [**McN**, §6.2] for more details.

Define $\overline{\mathrm{Fl}}_G$ to be the functor from $\mathbb{C}$-algebras to sets given by
(2.2.5)
$$\overline{\mathrm{Fl}}_G(R) = \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{c} \mathcal{E} \text{ a principal } \mathcal{I}\text{-bundle,} \\ \beta : \mathcal{E}_{| \operatorname{Spec}(R((x)))} \xrightarrow{\sim} G \times \operatorname{Spec}(R((x))) \text{ a trivialization} \end{array} \right\}.$$

The analogues of Proposition 2.2.2 and Lemma 2.2.4 for the nonconstant group scheme $\mathcal{I} \to \operatorname{Spec}(\mathscr{O})$ replacing the constant group scheme $G \times \operatorname{Spec}(\mathscr{O}) \to \operatorname{Spec}(\mathscr{O})$ are as follows.

Proposition 2.2.5. — *There is a canonical isomorphism* $\mathrm{Fl}_G \xrightarrow{\sim} \overline{\mathrm{Fl}}_G$.

Lemma 2.2.6. — *The group scheme I represents the functor*

$$R \mapsto \left\{ (\mathcal{E}, \beta, \gamma) \;\middle|\; \begin{array}{c} \mathcal{E} \textit{ a principal } \mathcal{I}\textit{-bundle,} \\ \beta : \mathcal{E} \xrightarrow{\sim} \mathcal{I} \times_{\operatorname{Spec}(\mathscr{O})} \operatorname{Spec}(R[[x]]) \textit{ a trivialization,} \\ \textit{and } \gamma : \mathcal{E} \xrightarrow{\sim} \mathcal{I} \times_{\operatorname{Spec}(\mathscr{O})} \operatorname{Spec}(R[[x]]) \textit{ a trivialization} \end{array} \right\}.$$

A proof of Proposition 2.2.5 can be found in [**Zh4**, Proposition 1.3.6]; see also [**He**, Proposition 4]. Once again, an essential step in the proof is the observation that any $\mathcal{I}$-bundle on $\operatorname{Spec}(R[[x]])$ can be trivialized on $\operatorname{Spec}(R'[[x]])$ for some étale cover $R \to R'$, which follows from Proposition 2.1.4 (applied to the group scheme $\mathcal{I} \times_{\operatorname{Spec}(\mathscr{O})} \operatorname{Spec}(R[[x]])$ this time).

Remark 2.2.7. —
1. The setting considered in [**Rc4**] also covers the construction of $\overline{\mathrm{Fl}}_G$ and Fl_G. Thus, as in Remark 2.2.3, one obtains in this way that the étale sheafification of the functor sending R to $\mathrm{L}G(R)/I(R)$ is already an fpqc sheaf, and hence that it coincides with Fl_G as defined in §2.2.1.
2. The descriptions of Gr_G and Fl_G given in Propositions 2.2.2 and 2.2.5 are in terms of bundles on a formal disc. These ind-schemes can also be described in terms of bundles on a curve, as follows. Let C be a reduced connected complex curve, and let 0 denote a smooth closed marked point on C. (The example we have in mind is when $C = \mathbb{A}^1_{\mathbb{C}}$, with 0 the origin; this explains our choice of notation.) Let $C^\circ := C \smallsetminus \{0\}$. For R a $\mathbb{C}$-algebra, we will denote by C_R and C°_R the products of C and C° respectively with $\operatorname{Spec}(R)$. It is a consequence of the Beauville–Laszlo theorem [**BLa**] that Gr_G represents the functor
$$R \mapsto \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{c} \mathcal{E} \text{ a principal } G\text{-bundle over } C_R, \\ \beta : \mathcal{E}_{|C^\circ_R} \xrightarrow{\sim} G \times C^\circ_R \text{ a trivialization} \end{array} \right\}.$$
See [**BR**, §1.7.1] for a brief discussion, and [**Zh4**, §1.4] or [**Rc4**, §3.2] for more details. A similar description of Fl_G (also treated in [**Zh4**]) can be given using the group scheme $\mathcal{G}$ defined in §2.2.3 below.

Example 2.2.8. — Consider once again the setting of Example 1.2.3. In this case the group scheme $\mathcal{I}$ admits a concrete description: it represents the functor sending an $\mathscr{O}$-algebra R to the group of n-tuples $(g_1, \dots, g_n) \in \mathrm{GL}(n, R)^n$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
R^n & \xrightarrow{\varepsilon_n^\vee(x)} & R^n & \xrightarrow{\varepsilon_{n-1}^\vee(x)} & \cdots & \xrightarrow{\varepsilon_2^\vee(x)} & R^n & \xrightarrow{\varepsilon_1^\vee(x)} & R^n \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle g_2}\downarrow & & & & {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle g_1} \\
R^n & \xrightarrow{\varepsilon_n^\vee(x)} & R^n & \xrightarrow{\varepsilon_{n-1}^\vee(x)} & \cdots & \xrightarrow{\varepsilon_2^\vee(x)} & R^n & \xrightarrow{\varepsilon_1^\vee(x)} & R^n.
\end{array}$$

The canonical morphism $\mathcal{I} \to G \times \mathrm{Spec}(\mathscr{O})$ sends an n-tuple $(g_1, \dots, g_n)$ to g_1.

In case x is invertible in R the datum of the n-tuple $(g_1, \dots, g_n)$ is equivalent to the datum of g_1, illustrating the fact that the morphism $\mathcal{I} \to G \times \mathrm{Spec}(\mathscr{O})$ restricts to an isomorphism over $\mathrm{Spec}(\mathscr{K})$. On the other hand, the restriction of this morphism to $\{0\} \subset \mathrm{Spec}(\mathscr{O})$ is far from an isomorphism. For instance, when $n = 2$, the fiber of $\mathcal{I}$ over $\{0\}$ consists of pairs of invertible matrices $(A, B) \in \mathrm{GL}(2, \mathbb{C})^2$ satisfying

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot A = B \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot B = A \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, this fiber is identified with the closed subgroup of $\mathrm{GL}(2, \mathbb{C})^2$ consisting of pairs of matrices of the form

$$\left(\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \right)$$

with α, δ in $\mathbb{C}^\times$ and $\beta, \gamma \in \mathbb{C}$.

Here is another description of $\mathcal{I}$ in this case, reminiscent of Example 2.2.1. Let $\mathfrak{S}_n$ be the symmetric group on $\{1, \dots, n\}$. For a permutation $\sigma \in \mathfrak{S}_n$, let $\mathsf{a}(\sigma) = |\{i \in \{1, \dots, n\} \mid i < \sigma(i)\}|$. For a matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in R^{n^2}$, we define the *deformed determinant* by

$$\det_x(A) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) x^{\mathsf{a}(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

We also define *deformed matrix multiplication* as follows: for matrices $A, B \in R^{n^2}$, we define $A \cdot_x B$ by

$$(A \cdot_x B)_{ij} = \begin{cases} \sum_{1 \leq k < j \text{ or } i < k \leq n} x a_{ik} b_{kj} + \sum_{j \leq k \leq i} a_{ik} b_{kj} & \text{if } i \geq j, \\ \sum_{1 \leq k \leq i \text{ or } j \leq k \leq n} a_{ik} b_{kj} + \sum_{i < k < j} x a_{ik} b_{kj} & \text{if } i < j. \end{cases}$$

As a mnemonic, the deformed determinant of A is the ordinary determinant of the matrix

$$\gamma_1(A) := \begin{pmatrix} a_{11} & x a_{12} & x a_{13} & \cdots & x a_{1n} \\ a_{21} & a_{22} & x a_{23} & \cdots & x a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & x a_{3n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix},$$

and deformed multiplication corresponds under γ_1 to the usual matrix multiplication; that is, we have $\gamma_1(A \cdot_x B) = \gamma_1(A)\gamma_1(B)$.

The map γ_1 can be generalized as follows. Let $\rho \in \mathfrak{S}_n$ be the permutation given by $\rho(i) = i+1$ for $1 \le i \le n-1$, and $\rho(n) = 1$. For $1 \le k \le n$, define $\gamma_k(A)$ by

$$\gamma_k(A)_{ij} = \begin{cases} xa_{ij} & \text{if } \rho^{k-1}(i) < \rho^{k-1}(j), \\ a_{ij} & \text{otherwise.} \end{cases}$$

One can show that $\det_x(A) = \det \gamma_k(A)$ for all $1 \le k \le n$, and that $\gamma_k(A \cdot_x B) = \gamma_k(A)\gamma_k(B)$. Moreover, we have

$$\varepsilon^\vee_{n+1-k}(x)\gamma_k(A) = \gamma_{k+1}(A)\varepsilon^\vee_{n+1-k}(x).$$

In fact, the map $A \mapsto (\gamma_1(A), \dots, \gamma_k(A))$ defines a bijection

$$\{A \in R^{n^2} \mid \det_x(A) \in R^\times\} \cong \mathcal{I}(R).$$

2.2.3. Global group schemes and global affine Grassmannians. — In §2.2.2 we have explained how to describe the ind-schemes Gr_G and Fl_G in terms of bundles on a formal disc. To go further we will need to discuss several "global" analogues of these ind-schemes, which involve bundles on a curve. The first example of such a construction (which is the main ingredient in the definition of the central functor) is explained in the present subsection.

2.2.3.1. *Global loop and arc groups.* — From now on, as in §1.3.3 we let $C = \mathbb{A}^1_{\mathbb{C}}$, with "marked" point 0, and set $C^\circ := C \smallsetminus \{0\}$. Recall that for y an R-point of C we have the schemes $\Gamma_y \subset C_R$, $\widehat{\Gamma}_y$ and $\widehat{\Gamma}^\circ_y := \widehat{\Gamma}_y \smallsetminus \Gamma_y$.

We will consider the smooth affine group scheme $\mathcal{G}$ over C obtained by gluing the Iwahori group scheme $\mathcal{I}$ over $\mathrm{Spec}(\mathscr{O})$ to the constant group scheme $G \times C^\circ$ using fpqc descent. This group scheme is endowed with a canonical morphism $\mathcal{G} \to G \times C$ (which is *not* a closed embedding). One may consider principal $\mathcal{G}$-bundles on (base changes of) C, as explained in §2.1.2. From now on, for any scheme $X \to C$, we let

$$\mathcal{E}^0_X := \text{the trivial } \mathcal{G}\text{-bundle over } X.$$

Remark 2.2.9. — Given a base scheme S, an effective Cartier divisor $S_0 \subset S$, a group scheme H over S, and a closed subgroup $K \subset H \times_S S_0$, Mayeux–Richarz–Romagny define in [**MRR**] a scheme $\mathcal{H}$ over S which they call the *Néron blowup of H in K along S_0*. This construction is based on (a generalization of) the theory of dilatations. This scheme has the property that $\mathcal{H} \times_S S_0$ is an effective Cartier divisor on $\mathcal{H}$, and is universal for this property in an appropriate sense. In case H is a *flat* group scheme, $\mathcal{H}$ is again a group scheme. As explained in [**MRR**, Example 3.3], the group scheme $\mathcal{G}$ considered above can also be described as the Néron blowup of $G \times \mathbb{A}^1$ in B along the divisor $\{0\} \subset \mathbb{A}^1$.

We define two associated "loop" and "arc" groups as the following functors from $\mathbb{C}$-algebras to sets:

$$\mathcal{L}\mathcal{G} : R \mapsto \{(y, \gamma) \mid y \in C(R) \text{ and } \gamma \in \mathcal{G}(\widehat{\Gamma}^\circ_y)\},$$
$$\mathcal{L}^+\mathcal{G} : R \mapsto \{(y, \gamma) \mid y \in C(R) \text{ and } \gamma \in \mathcal{G}(\widehat{\Gamma}_y)\}.$$

Remark 2.2.10. — By definition, $\mathcal{L}\mathcal{G}$ and $\mathcal{L}^+\mathcal{G}$ admit canonical maps to C. Hence they can can also be described as functors from $\Gamma(C, \mathscr{O}_C)$-algebras to sets. In these terms, $\mathcal{L}^+\mathcal{G}$ sends a $\mathbb{C}[x]$-algebra R to the set of points of $\mathcal{G} \times C$ over the completion of $(C \times C) \times_C \mathrm{Spec}(R) = C \times \mathrm{Spec}(R)$ at the diagonal $(\Delta C) \times_C \mathrm{Spec}(R)$. (Here the fiber product $(C \times C) \times_C \mathrm{Spec}(R)$ is taken with respect to projection onto the second factor $C \times C \to C$.) The functor $\mathcal{L}\mathcal{G}$ admits a similar description. See [**He**, Example (2) on p. 504] or [**HR1**, §3.1] for this point of view.

For the proof of Lemma 2.2.12 below, we will need a family of "finite-type" variants of $\mathcal{L}^+\mathcal{G}$, defined as follows. For any integer $m \geq 1$ and any $y \in C(R)$, let $\Gamma_y^{(m)}$ denote the m-th nilpotent thickening of the closed subscheme $\Gamma_y \subset C_R$, i.e., $\Gamma_y^{(m)} = \mathrm{Spec}(R[x]/((x-y)^m))$, cf. §2.1.3. We then denote by $\mathcal{L}_m^+\mathcal{G}$ the functor from $\mathbb{C}$-algebras to sets defined by

$$\mathcal{L}_m^+\mathcal{G} : R \mapsto \left\{(y, \gamma) \mid y \in C(R) \text{ and } \gamma \in \mathcal{G}(\Gamma_y^{(m)})\right\}.$$

Lemma 2.2.11. — *The functor $\mathcal{L}_m^+\mathcal{G}$ is represented by a smooth affine group scheme of finite type over C. Moreover, if $m \geq l$, the natural homomorphism $\mathcal{L}_m^+\mathcal{G} \to \mathcal{L}_l^+\mathcal{G}$ is smooth and surjective, and the fiber of its kernel over any closed point $x \in C$ is unipotent.*

Proof. — As in Remark 2.2.10, one can also describe $\mathcal{L}_m^+\mathcal{G}$ as the functor from $\mathbb{C}[x]$-algebras to sets sending R to the set of points of $\mathcal{G} \times C$ over the m-th nilpotent thickening in $C \times \mathrm{Spec}(R)$ of the closed subscheme $(\Delta C) \times_C \mathrm{Spec}(R)$. Then, as explained in [**HR1**, Proof of Lemma 3.2], if we denote by $D^{(m)}$ the m-th nilpotent thickening of ΔC in $C \times C$, it is not difficult to check that $\mathcal{L}_m^+\mathcal{G}$ is the Weil restriction of scalars of $(\mathcal{G} \times C) \times_{C \times C} D^{(m)}$ along the map $D^{(m)} \to C$ induced by the second projection. By [**BLR**, §7.6, Theorem 4], this functor is represented by an affine scheme, and by [**BLR**, §7.6, Proposition 5], this scheme is smooth (and in particular of finite type). (Alternatively, smoothness can also be deduced from the fact that the morphism $\mathcal{L}_m^+\mathcal{G} \to \mathcal{L}_1^+\mathcal{G} = \mathcal{G}$ is smooth, which we prove below.)

Now we fix $m \geq l$, and consider the natural map $\mathcal{L}_m^+\mathcal{G} \to \mathcal{L}_l^+\mathcal{G}$. This map is surjective by the infinitesimal lifting criterion for smoothness (see [**SP**, Tag 02H6]), since $\mathcal{G}$ is smooth. It is also of finite type since $\mathcal{L}_m^+\mathcal{G}$ is of finite type. We will prove that this map is smooth in case $m \leq 2l$, which of course will be sufficient to conclude that it is smooth in general. By the same lifting criterion, to prove this property it suffices to prove that for any $\mathbb{C}$-algebra R and any ideal $I \subset R$ of square zero, given a commutative diagram

$$\begin{tikzcd} \mathcal{L}_m^+\mathcal{G} \arrow[d] & \mathrm{Spec}(R/I) \arrow[l] \arrow[d] \\ \mathcal{L}_l^+\mathcal{G} & \mathrm{Spec}(R) \arrow[l] \end{tikzcd}$$

there exists a map $\mathrm{Spec}(R) \to \mathcal{L}_m^+\mathcal{G}$ that makes both triangles commute. The lower arrow consists of a pair (y, γ) with $y \in C(R) = R$ and $\gamma \in \mathcal{G}(R[x]/(x-y)^l)$. Similarly, the upper arrow consists of a pair (y', γ') with $y' \in C(R/I) = R/I$ and

$\gamma' \in \mathcal{G}((R/I)[x]/(x-y')^m)$. The commutativity means that y' is the image of y and the images of γ and γ' in $\mathcal{G}((R/I)[x]/(x-y')^l)$ coincide; this common image will be denoted γ''. To conclude we have to prove that there exists $\tilde{\gamma} \in \mathcal{G}(R[x]/(x-y)^m)$ whose image in $\mathcal{G}(R[x]/(x-y)^l)$, resp. $\mathcal{G}((R/I)[x]/(x-y')^m)$, is γ, resp. γ'. Since $\mathcal{G}$ is smooth, hence formally smooth, the natural map

$$\mathcal{G}(R[x]/(x-y)^m) \to \mathcal{G}((R/I)[x]/(x-y')^l)$$

is surjective; choosing a preimage of γ'' and multiplying by its inverse, we can (and will) assume that γ'' is the image of the unit section $e \in \mathcal{G}(\mathbb{C}[x])$ of $\mathcal{G}$.

Consider the conormal module $\omega_{\mathcal{G}}$ associated with e in the sense of [**DG**, II, §4, no. 3]. This is a $\mathbb{C}[x]$-module, which by [**DG**, II, §4, Proposition in 3.4] and [**SP**, Tag 02G1] is finite locally free (in particular, projective). Note also that for any $\mathbb{C}[x]$-algebra S, the similar conormal module for the unit section of the group scheme

$$\mathrm{Spec}(S) \times_{\mathrm{Spec}(\mathbb{C}[x])} \mathcal{G}$$

is $S \otimes_{\mathbb{C}[x]} \omega_{\mathcal{G}}$ by [**DG**, I, §4, Lemme 1.6].

First we consider the case $S = R[x]/((x-y)^l)$, and the conormal module $R[x]/((x-y)^l) \otimes_{\mathbb{C}[x]} \omega_{\mathcal{G}}$. The ideal $I \cdot R[x]/((x-y)^l) \subset R[x]/((x-y)^l)$ is of square 0; in view of [**DG**, II, §4, Proposition in 3.2], we therefore have a canonical identification of

$$\begin{aligned}\mathrm{Hom}_{R[x]/((x-y)^l)}(R[x]/((x-y)^l) \otimes_{\mathbb{C}[x]} \omega_{\mathcal{G}}, I \cdot R[x]/((x-y)^l))& \\ \cong \mathrm{Hom}_{\mathbb{C}[x]}(\omega_{\mathcal{G}}, I \cdot R[x]/((x-y)^l))&\end{aligned}$$

with the kernel of the group morphism

$$\mathcal{G}\big(R[x]/((x-y)^l)\big) \to \mathcal{G}\big((R/I)[x]/((x-y')^l)\big).$$

In particular, γ belongs to this kernel, hence corresponds to a certain morphism $\phi_1 : \omega_{\mathcal{G}} \to I \cdot R[x]/((x-y)^l)$. We will denote by ϕ_1' the composition of ϕ_1 with the embedding $I \cdot R[x]/((x-y)^l) \hookrightarrow R[x]/((x-y)^l)$.

Next, consider the case $S = (R/I)[x]/((x-y')^m)$. The ideal $((x-y')^l) \subset (R/I)[x]/((x-y')^m)$ is of square 0 since $2l \geq m$; we therefore have a canonical identification of

$$\begin{aligned}\mathrm{Hom}_{(R/I)[x]/((x-y')^m)}((R/I)[x]/((x-y')^m) \otimes_{\mathbb{C}[x]} \omega_{\mathcal{G}}, ((x-y')^l))& \\ \cong \mathrm{Hom}_{\mathbb{C}[x]}(\omega_{\mathcal{G}}, ((x-y')^l))&\end{aligned}$$

with the kernel of the group morphism

$$\mathcal{G}\big((R/I)[x]/((x-y')^m)\big) \to \mathcal{G}\big((R/I)[x]/((x-y')^l)\big).$$

In particular, γ' belongs to this kernel, hence corresponds to a certain morphism $\phi_2 : \omega_{\mathcal{G}} \to ((x-y')^l)$. We will denote by ϕ_2' the composition of ϕ_2 with the *opposite* of the embedding $((x-y')^l) \hookrightarrow (R/I)[x]/((x-y')^m)$.

Finally we consider the case $S = R[x]/\big(((x-y)^m) + I \cdot (x-y)^l R[x]\big)$. The ideal $((x-y)^l) + (I) \subset R[x]/\big(((x-y)^m) + I \cdot (x-y)^l R[x]\big)$ is of square 0; we therefore have a canonical identification of $\mathrm{Hom}_{\mathbb{C}[x]}(\omega_{\mathcal{G}}, ((x-y)^l) + (I))$ with the kernel of the group morphism

$$\mathcal{G}\Big(R[x]/\big(((x-y)^m) + I \cdot (x-y)^l R[x]\big)\Big) \to \mathcal{G}\big((R/I)[x]/((x-y')^l)\big).$$

Now we have an exact sequence

$$0 \to R[x]/\big(((x-y)^m) + I \cdot (x-y)^l R[x]\big) \xrightarrow{f} \\ R[x]/((x-y)^l) \oplus (R/I)[x]/((x-y')^m) \xrightarrow{g} (R/I)[x]/((x-y')^m)$$

where f is induced by the obvious projections, and g by the maps considered in the definitions of ϕ_1' and ϕ_2'. (To check this, one can use that $R[x]/((x-y)^l)$ identifies with R^l via the coefficients on the polynomials $(x-y)^i$, $(R/I)[x]/((x-y')^m)$ identifies with $(R/I)^m$ via the coefficients on the polynomials $(x-y')^j$, and similar variants for the first and third terms in the sequence.) The morphisms ϕ_1' and ϕ_2' define a morphism of $\mathbb{C}[x]$-modules from $\omega_{\mathcal{G}}$ to the second term in the sequence, whose composition with g vanishes. By projectivity, this morphism factors through a morphism $\omega_{\mathcal{G}} \to R[x]/\big(((x-y)^m) + I \cdot (x-y)^l R[x]\big)$, which takes values in the ideal $((x-y)^l) + (I) \subset R[x]/\big(((x-y)^m) + I \cdot (x-y)^l R[x]\big)$. By the comments above this morphism defines a point

$$\tilde{\gamma}' \in \mathcal{G}\Big(R[x]/\big(((x-y)^m) + I \cdot (x-y)^l R[x]\big)\Big)$$

whose image in $\mathcal{G}(R[x]/((x-y)^l))$, resp. $(R/I)[x]/((x-y')^m)$, is γ, resp. γ'.

By formal smoothness the natural morphism

$$\mathcal{G}(R[x]/(x-y)^m) \to \mathcal{G}\Big(R[x]/\big(((x-y)^m) + I \cdot (x-y)^l R[x]\big)\Big)$$

is surjective; choosing a preimage $\tilde{\gamma}$ of $\tilde{\gamma}'$ provides the desired point.

Finally, to check that each fiber of the surjection $\mathcal{L}_m^+\mathcal{G} \to \mathcal{L}_l^+\mathcal{G}$ is unipotent one can use an appropriate embedding of $\mathcal{G}$ as a closed subscheme of some $\mathrm{GL}_{n,C}$. (The existence of such an embedding follows e.g. from the general result discussed in §10.1.3.) This reduces the proof to the case of $\mathrm{GL}_{n,C}$, which is well known and easy. ☐

Lemma 2.2.12. — *The functor $\mathcal{L}\mathcal{G}$ is represented by an ind-affine group ind-scheme over C, and the functor $\mathcal{L}^+\mathcal{G}$ is represented by a flat affine group scheme over C. Moreover, $\mathcal{L}\mathcal{G}$ is canonically identified with the functor from $\mathbb{C}$-algebras to sets defined by*

$$R \mapsto \left\{ (y, \mathcal{E}, \beta, \gamma) \;\middle|\; \begin{array}{c} y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma}_y, \\ \beta : \mathcal{E}_{|\widehat{\Gamma}_y^\circ} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma}_y^\circ} \text{ and } \gamma : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma}_y} \text{ trivializations} \end{array} \right\},$$

and $\mathcal{L}^+\mathcal{G}$ is canonically identified with the functor from $\mathbb{C}$-algebras to sets defined by

$$R \mapsto \left\{ (y, \mathcal{E}, \beta, \gamma) \;\middle|\; \begin{array}{c} y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma}_y, \\ \beta : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma}_y} \text{ and } \gamma : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma}_y} \text{ trivializations} \end{array} \right\}.$$

The group operation is given by formulas similar to those in (2.2.4).

Proof. — For the fact that $\mathcal{L}\mathcal{G}$ and $\mathcal{L}^+\mathcal{G}$ are represented by (ind-)schemes, see [**He**, Proposition 2], [**PZ**, §6.2.4], or [**HR1**, Lemma 3.2]. The claim that $\mathcal{L}^+\mathcal{G}$ is flat over C is also found in [**HR1**, Lemma 3.2]: it follows from the observation that

$$\mathcal{L}^+\mathcal{G} \cong \varprojlim_m \mathcal{L}^+_m\mathcal{G},$$

along with the fact from Lemma 2.2.11 that each $\mathcal{L}^+_m\mathcal{G}$ is flat.

The moduli descriptions of $\mathcal{L}\mathcal{G}$ and $\mathcal{L}^+\mathcal{G}$ given in the statement are routine, following the pattern of Lemma 2.2.4. □

Example 2.2.13. — In the spirit of Examples 2.2.1 and 2.2.8, let us describe the $\mathbb{C}$-points of $\mathcal{L}\mathcal{G}$ and $\mathcal{L}^+\mathcal{G}$ explicitly in the case where $G = \mathrm{GL}(n)$. Given a complex number $y \in C$, let $\mathscr{O}_y := \mathbb{C}[[x-y]]$ be the completion of the local ring of $\mathscr{O}(C)$ at y, and let $\mathscr{K}_y := \mathbb{C}((x-y))$ be its fraction field. A $\mathbb{C}$-point of $\mathcal{L}\mathcal{G}$ is a tuple $(y, g_1, \dots, g_n)$ with $y \in C$ and $g_1, \dots, g_n \in \mathrm{GL}(n, \mathscr{K}_y)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{K}_y^n \arrow[r, "\varepsilon_n^\vee(x)"] \arrow[d, "g_1"'] & \mathscr{K}_y^n \arrow[r, "\varepsilon_{n-1}^\vee(x)"] \arrow[d, "g_2"'] & \cdots \arrow[r, "\varepsilon_2^\vee(x)"] & \mathscr{K}_y^n \arrow[r, "\varepsilon_1^\vee(x)"] \arrow[d, "g_n"'] & \mathscr{K}_y^n \arrow[d, "g_1"] \\
\mathscr{K}_y^n \arrow[r, "\varepsilon_n^\vee(x)"] & \mathscr{K}_y^n \arrow[r, "\varepsilon_{n-1}^\vee(x)"] & \cdots \arrow[r, "\varepsilon_2^\vee(x)"] & \mathscr{K}_y^n \arrow[r, "\varepsilon_1^\vee(x)"] & \mathscr{K}_y^n.
\end{tikzcd}$$

Of course, since x is invertible in $\mathscr{K}_y$, g_1 alone determines the other maps. We see explicitly that $\mathcal{L}\mathcal{G} \cong \mathrm{GL}(n, \mathscr{K}) \times C$.

Similarly, a $\mathbb{C}$-point of $\mathcal{L}^+\mathcal{G}$ is a tuple $(y, g_1, \dots, g_n)$ with $y \in C$ and $g_1, \dots, g_n \in \mathrm{GL}(n, \mathscr{O}_y)$ such that a diagram similar to the one above (with $\mathscr{K}_y$ replaced by $\mathscr{O}_y$) commutes. If $y \neq 0$, then x is invertible in $\mathscr{O}_y$, and g_1 alone determines the other maps. Thus, the fiber of $\mathcal{L}^+\mathcal{G} \to C$ over any point $y \neq 0$ can be identified with $\mathrm{GL}(n, \mathscr{O})$. On the other hand, its fiber over $0 \in C$ is identified with the group I from Example 2.2.1.

The observations in the preceding example can be generalized to arbitrary G. First, we have

$$\mathcal{L}\mathcal{G} \cong \mathrm{L}G \times C.$$

Next, with respect to the structure map $\mathcal{L}^+\mathcal{G} \to C$, denote by

$$(\mathcal{L}^+\mathcal{G})_{\underline{0}}, \qquad \text{resp.} \qquad (\mathcal{L}^+\mathcal{G})_{|C^\circ},$$

the preimage of $\{0\}$, resp. C°. We then have canonical isomorphisms

$$(\mathcal{L}^+\mathcal{G})_{\underline{0}} \cong I, \qquad (\mathcal{L}^+\mathcal{G})_{|C^\circ} \cong \mathrm{L}^+G \times C^\circ.$$

2.2.3.2. *Central affine Grassmannian.* — We define the *central affine Grassmannian* (attached to $\mathcal{G}$) as the functor from $\mathbb{C}$-algebras to sets defined by:

$$\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}(R) = \left\{ (y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{c} y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma}_y, \\ \beta : \mathcal{E}_{|\widehat{\Gamma}^\circ_y} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma}^\circ_y} \text{ a trivialization} \end{array} \right\}. \tag{2.2.6}$$

Remark 2.2.14. — As in Remark 2.2.7, let us note that the definition of $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ considered above involves a $\mathcal{G}$-bundle on $\widehat{\Gamma}_y$. However, the Beauville–Laszlo descent theorem (in the form stated e.g. in [**Zh4**, Theorem 1.4.3]) implies that $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}(R)$ also classifies data $(y, \mathcal{E}', \beta')$ where $y \in C(R)$, $\mathcal{E}'$ is a $\mathcal{G}$-bundle on C_R, and $\beta' : \mathcal{E}'_{|C_R \smallsetminus \Gamma_y} \xrightarrow{\sim} \mathcal{E}^0_{C_R \smallsetminus \Gamma_y}$ is a trivialization.

The analogue of Proposition 2.2.2 in this setting is then the following claim.

Proposition 2.2.15. — *The functor $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ is represented by an ind-projective ind-scheme over C. The obvious morphism*

$$\mathcal{L}\mathcal{G} \to \mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$$

is an étale locally trivial principal $\mathcal{L}^+\mathcal{G}$-bundle (in the sense spelled out in §2.1.4.2); in particular it factors through an isomorphism

$$(\mathcal{L}\mathcal{G}/\mathcal{L}^+\mathcal{G})_{\mathrm{fppf}} \xrightarrow{\sim} \mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}.$$

In this statement $(\mathcal{L}\mathcal{G}/\mathcal{L}^+\mathcal{G})_{\mathrm{fppf}}$ is the fppf sheafification of the presheaf sending a $\mathbb{C}$-algebra R to $\mathcal{L}\mathcal{G}(R)/\mathcal{L}^+\mathcal{G}(R)$. As in Remark 2.2.3, Proposition 2.2.15 shows that $(\mathcal{L}\mathcal{G}/\mathcal{L}^+\mathcal{G})_{\mathrm{fppf}}$ is also the étale sheafification of the presheaf above, and that this functor is in fact an fpqc sheaf.

Proof. — By Proposition 10.1.9, there exists a vector bundle $\mathcal{E}$ over C and a closed immersion of group schemes $\mathcal{G} \hookrightarrow \mathrm{GL}(\mathcal{E})$ such that the quotient $\mathrm{GL}(\mathcal{E})/\mathcal{G}$ is quasi-affine over C. Here, since $C = \mathbb{A}^1$, this vector bundle must be trivial. One can therefore apply [**HR1**, Corollary 3.11] to obtain the representability of $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$. (See Remark 2.2.14 for the comparison between our conventions and those of [**HR1**]. See also [**He**, Proposition 2], [**Rc2**, Lemma 2.8] and [**PZ**, Proposition 6.5] for earlier references for this claim.) The fact that this ind-scheme is ind-projective is an application of [**Rc2**, Theorem 2.19]. For the second sentence, see [**He**, Proposition 4] or [**HR1**, Lemma 3.4(ii)]. (Once again, the key point here is to show that given $y \in C(R)$, for any $\mathcal{G}$-bundle $\mathcal{E}$ on $\widehat{\Gamma}_y$ there exists an étale cover $R \to R'$ such that the pull-back of $\mathcal{E}$ to $\widehat{\Gamma}_{y'}$ is trivial, where $y' \in C(R')$ is the image of y. This follows from Proposition 2.1.4.) □

As above, we denote by

$$(\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}})_{\underline{0}}, \quad \text{resp.} \quad (\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}})_{|C^\circ},$$

the preimage of $\{0\}$, resp. of C°, with respect to the structure map $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}} \to C$.

By construction, we have $\mathcal{G}_{|\operatorname{Spec}(\mathscr{O})} \cong \mathcal{I}$, so a principal $\mathcal{G}$-bundle over $\widehat{\Gamma}_0$ is the same as a principal $\mathcal{I}$-bundle. In view of this observation, and comparing (2.2.6) with (2.2.5), we obtain a canonical identification

$$(\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}})_{\underline{0}} \cong \mathrm{Fl}_G. \tag{2.2.7}$$

On the other hand, if $y \in C^\circ$, then $\mathcal{G}_{|\widehat{\Gamma}_y} \cong G \times \widehat{\Gamma}_y$. Using the (additive) group structure of $C \cong \mathbb{A}^1$, we see that for any $y \in C(R)$, there is a canonical automorphism of C_R (as an R-scheme) transforming y into $\{0\} \times \operatorname{Spec}(R)$. Via this automorphism, we can identify $\widehat{\Gamma}_y$ with $\mathrm{Spec}(R[[x]])$. Then any principal $\mathcal{G}$-bundle over $\widehat{\Gamma}_y$ can be regarded as a principal G-bundle over $\mathrm{Spec}(R[[x]])$. Combining these observations, we obtain a canonical isomorphism

$$(\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}})_{|C^\circ} \cong \mathrm{Gr}_G \times C^\circ. \tag{2.2.8}$$

Example 2.2.16. — Let us describe $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ explicitly in the case of $G = \mathrm{GL}(n)$. Let $y \in C$ be a complex number, and let $\mathscr{O}_y$ and $\mathscr{K}_y$ be as in Example 2.2.13. A *y-lattice* is a torsion-free finitely generated $\mathscr{O}_y$-submodule Λ of $\mathscr{K}_y^n$ such that $\mathscr{K}_y \cdot \Lambda = \mathscr{K}_y^n$. A *y-lattice chain* $\underline{\Lambda}$ is a sequence of y-lattices

$$\underline{\Lambda} = (\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset x\Lambda_1)$$

with the following property: there should exist some $\mathscr{K}_y$-basis $(f_1, \dots, f_n)$ of $\mathscr{K}_y^n$ such that for $1 \le k \le n$, Λ_k is given by

$$\Lambda_k = \text{the } \mathscr{O}_y\text{-span of } \{f_1, f_2, \dots, f_{n+1-k}, xf_{n+2-k}, xf_{n+3-k}, \dots, xf_n\}.$$

Of course, when $y \neq 0$, x is invertible in $\mathscr{O}_y$, and this definition implies that $\Lambda_1 = \Lambda_2 = \cdots = \Lambda_n$. When $y = 0$, on the other hand, our condition is equivalent to requiring that $\dim(\Lambda_i/\Lambda_{i+1}) = 1$, as in Example 2.2.1.

The group $(\mathcal{L}\mathcal{G})_y$ acts transitively on the set of y-lattice chains, and the group $(\mathcal{L}^+\mathcal{G})_y$ is the stabilizer of the y-lattice chain

$$\mathscr{O}_y^n \supset (x\mathscr{O}_y) \times \mathscr{O}_y^{n-1} \supset (x\mathscr{O}_y)^2 \times \mathscr{O}_y^{n-2} \supset \cdots \supset (x\mathscr{O}_y)^{n-1} \times \mathscr{O}_y \supset (x\mathscr{O}_y)^n.$$

We can thus identify $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}(\mathbb{C})$ with the set of pairs $(y, \underline{\Lambda})$ where $y \in C$ and $\underline{\Lambda}$ is a y-lattice chain.

2.2.4. Example: the first fundamental coweight for $\mathrm{GL}(n)$**.** — In view of the constructions studied in the rest of this book, it is an important problem to understand the "global Schubert varieties," defined as the closures of the locally closed subvarieties

$$\mathrm{Gr}^\lambda_G \times C^\circ \subset \mathrm{Gr}_G \times C^\circ \overset{(2.2.8)}{\cong} (\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}})_{|C^\circ} \subset \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$$

for $\lambda \in \mathbf{X}^\vee_+$. Unfortunately, in practice, describing these global Schubert varieties explicitly turns out to be difficult, even in the case when the stratum $\mathrm{Gr}^\lambda_G \subset \mathrm{Gr}_G$ is "simple" (e.g. when it is closed).

One case when such a description is possible, however, is the setting considered in Example 1.2.3, i.e. the case of the dominant coweight $\varepsilon_1^\vee$ for $\mathrm{GL}(n)$. We rely on the description of $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ for $\mathrm{GL}(n)$ from Example 2.2.16 as the set of pairs $(y, \underline{\Lambda})$, where $y \in C$ and $\underline{\Lambda}$ is a y-lattice chain. Let X be the closed subset given by

$$\begin{aligned} X = \{(y, \Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset x\Lambda_1) \in \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}} \mid &\text{ for each } k, \\ (x(x-y)\mathscr{O}_y)^{k-1} \times ((x-y)\mathscr{O}_y)^{n-k+1} \subset \Lambda_k \subset (x\mathscr{O}_y)^{k-1} &\times \mathscr{O}_y^{n-k+1} \\ \text{and } \dim(((x\mathscr{O}_y)^{k-1} \times \mathscr{O}_y^{n-k+1})/\Lambda_k) &= 1\}. \end{aligned}$$

Comparing this with the description of $\mathrm{Gr}^{\varepsilon_1^\vee}_{\mathrm{GL}(n)}$ from Example 1.2.3, we see that

$$X_{|C^\circ} = \mathrm{Gr}^{\varepsilon_1^\vee}_{\mathrm{GL}(n)} \times C^\circ.$$

We will see below that X is irreducible, so in fact $X = \overline{\mathrm{Gr}^{\varepsilon_1^\vee}_{\mathrm{GL}(n)} \times C^\circ}$.

For an alternative description of X, consider the canonical identification

$$((x\mathscr{O}_y)^{k-1} \times \mathscr{O}_y^{n-k+1})/((x(x-y)\mathscr{O}_y)^{k-1} \times ((x-y)\mathscr{O}_y)^{n-k+1}) \cong \mathbb{C}^n. \tag{2.2.9}$$

Under this identification, the inclusion map $(x\mathscr{O}_y)^k \times \mathscr{O}_y^{n-k} \hookrightarrow (x\mathscr{O}_y)^{k-1} \times \mathscr{O}_y^{n-k+1}$ induces a map of quotient spaces

$$\frac{(x\mathscr{O}_y)^k \times \mathscr{O}_y^{n-k}}{(x(x-y)\mathscr{O}_y)^k \times ((x-y)\mathscr{O}_y)^{n-k}} \to \frac{(x\mathscr{O}_y)^{k-1} \times \mathscr{O}_y^{n-k+1}}{(x(x-y)\mathscr{O}_y)^{k-1} \times ((x-y)\mathscr{O}_y)^{n-k+1}}$$

which is identified under (2.2.9) with

$$\varepsilon_k^\vee(y) : \mathbb{C}^n \to \mathbb{C}^n.$$

Now consider a point $(y, \Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_n \supset x\Lambda_1)$ of X. Each lattice $\Lambda_k \subset (x\mathscr{O}_y)^{k-1} \times \mathscr{O}_y^{n-k+1}$ corresponds under (2.2.9) to a hyperplane $H_k \subset \mathbb{C}^n$, and the condition $\Lambda_k \supset \Lambda_{k+1}$ is equivalent to the condition that $H_k \supset \varepsilon_k^\vee(y)(H_{k+1})$. To summarize, we have identified

$$\begin{aligned}X \cong \{(y, H_1, \ldots, H_n) \in C \times (\mathbb{P}^{n-1})^n \mid \\ \varepsilon_1^\vee(y)(H_2) \subset H_1,\ \ldots,\ \varepsilon_{n-1}^\vee(y)(H_n) \subset H_{n-1},\ \varepsilon_n^\vee(y)(H_1) \subset H_n\},\end{aligned} \tag{2.2.10}$$

where $\mathbb{P}^{n-1}$ is the projective space of hyperplanes in $\mathbb{C}^n$, with the natural action of $\mathrm{GL}(n)$. (See [**ARi**, §8.1] for additional discussion and references).

Let $f : X \to C$ be the restriction of the structure map $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}} \to C$. In terms of (2.2.10), the set $f^{-1}(0)$ corresponds to

$$\begin{aligned}\{(H_1, \ldots, H_n) \in (\mathbb{P}^{n-1})^n \mid \\ \varepsilon_1^\vee(0)(H_2) \subset H_1,\ \ldots,\ \varepsilon_{n-1}^\vee(0)(H_n) \subset H_{n-1},\ \varepsilon_n^\vee(0)(H_1) \subset H_n\}.\end{aligned} \tag{2.2.11}$$

On the other hand, $f^{-1}(0)$ is also identified with an I-stable subset of $(\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}})_{\underline{0}} = \mathrm{Fl}_{\mathrm{GL}(n)}$. Let us describe the I-action on $f^{-1}(0)$ in terms of (2.2.11). For any $i \in \{1, \ldots, n\}$, consider the group homomorphism

$$\varrho_i : I \to B$$

sending a matrix $g = (g_{i,j})_{i,j=1,\ldots,n}$ to the matrix

$$\left(\begin{array}{cccccc|ccccc}
\overline{g_{1,1}} & 0 & 0 & \cdots & 0 & & & & & & \\
0 & \overline{g_{2,2}} & 0 & \cdots & 0 & & & & & & \\
\vdots & & \ddots & & \vdots & & & & 0 & & \\
\vdots & & & \overline{g_{i-2,i-2}} & 0 & & & & & & \\
0 & \cdots & & 0 & \overline{g_{i-1,i-1}} & & & & & & \\
\hline
& & & & & & \overline{g_{i,i}} & 0 & 0 & \cdots & 0 \\
& & & & & & \overline{g_{i+1,i}} & \overline{g_{i+1,i+1}} & 0 & \cdots & 0 \\
& & 0 & & & & \vdots & & \ddots & & \vdots \\
& & & & & & \vdots & & & \overline{g_{n-1,n-1}} & 0 \\
& & & & & & \overline{g_{n,i}} & \cdots & & \overline{g_{n,n-1}} & \overline{g_{n,n}}
\end{array}\right),$$

where for P in $\mathscr{O}$ we write $\overline{P}$ for the image of P in $\mathscr{O}/x\mathscr{O} = \mathbb{C}$. We have a natural action of B^n of $(\mathbb{P}^{n-1})^n$. Under (2.2.10), the action of an element $g \in I$ on $f^{-1}(0)$ corresponds to the action of $(\varrho_1(g), \dots, \varrho_n(g)) \in B^n$ on (2.2.11).

This description shows that $f^{-1}(0)$ contains the unique 0-dimensional I-orbit in the connected component of $\mathrm{Fl}_{\mathrm{GL}(n)}$ containing $x^{\varepsilon_1^\vee} I$. In terms of (2.2.11), this orbit corresponds to the point

$$(\ker(e_1^*), \dots, \ker(e_n^*)) \in (\mathbb{P}^{n-1})^n, \tag{2.2.12}$$

where $(e_1, \dots, e_n)$ is the natural basis of $\mathbb{C}^n$ and $(e_1^*, \dots, e_n^*)$ is the dual basis.

To obtain a better understanding of the geometry of X around the special fiber $f^{-1}(0)$, one can introduce the morphism

$$u : \mathbb{A}^n \to X$$

defined as follows. Given $\underline{x} = (x_1, \dots, x_n) \in \mathbb{A}^n$ and $i \in \{1, \dots, n\}$, we define the hyperplane

$$\begin{aligned} H_i(\underline{x}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid (x_i \cdots x_n) z_1 + (x_i \cdots x_n x_1) z_2 + \cdots \\ + (x_i \cdots x_n x_1 \cdots x_{i-2}) z_{i-1} + z_i + x_i z_{i+1} + \cdots + (x_i \cdots x_{n-1}) z_n = 0\}. \end{aligned}$$

(In this equation, the first terms are omitted if $i = 1$, and the last terms are omitted if $i = n$.) Then we set

$$u(x_1, \dots, x_n) = \left(\prod_{i=1}^n x_i, H_1(x_1, \dots, x_n), \dots, H_n(x_1, \dots, x_n) \right).$$

One can check that u is an open embedding, see [**ARi**, Lemma 8.1]. Moreover, since its image contains the I-fixed point (2.2.12) (in fact this point is $u(0)$), which lies in the closure of all the I-orbits in its connected component of Fl_G, this image must meet all the I-orbits in $f^{-1}(0)$. In particular, since the singular locus and the irreducible components of X are I-stable, it follows that X is smooth and irreducible, see [**ARi**, Corollary 8.2].

2.3. Iterated global affine Grassmannians

2.3.1. Overview. — The next tool that we need to introduce, and which will play an important role in Chapter 3, is an ind-scheme known as the *Beĭlinson–Drinfeld affine Grassmannian.* This space, denoted by $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$, satisfies

$$(\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}})_{|C^\circ} \cong \mathrm{Gr}_G \times \mathrm{Fl}_G \times C^\circ, \qquad (\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}})_{\underline{0}} \cong \mathrm{Fl}_G,$$

where we use the same notational conventions as in §2.2.3. (The definition will be given below, as an instance of a more general construction. The idea of this construction goes back to Beĭlinson–Drinfeld [**BD**], but will be developed here for nonconstant group schemes over C.)

Unfortunately, we immediately run into a technical obstacle involving group actions. Convolution on Gr_G or on Fl_G typically involves equivariance for the groups L^+G and I. The global analogues of these groups are $\mathrm{L}^+G \times C$ and $\mathcal{L}^+\mathcal{G}$, respectively. Unfortunately, neither $\mathrm{L}^+G \times C$ nor $\mathcal{L}^+\mathcal{G}$ acts on $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$!

Below, we introduce a new group scheme, denoted by $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$, that acts on both $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ and $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$, as well as on further generalizations (called *iterated global affine Grassmannians*). This group scheme lets us define global twisted products and global convolution in great generality.

2.3.2. Beĭlinson–Drinfeld group schemes. — For y an R-point of C, we may consider the closed subscheme $\Gamma_0 \cup \Gamma_y \subset C_R$. Let $\widehat{\Gamma_0 \cup \Gamma_y}$ denote the completion of C_R along this subscheme, and set $(\widehat{\Gamma_0 \cup \Gamma_y})^\circ := \widehat{\Gamma_0 \cup \Gamma_y} \smallsetminus (\Gamma_0 \cup \Gamma_y)$. The *Beĭlinson–Drinfeld loop and arc groups* are the functors from $\mathbb{C}$-algebras to sets given by

$$\mathcal{L}\mathcal{G}^{\mathrm{BD}}(R) = \left\{ (y,\gamma) \,\middle|\, y \in C(R) \text{ and } \gamma \in \mathcal{G}\big((\widehat{\Gamma_0 \cup \Gamma_y})^\circ\big) \right\},$$
$$\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}(R) = \left\{ (y,\gamma) \,\middle|\, y \in C(R) \text{ and } \gamma \in \mathcal{G}\big(\widehat{\Gamma_0 \cup \Gamma_y}\big) \right\}.$$

Lemma 2.3.1. — *The functor $\mathcal{L}\mathcal{G}^{\mathrm{BD}}$ is represented by an ind-affine group ind-scheme over C, and the functor $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ is represented by a flat affine group scheme over C. Moreover, $\mathcal{L}\mathcal{G}^{\mathrm{BD}}$ is canonically identified with the functor*

$$R \mapsto \left\{ (y,\mathcal{E},\beta,\gamma) \,\middle|\, \begin{array}{c} y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma_0 \cup \Gamma_y}, \\ \beta : \mathcal{E}_{|(\widehat{\Gamma_0 \cup \Gamma_y})^\circ} \xrightarrow{\sim} \mathcal{E}^0_{(\widehat{\Gamma_0 \cup \Gamma_y})^\circ},\ \gamma : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_0 \cup \Gamma_y}} \text{ trivializations} \end{array} \right\},$$

and $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ is canonically identified with the functor

$$R \mapsto \left\{ (y,\mathcal{E},\beta,\gamma) \,\middle|\, \begin{array}{c} y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma_0 \cup \Gamma_y}, \\ \beta : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_0 \cup \Gamma_y}} \text{ and } \gamma : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_0 \cup \Gamma_y}} \text{ trivializations} \end{array} \right\}.$$

The group operation is given by formulas similar to those in (2.2.4). The proof is essentially identical to that of Lemma 2.2.12, and will be omitted. We remark that the proof of flatness for $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ involves writing it as an inverse limit of smooth (and hence finite type) group schemes over C. Specifically, for any integer $m \geq 1$, one can define a functor from $\mathbb{C}$-algebras to sets by

$$\mathcal{L}^+_m\mathcal{G}^{\mathrm{BD}}(R) = \left\{ (y,\gamma) \mid y \in C(R) \text{ and } \gamma \in \mathcal{G}\big((\Gamma_0 \cup \Gamma_y)^{(m)}\big) \right\},$$

where $(\Gamma_0 \cup \Gamma_y)^{(m)}$ is the m-th nilpotent thickening of the closed subscheme $\Gamma_0 \cup \Gamma_y \subset C_R$. These functors satisfy the following analogue of Lemma 2.2.11. Once again, the proof of this statement is analogous to that of Lemma 2.2.11, and is therefore omitted.

Lemma 2.3.2. — *The functor $\mathcal{L}^+_m\mathcal{G}^{\mathrm{BD}}$ is represented by a smooth group scheme of finite type over C. Moreover, if $m \geq l$, the natural homomorphism $\mathcal{L}^+_m\mathcal{G}^{\mathrm{BD}} \to \mathcal{L}^+_l\mathcal{G}^{\mathrm{BD}}$ is smooth and surjective, and the fiber of its kernel over any closed point $x \in C$ is unipotent.*

The flatness of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ then follows from the observation that

$$\mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \cong \varprojlim_m \mathcal{L}^+_m\mathcal{G}^{\mathrm{BD}}.$$

From the moduli descriptions in Lemma 2.3.1, we see that there are obvious C-group scheme homomorphisms

$$\mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \to I \times C, \tag{2.3.1}$$

$$\mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \to \mathcal{L}^+\mathcal{G} \tag{2.3.2}$$

given by restricting the bundle and its trivializations to $\widehat{\Gamma}_0$ and to $\widehat{\Gamma}_y$, respectively.

2.3.3. Beĭlinson–Drinfeld affine Grassmannian. — As for $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ (see (2.2.6)), one can now define the *Beĭlinson–Drinfeld affine Grassmannian* (attached to $\mathcal{G}$) as the functor from $\mathbb{C}$-algebras to sets given by

$$\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}(R) = \left\{ (y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{c} y \in C(R),\, \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle on } \widehat{\Gamma_0 \cup \Gamma}_y, \\ \beta : \mathcal{E}_{|\widehat{\Gamma_0\cup\Gamma_y}^\circ} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_0\cup\Gamma_y}^\circ} \text{ a trivialization} \end{array} \right\}. \tag{2.3.3}$$

Remark 2.3.3. — As in Remarks 2.2.7 and 2.2.14, we may again invoke the Beauville–Laszlo descent theorem to describe the functor $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$ in terms of principal $\mathcal{G}$-bundles over curves rather than formal neighborhoods. Specifically, $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}(R)$ also classifies data $(y, \mathcal{E}', \beta')$ where $y \in C(R)$, $\mathcal{E}'$ is a principal $\mathcal{G}$-bundle over C_R, and $\beta' : \mathcal{E}'_{|C_R \smallsetminus (\Gamma_0 \cup \Gamma_y)} \xrightarrow{\sim} \mathcal{E}^0_{C_R \smallsetminus (\Gamma_0 \cup \Gamma_y)}$ is a trivialization.

The next statement is the counterpart of Proposition 2.2.15 in our present setting.

Proposition 2.3.4. — *The functor $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$ is represented by an ind-scheme over C. The obvious morphism*

$$\mathcal{L}\mathcal{G}^{\mathrm{BD}} \to \mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$$

is an étale locally trivial principal $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-bundle; in particular, it factors through an isomorphism

$$(\mathcal{L}\mathcal{G}^{\mathrm{BD}}/\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{\mathrm{fppf}} \xrightarrow{\sim} \mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}.$$

Proof. — The proof is similar to that of Proposition 2.2.15. In view of Remark 2.3.3, the representability is proved in an application of [**HR1**, Corollary 3.10] (applied to the noetherian ring $\mathcal{O} = \mathbb{C}[x]$, the $\mathcal{O}$-curve $X = C \times C$, and the divisor $(\Delta C) \cup (\{0\} \times C)$). See also [**PZ**, p. 231]. For the second sentence, see [**HR1**, Lemma 3.4(ii)]. (As always, the key point is to show that given $y \in C(R)$, any principal $\mathcal{G}$-bundle over $\widehat{\Gamma_0 \cup \Gamma}_y$ can be trivialized on $\widehat{\Gamma_0 \cup \Gamma}_{y'}$ for some étale cover $R \to R'$, which follows from Proposition 2.1.4.) □

The following statement is claimed in various sources, but we were not able to locate a reference where it is actually proved. The argument below was kindly explained to us by T. Richarz.

Proposition 2.3.5. — *The ind-scheme $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$ is ind-proper over C.*

Proof. — Consider the functor $\mathbf{Gr}_{G_C}^{\mathrm{BD}}$ which is defined as for $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$, but replacing the group scheme $\mathcal{G}$ by the constant group scheme $G_C := G \times C$. (This functor, along with several variants, will be studied more thoroughly in Section 2.5 below.) This functor is representable by an ind-projective ind-scheme by [**HR1**, Corollary 3.11]. The natural morphism $\mathcal{G} \to G_C$ induces a morphism of ind-schemes

$$\mathbf{Gr}^{\mathrm{BD}}_{\mathcal{G}} \to \mathbf{Gr}^{\mathrm{BD}}_{G_C}.$$

We claim that this morphism is an étale locally trivial fibration with fiber the projective scheme G/B. This will conclude the proof, since properness is fpqc local on the base (see Remark 2.1.10(2)). By Proposition 2.2.15 and Proposition 2.3.4, combined with the observation that $\mathcal{L}\mathcal{G}^{\mathrm{BD}} \cong \mathcal{L}G^{\mathrm{BD}}_C$, we see that this morphism is étale locally isomorphic to the projection

$$(\mathcal{L}^+G^{\mathrm{BD}}_C/\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{\mathrm{fppf}} \times_C \mathbf{Gr}^{\mathrm{BD}}_{G_C} \to \mathbf{Gr}^{\mathrm{BD}}_{G_C},$$

where $\mathcal{L}^+G^{\mathrm{BD}}_C$ is the counterpart of $\mathcal{L}^+\mathcal{G}$ for the group scheme G_C. Next, using (2.3.1) and the analogous map $\mathcal{L}^+G^{\mathrm{BD}}_C \to \mathrm{L}^+G \times C$, it is easy to see that

$$(\mathcal{L}^+G^{\mathrm{BD}}_C/\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{\mathrm{fppf}} \cong \mathrm{L}^+G/I \times C \cong G/B \times C,$$

which shows the desired claim. □

We will see later (see Lemma 2.3.10) that this ind-scheme is in fact ind-projective over C.

2.3.4. Iterated global affine Grassmannians. — We will now define a family of "iterated" generalizations of $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ and $\mathbf{Gr}^{\mathrm{BD}}_{\mathcal{G}}$ that depend on some labels $S_1, \dots, S_n$. Each label S_i is allowed to be one of the following four symbols:

$$\varnothing, \qquad \underline{0}, \qquad \underline{\mathsf{y}}, \qquad \underline{\mathsf{y}}\underline{\cup}\underline{0}.$$

Let R be a $\mathbb{C}$-algebra, and let $y \in C(R)$. If S is one of the four symbols above, we define $\widehat{\Gamma}^\circ_S$ as follows:

$$\widehat{\Gamma}^\circ_S = \begin{cases} \widehat{\Gamma_0 \cup \Gamma_y} & \text{if } S = \varnothing, \\ \widehat{\Gamma_0 \cup \Gamma_y} \smallsetminus \Gamma_0 & \text{if } S = \underline{0}, \\ \widehat{\Gamma_0 \cup \Gamma_y} \smallsetminus \Gamma_y & \text{if } S = \underline{\mathsf{y}}, \\ \widehat{\Gamma_0 \cup \Gamma_y} \smallsetminus (\Gamma_0 \cup \Gamma_y) & \text{if } S = \underline{\mathsf{y}}\underline{\cup}\underline{0}. \end{cases}$$

Given a sequence of labels $\underline{S} = (S_1, \dots, S_n)$, we define $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$ to be the functor from $\mathbb{C}$-algebras to sets given by

$$(2.3.4) \quad \mathbf{Gr}_{\mathcal{G}}(\underline{S})(R) = \left\{ (y, \mathcal{E}^1, \dots, \mathcal{E}^n, \beta_1, \dots, \beta_n) \;\middle|\; \begin{array}{l} y \in C(R),\ \mathcal{E}^1, \dots, \mathcal{E}^n \text{ are principal } \mathcal{G}\text{-bundles over } \widehat{\Gamma_0 \cup \Gamma_y}, \\ \text{and for } 1 \le i \le n,\ \beta_i : \mathcal{E}^i_{|\widehat{\Gamma}^\circ_{S_i}} \xrightarrow{\sim} \mathcal{E}^{i-1}_{|\widehat{\Gamma}^\circ_{S_i}} \text{ is an isomorphism} \end{array} \right\}.$$

(When $i = 1$, in the last condition one should interpret $\mathcal{E}^0_{|\widehat{\Gamma}^\circ_{S_1}}$ as $\mathcal{E}^0_{\widehat{\Gamma}^\circ_{S_1}}$; in other words β_1 is a trivialization of $\mathcal{E}^1_{|\widehat{\Gamma}^\circ_{S_1}}$.) There is an action map

$$\mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{S})$$

given on R-points by the formula

$$(2.3.5)\quad (y,\mathcal{E}',\beta',\gamma')\cdot(y,\mathcal{E}^1,\dots,\mathcal{E}^n,\beta_1,\dots,\beta_n) \\ = (y,\mathcal{E}^1,\dots,\mathcal{E}^n,\beta'_{|\widehat{\Gamma}^\circ_{S_1}}\circ(\gamma')^{-1}_{|\widehat{\Gamma}^\circ_{S_1}}\circ\beta_1,\beta_2,\dots,\beta_n),$$

where we use the identification from Lemma 2.3.1. We will see below (see Proposition 2.3.11) that the functor $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$ is represented by an ind-proper ind-scheme of ind-finite type.

For $n=1$, the functors (2.3.4) are described by the following lemma.

Lemma 2.3.6. — *We have*

$$\mathbf{Gr}_{\mathcal{G}}(\varnothing)\cong C,\quad \mathbf{Gr}_{\mathcal{G}}(\underline{0})\cong \mathrm{Fl}_G\times C,\quad \mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}})\cong \mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}},\quad \mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}}\cup\underline{0})\cong \mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}.$$

In particular, these functors are represented by ind-proper ind-schemes over C.

Proof. — An R-point of $\mathbf{Gr}_{\mathcal{G}}(\varnothing)$ consists of an R-point y of C, a principal $\mathcal{G}$-bundle over $\widehat{\Gamma_0\cup\Gamma_y}$, and a trivialization of this $\mathcal{G}$-bundle. Up to isomorphism, for a fixed y there is only one such datum; so the first isomorphism is clear.

Next, there is an obvious map $\mathbf{Gr}_{\mathcal{G}}(\underline{0})\to \mathrm{Fl}_G\times C$ given by

$$(2.3.6)\qquad (y,\mathcal{E},\beta)\mapsto \big((\mathcal{E}_{|\widehat{\Gamma}_0},\beta_{|\widehat{\Gamma}^\circ_0}),y\big).$$

To show that this is an isomorphism, we must describe the inverse map. Given $(\mathcal{E},\beta)\in \mathrm{Fl}_G(R)$ and $y\in C(R)$, using the Beauville–Laszlo descent theorem (see [**Zh4**, Theorem 1.4.3]) one can obtain a principal $\mathcal{G}$-bundle $\overline{\mathcal{E}}$ by gluing $\mathcal{E}$ (on $\widehat{\Gamma}_0$) with the trivial principal bundle on C°_R using β; then the restriction $\tilde{\mathcal{E}}$ of $\overline{\mathcal{E}}$ to $\widehat{\Gamma_0\cup\Gamma_y}$ comes with an isomorphism $\tilde{\beta}:\tilde{\mathcal{E}}_{|\widehat{\Gamma_0\cup\Gamma_y}\smallsetminus\Gamma_0}\xrightarrow{\sim}\mathcal{E}^0_{\widehat{\Gamma_0\cup\Gamma_y}\smallsetminus\Gamma_0}$. It is easily checked that the map

$$((\mathcal{E},\beta),y)\mapsto (y,\tilde{\mathcal{E}},\tilde{\beta})$$

is inverse to (2.3.6).

The proof that $\mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}})\cong \mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ is very similar and will be omitted. Finally, the isomorphism $\mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}}\cup\underline{0})\cong\mathbf{Gr}_{\mathcal{G}}^{\mathrm{BD}}$ is clear from the definition. □

The same reasoning as for the first isomorphism in Lemma 2.3.6 shows that if $S_i=\varnothing$, then there is a canonical isomorphism

$$(2.3.7)\qquad \mathbf{Gr}_{\mathcal{G}}(S_1,\dots,S_{i-1},\varnothing,S_{i+1},\dots,S_n)\cong \mathbf{Gr}_{\mathcal{G}}(S_1,\dots,S_{i-1},S_{i+1},\dots,S_n).$$

Remark 2.3.7. — The second isomorphism in Lemma 2.3.6 shows that the $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-action on $\mathbf{Gr}_{\mathcal{G}}(\underline{0})$ factors through the map $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}\to I\times C$ from (2.3.1). Similarly, the $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-action on $\mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}})$ factors through the map $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}\to\mathcal{L}^+\mathcal{G}$ of (2.3.2).

There is an obvious "union" operation on the set of symbols $\varnothing,\underline{0},\underline{\mathrm{y}},\underline{\mathrm{y}}\cup\underline{0}$. For $1\le i\le j\le n$, we define the *convolution map* to be the map

$$(2.3.8)\qquad \mu_{i,j}:\mathbf{Gr}_{\mathcal{G}}(S_1,\dots,S_n)\to \mathbf{Gr}_{\mathcal{G}}(S_1,\dots,S_{i-1},S_i\cup\dots\cup S_j,S_{j+1},\dots,S_n)$$

that sends $(y,\mathcal{E}_1,\dots,\mathcal{E}_n,\beta_1,\dots,\beta_n)$ to the point

$$(y,\mathcal{E}_1,\dots,\mathcal{E}_{i-1},\mathcal{E}_j,\mathcal{E}_{j+1},\dots\mathcal{E}_n,\beta_1,\dots,\beta_{i-1},\beta'_j,\beta_{j+1},\dots,\beta_n)$$

where β'_j is given by the following formula (in which we set $S' = S_i \cup \cdots \cup S_j$):

$$\beta'_j = \beta_{j|\widehat{\Gamma}^\circ_{S'}} \circ \beta_{j-1|\widehat{\Gamma}^\circ_{S'}} \circ \cdots \circ \beta_{i|\widehat{\Gamma}^\circ_{S'}}.$$

Remark 2.3.8. — There are closely related functors $\mathbf{Gr}'_{\mathcal{G}}(S_1, \ldots, S_n)$ that are defined similarly, except that every mention of $\widehat{\Gamma_0 \cup \Gamma_y}$ is replaced by C_R. For instance, we have

$$\mathbf{Gr}'_{\mathcal{G}}(\underline{y})(R) = \left\{ (y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{c} y \in C(R),\, \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } C_R, \\ \beta : \mathcal{E}_{|C_R \smallsetminus \Gamma_y} \xrightarrow{\sim} \mathcal{E}^0_{C_R \smallsetminus \Gamma_y} \text{ a trivialization} \end{array} \right\}.$$

(It is left to the reader to write down the definition of $\mathbf{Gr}'_{\mathcal{G}}(S_1, \ldots, S_n)$ carefully in general.)

For any $\underline{S} = (S_1, \ldots, S_n)$, there is an isomorphism of functors

$$\mathbf{Gr}'_{\mathcal{G}}(\underline{S}) \xrightarrow{\sim} \mathbf{Gr}_{\mathcal{G}}(\underline{S}).$$

In fact, there is an obvious map $\mathbf{Gr}'_{\mathcal{G}}(\underline{S}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{S})$ given by restricting all $\mathcal{G}$-bundles and maps from C_R to $\widehat{\Gamma_0 \cup \Gamma_y}$. To construct a map in the opposite direction, suppose we have $(y, \mathcal{E}^1, \ldots, \mathcal{E}^n, \beta_1, \ldots, \beta_n) \in \mathbf{Gr}_{\mathcal{G}}(\underline{S})(R)$. Then each $\mathcal{E}^i$ is equipped with a trivialization over $(\widehat{\Gamma_0 \cup \Gamma_y})^\circ$ given by

$$\beta_{1|(\widehat{\Gamma_0 \cup \Gamma_y})^\circ} \circ \cdots \circ \beta_{i|(\widehat{\Gamma_0 \cup \Gamma_y})^\circ} : \mathcal{E}^i_{|(\widehat{\Gamma_0 \cup \Gamma_y})^\circ} \to \mathcal{E}^0_{(\widehat{\Gamma_0 \cup \Gamma_y})^\circ}.$$

Let $\tilde{\mathcal{E}}^i$ be the principal $\mathcal{G}$-bundle on C_R obtained by gluing $\mathcal{E}^i$ to the trivial bundle on $C_R \smallsetminus (\Gamma_0 \cup \Gamma_y)$ along $(\widehat{\Gamma_0 \cup \Gamma_y})^\circ$. (Once again, this gluing is possible thanks to the Beauville–Laszlo descent theorem [**Zh4**, Theorem 1.4.3].) Similarly, let $\tilde{\beta}_i$ be the map obtained by gluing β_i to the identity map of the trivial bundle along $(\widehat{\Gamma_0 \cup \Gamma_y})^\circ$. Then $(y, \tilde{\mathcal{E}}^1, \ldots, \tilde{\mathcal{E}}^n, \tilde{\beta}_1, \ldots, \tilde{\beta}_n)$ is an R-point of $\mathbf{Gr}'_{\mathcal{G}}(\underline{S})$, and our construction defines a map $\mathbf{Gr}_{\mathcal{G}}(\underline{S}) \to \mathbf{Gr}'_{\mathcal{G}}(\underline{S})$ that is inverse to the map mentioned above.

It might seem more appealing to work with the functors $\mathbf{Gr}'_{\mathcal{G}}(S_1, \ldots, S_n)$ rather than $\mathbf{Gr}_{\mathcal{G}}(S_1, \ldots, S_n)$. However, in the present book there is a compelling reason to stick to $\mathbf{Gr}_{\mathcal{G}}(S_1, \ldots, S_n)$: see Remark 2.3.15 below for details.

2.3.5. Principal bundles and representability. — The goal of this subsection is to prove that $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$ is represented by an ind-scheme. As a tool for the proof, we introduce a functor $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \ldots, S_n)$, defined as follows (here $S_1, \ldots, S_n$ are labels as in §2.3.4):

$$\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \ldots, S_n)(R) = \{(y, \mathcal{E}^1, \ldots, \mathcal{E}^n, \beta_1, \ldots, \beta_n, \gamma) \mid \\ \left. \begin{array}{c} y \in C(R),\, \mathcal{E}^1, \ldots, \mathcal{E}^n \text{ principal } \mathcal{G}\text{-bundles over } \widehat{\Gamma_0 \cup \Gamma_y}; \\ \text{for } 1 \le i \le n,\, \beta_i : \mathcal{E}^i_{|\widehat{\Gamma}^\circ_{S_i}} \xrightarrow{\sim} \mathcal{E}^{i-1}_{|\widehat{\Gamma}^\circ_{S_i}} \text{ an isomorphism}; \\ \gamma : \mathcal{E}^n \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_0 \cup \Gamma_y}} \text{ a trivialization} \end{array} \right\}.$$

(Here again, for $i = 1$, β_1 is understood to be a trivialization.) There is a right action of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ on $\mathbf{Gr}_{\mathcal{G}}^{(\infty)}(S_1, \dots, S_n)$ given by

$$(y, \mathcal{E}^1, \dots, \mathcal{E}^n, \beta_1, \dots, \beta_n, \gamma) \cdot (y, \mathcal{E}', \beta', \gamma') \\ = (y, \mathcal{E}^1, \dots, \mathcal{E}^n, \beta_1, \dots, \beta_n, \gamma' \circ (\beta')^{-1} \circ \gamma), \tag{2.3.9}$$

where $(y, \mathcal{E}', \beta', \gamma') \in \mathcal{L}^+\mathcal{G}^{\mathrm{BD}}(R)$ as in the description in Lemma 2.3.1. There is also a left action of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ on $\mathbf{Gr}_{\mathcal{G}}^{(\infty)}(S_1, \dots, S_n)$ given by a formula similar to (2.3.5).

There is an obvious map

$$p : \mathbf{Gr}_{\mathcal{G}}^{(\infty)}(S_1, \dots, S_n) \to \mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n) \tag{2.3.10}$$

which is given an R-points by forgetting γ. This map can be seen as a special case of a more general map

$$p_i : \mathbf{Gr}_{\mathcal{G}}^{(\infty)}(S_1, \dots, S_i) \times_C \mathbf{Gr}_{\mathcal{G}}(S_{i+1}, \dots, S_n) \to \mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n), \tag{2.3.11}$$

that is given by the following formula on R-points:

$$\big((y, \mathcal{E}^1, \dots, \mathcal{E}^i, \beta_1, \dots, \beta_i, \gamma), (y, \mathcal{E}^{i+1}, \dots, \mathcal{E}^n, \beta_{i+1}, \dots, \beta_n)\big) \\ \mapsto (y, \mathcal{E}^1, \dots, \mathcal{E}^n, \beta_1, \dots, \beta_i, \gamma^{-1}_{|\widehat{\Gamma}^{\circ}_{S_{i+1}}} \circ \beta_{i+1}, \beta_{i+2}, \dots, \beta_n).$$

We equip the domain of (2.3.11) with a right $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-action by combining the right action from (2.3.9) with the inverse of the left action from (2.3.5). That is, we set

$$(u, v) \cdot g = (ug, g^{-1}v), \qquad \begin{cases} u \in \mathbf{Gr}_{\mathcal{G}}^{(\infty)}(S_1, \dots, S_i)(R), \quad v \in \mathbf{Gr}_{\mathcal{G}}(S_{i+1}, \dots, S_n)(R), \\ g \in \mathcal{L}^+\mathcal{G}^{\mathrm{BD}}(R). \end{cases}$$

The map p_i in (2.3.11) is then $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-invariant, i.e., it intertwines this action with the trivial $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-action on $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$.

Lemma 2.3.9. — *The map p from* (2.3.10)*, resp. the map p_i from* (2.3.11)*, is an étale locally trivial torsor for the sheaf of groups $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \times_C \mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$.*

Proof. — We will prove the claim for (2.3.11). The proof for (2.3.10) is essentially identical. Let $\underline{S}' = (S_1, \dots, S_i)$ and $\underline{S}'' = (S_{i+1}, \dots, S_n)$, and let $\underline{S} = (S_1, \dots, S_n)$. It is straightforward to check that the map

$$\mathbf{Gr}_{\mathcal{G}}^{(\infty)}(\underline{S}') \times_C \mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'') \to \\ \big(\mathbf{Gr}_{\mathcal{G}}^{(\infty)}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')\big) \times_{\mathbf{Gr}_{\mathcal{G}}(\underline{S})} \big(\mathbf{Gr}_{\mathcal{G}}^{(\infty)}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')\big)$$

given by the action and projection morphisms (as in the definition of torsors in §2.1.2) is an isomorphism of functors, so what we must show is that p_i admits sections étale locally. Let $(y, \mathcal{E}^1, \dots, \mathcal{E}^n, \beta_1, \dots, \beta_n)$ be an R-point of $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$. For any ring homomorphism $R \to R'$, let $(y_{R'}, \mathcal{E}^1_{R'}, \dots \mathcal{E}^n_{R'}, \beta_{1,R'}, \dots, \beta_{n,R'})$ be the corresponding R'-point. By Proposition 2.1.4, we may choose $R \to R'$ so that it is an étale cover, and so that $\mathcal{E}^i_{R'}$ is trivial. That is, there exists an isomorphism

$$\gamma : \mathcal{E}^i_{R'} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_0 \cup \Gamma_{y_{R'}}}}.$$

Let

$$\beta'_{i+1,R'} := \gamma_{|\widehat{\Gamma}^{\circ}_{S_{i+1}}} \circ \beta_{i+1,R'} : \mathcal{E}^{i+1}_{|\widehat{\Gamma}^{\circ}_{S_{i+1}}} \xrightarrow{\sim} \mathcal{E}^{0}_{\widehat{\Gamma}^{\circ}_{S_i}}.$$

Then the point

$$\begin{aligned}\big((y_{R'}, \mathcal{E}^1_{R'}, \dots \mathcal{E}^i_{R'}, \beta_{1,R'}, \dots, \beta_{i,R'}, \gamma), (y_{R'}, \mathcal{E}^{i+1}_{R'}, \dots \mathcal{E}^n_{R'}, \beta'_{i+1,R'}, \beta_{i+2,R'}, \dots, \beta_{n,R'})\big)\\ \in (\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_i) \times_C \mathbf{Gr}_{\mathcal{G}}(S_{i+1}, \dots, S_n))(R)\end{aligned}$$

gives us the desired section of p_i. □

In the case of $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{\mathsf{y}} \cup \underline{0})$, the right action of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ extends to an action of $\mathcal{L}\mathcal{G}^{\mathrm{BD}}$: indeed, Lemma 2.3.1 implies that

$$\mathbf{Gr}^{(\infty)}(\underline{\mathsf{y}} \cup \underline{0}) \cong \mathcal{L}\mathcal{G}^{\mathrm{BD}}. \tag{2.3.12}$$

Lemma 2.3.10. — *For any $S \in \{\varnothing, \underline{0}, \underline{\mathsf{y}}, \underline{\mathsf{y}} \cup \underline{0}\}$, the ind-scheme $\mathbf{Gr}_{\mathcal{G}}(S)$ admits a presentation*

$$\mathbf{Gr}_{\mathcal{G}}(S) = \mathrm{colim}_m \mathbf{Gr}_{\mathcal{G}}(S)_m$$

indexed by $\mathbb{Z}_{\geq 0}$ such that the $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-action stabilizes each $\mathbf{Gr}_{\mathcal{G}}(S)_m$, and each of these schemes is projective and admits an $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-equivariant line bundle which is relatively ample for the structure morphism $\mathbf{Gr}_{\mathcal{G}}(S)_m \to C$.

Proof. — As in the proof of Proposition 2.2.15 there exists a closed immersion $\mathcal{G} \to \mathrm{GL}_{n,C}$ for some $n \geq 0$ such that the quotient $\mathrm{GL}_{n,C}/\mathcal{G}$ is quasi-affine, hence a morphism of ind-schemes $\mathbf{Gr}_{\mathcal{G}}(S) \to \mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)$ which is representable by a locally closed immersion, see [**HR1**, Proposition 3.10]. Here $\mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)$ is defined as for $\mathbf{Gr}_{\mathcal{G}}(S)$, but with the group scheme $\mathcal{G}$ replaced by $\mathrm{GL}_{n,C}$. This ind-scheme is studied in [**HR1**, Lemma 3.8]; the construction explained there shows that it admits a presentation

$$\mathbf{Gr}_{\mathrm{GL}_{n,C}}(S) = \mathrm{colim}_m \mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)_m$$

where each scheme $\mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)_m$ is a Quot scheme, and in particular admits a canonical closed immersion into a Grassmannian scheme. Pulling back the natural relatively ample line bundle on this Grassmannian scheme we obtain a relatively ample line bundle on $\mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)_m$. We then have a presentation

$$\mathbf{Gr}_{\mathcal{G}}(S) = \mathrm{colim}_m \mathbf{Gr}_{\mathcal{G}}(S)_m \text{ where } \mathbf{Gr}_{\mathcal{G}}(S)_m := \mathbf{Gr}_{\mathcal{G}}(S) \times_{\mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)} \mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)_m.$$

Since $\mathbf{Gr}_{\mathcal{G}}(S)$ is ind-proper over C (see Lemma 2.3.6), each scheme $\mathbf{Gr}_{\mathcal{G}}(S)_m$ is proper over C, which implies that the locally closed immersion

$$\mathbf{Gr}_{\mathcal{G}}(S)_m \to \mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)_m$$

is proper by [**SP**, Tag 01W6], hence a closed immersion by [**SP**, Tag 04XV]. Hence $\mathbf{Gr}_{\mathcal{G}}(S)_m$ is projective over C, and restricting the relatively ample line bundle on $\mathbf{Gr}_{\mathrm{GL}_{n,C}}(S)_m$ considered above we obtain a relatively ample line bundle on $\mathbf{Gr}_{\mathcal{G}}(S)_m$, which can be seen to be $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-equivariant. □

Proposition 2.3.11. — *The functor $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$ is represented by an ind-proper ind-scheme over C. More precisely, it admits a presentation*

$$\mathbf{Gr}_{\mathcal{G}}(\underline{S}) = \mathrm{colim}_m \mathbf{Gr}_{\mathcal{G}}(\underline{S})_m$$

indexed by integers $m \geq 0$, where each $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m$ is an $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-stable closed subscheme that is proper (in particular, of finite type) over C. Moreover, for any m the (left) $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-action on $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m$ factors through an action of the smooth group scheme of finite type $\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}$ for some $j_m \geq 1$.

Proof. — Write $\underline{S} = (S_1, \dots, S_n)$. We proceed by induction on n.

Suppose first that $n = 1$. The functors to consider are those listed in Lemma 2.3.6. If $S_1 = \varnothing$, then the claims about $\mathbf{Gr}_{\mathcal{G}}(\varnothing) \cong C$ are obvious. For the remaining cases, representability holds by Propositions 2.2.5, 2.2.15, and 2.3.4. The rest of the proposition just records some observations that can be made in the course of proving representability: see the references to [**HR1, He, PZ**] given in those proofs for further details.

Suppose now that $n > 1$, and assume that $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_{n-1})$ is known to be represented by an ind-scheme. As explained in §2.1.2, the torsor (see Lemma 2.3.9)

$$\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_{n-1}) \to \mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_{n-1}) \tag{2.3.13}$$

is automatically an étale locally trivial principal $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-bundle; in particular, $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_{n-1})$ is represented by an ind-scheme.

Recall from Lemma 2.3.1 that $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ is flat and affine (and hence quasi-compact) over C. Combining this with Lemma 2.3.10, we see that the assumptions of Remark 2.1.10(1) are satisfied; so by Proposition 2.1.9 the associated bundle

$$\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_{n-1}) \times_C^{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}} \mathbf{Gr}_{\mathcal{G}}(S_n)$$

exists as an ind-scheme. On the other hand, by another instance of Lemma 2.3.9, we have already identified the quotient of $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_{n-1}) \times_C \mathbf{Gr}_{\mathcal{G}}(S_n)$ by the right $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-action: it is $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$. We conclude that

$$\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n) \cong \mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_{n-1}) \times_C^{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}} \mathbf{Gr}_{\mathcal{G}}(S_n).$$

In particular, $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$ is represented by an ind-scheme. More precisely, let $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_{n-1})_m$ be the preimage of $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_{n-1})_m$ under (2.3.13), and then set

$$\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)_m := \mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_{n-1})_m \times_C^{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}} \mathbf{Gr}_{\mathcal{G}}(S_n)_m.$$

Proposition 2.1.9 tells us that $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$ is the colimit of the schemes $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)_m$ with closed immersions as transition maps. Since the schemes $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_{n-1})_m$ and $\mathbf{Gr}_{\mathcal{G}}(S_n)_m$ are each proper over C, the same holds for $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)_m$ by Remark 2.1.10(2). □

Remark 2.3.12. — Note that a morphism between ind-proper ind-schemes is automatically ind-proper by [**SP**, Tag 01W6]. In particular, the morphism $\mu_{i,j}$ of (2.3.8) is ind-proper.

In the course of the preceding proof, we have also established the following claim:

Proposition 2.3.13. — *The functor* $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{S})$ *is represented by an ind-scheme, and the map* $p : \mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{S}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{S})$ *makes* $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{S})$ *into a principal* $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$*-bundle over* $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$.

Finally, since the domain and codomain of (2.3.11) are now both known to be representable, we have the following immediate consequence of Lemma 2.3.9.

Corollary 2.3.14. — *Let* $\underline{S} = (S_1, \dots, S_n)$*. For any* $i \in \{1, \dots, n-1\}$*, the associated bundle* $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_i) \times_C^{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}} \mathbf{Gr}_{\mathcal{G}}(S_{i+1}, \dots, S_n)$ *exists as an ind-scheme, and we have*

$$\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n) \cong \mathbf{Gr}^{(\infty)}_{\mathcal{G}}(S_1, \dots, S_i) \times_C^{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}} \mathbf{Gr}_{\mathcal{G}}(S_{i+1}, \dots, S_n).$$

Remark 2.3.15. — One can write down a similar functor $\mathbf{Gr}^{(\infty)\prime}_{\mathcal{G}}(S_1, \dots, S_n)$ starting from the functor $\mathbf{Gr}'_{\mathcal{G}}(S_1, \dots, S_n)$ mentioned in Remark 2.3.8. An R-point of this functor would include the datum of a trivialization $\gamma : \mathcal{E}^n \to \mathcal{E}^0_{C_R}$. It is tempting to guess that this functor is a principal bundle for the group $\mathcal{L}^+_C\mathcal{G}$ given by

$$R \mapsto \left\{ (y, \mathcal{E}, \beta, \gamma) \;\middle|\; \begin{array}{c} y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } C_R, \\ \beta : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{C_R} \text{ and } \gamma : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{C_R} \text{ trivializations} \end{array} \right\}.$$

Unfortunately, $\mathcal{L}^+_C\mathcal{G}$ is only a group ind-scheme, and not a group scheme, over C. The literature does not seem to treat torsors and associated bundles for such objects, and thus it is unclear (at least to us) whether a version of Lemma 2.3.9 holds for $\mathbf{Gr}'_{\mathcal{G}}(S_1, \dots, S_n)$. Since this lemma is essential to all the results in this subsection, we do not know how to proceed with the study of $\mathbf{Gr}'_{\mathcal{G}}(S_1, \dots, S_n)$. In this book, we avoid this problem by working exclusively with $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$.

2.3.6. More principal bundles over $\mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$. — Occasionally, we will need variants of $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{S})$, denoted by

$$\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{S}) \qquad \mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathrm{y}}}(\underline{S}).$$

We define $\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(S_1, \dots, S_n)(R)$ to be the set of tuples $(y, \mathcal{E}^1, \dots, \mathcal{E}^n, \beta_1, \dots, \beta_n, \gamma)$ where $(y, \mathcal{E}^1, \dots, \mathcal{E}^n, \beta_1, \dots, \beta_n) \in \mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n)(R)$, and

$$\gamma : \mathcal{E}^n_{|\widehat{\Gamma}_0} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma}_0}$$

is a trivialization. The definition of $\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathrm{y}}}(S_1, \dots, S_n)(R)$ is the same, except that γ should be a trivialization

$$\gamma : \mathcal{E}^n_{|\widehat{\Gamma}_y} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma}_y}.$$

Again, there are obvious right actions of $I \times C$ on $\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{S})$ and of $\mathcal{L}^+\mathcal{G}$ on $\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathrm{y}}}(\underline{S})$, and projection maps

$$p_{\underline{0}} : \mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{S}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{S}),$$
$$p_{\underline{\mathrm{y}}} : \mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathrm{y}}}(\underline{S}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{S}).$$

By Lemmas 2.2.4 and 2.2.12, we have the following special cases, in the spirit of Lemma 2.3.6 and (2.3.12):

$$\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{0}) \cong \mathrm{L}G \times C, \qquad \mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathrm{y}}}(\underline{\mathrm{y}}) \cong \mathcal{L}\mathcal{G}. \tag{2.3.14}$$

The proof of the next statement closely follows the pattern of Proposition 2.3.13, and will be omitted.

Proposition 2.3.16. — 1. *The functor $\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{S})$ is represented by an ind-scheme over C, and the map $p_{\underline{0}} : \mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{S}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{S})$ is a principal $I \times C$-bundle over $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$.*
2. *The functor $\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathrm{y}}}(\underline{S})$ is represented by an ind-scheme over C, and the map $p_{\underline{\mathrm{y}}} : \mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathrm{y}}}(\underline{S}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{S})$ is a principal $\mathcal{L}^+\mathcal{G}$-bundle over $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$.*

Example 2.3.17. — Consider the morphism $\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{\mathrm{y}}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}})$. It is easily seen that the restriction of this morphism to C° identifies with the projection $\mathrm{Gr}_G \times I \times C^\circ \to \mathrm{Gr}_G \times C^\circ$, while its restriction to 0 identifies with the map $\mathrm{L}G \to \mathrm{Fl}_G$.

2.3.7. Special and generic fibers. — In the preceding subsections, we have introduced a number of ind-schemes Y equipped with maps $Y \to C$, including $\mathcal{L}\mathcal{G}$, $\mathcal{L}^+\mathcal{G}$, $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$, $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{S})$, etc. Following the conventions introduced in §2.2.3, for each such ind-scheme Y, we denote by

$$Y_{\underline{0}}, \qquad \text{resp.} \qquad Y_{|C^\circ},$$

the preimage of $\{0\}$, resp. C°, in Y. Our goal in this subsection is to determine the schemes $Y_{\underline{0}}$ (called "special fibers") and $Y_{|C^\circ}$ (called "generic fibers"). The results are summarized in Table 2.3.1. (Note that the first column of these table relies on identifications from Lemma 2.3.6 and from (2.3.12) and (2.3.14).)

Many of the entries in this table have been discussed in earlier sections. Specifically, the special and generic fibers for $\mathcal{L}\mathcal{G}$, $\mathcal{L}^+\mathcal{G}$, and $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ were described in §2.2.3. The special and generic fibers for $\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{0}) \cong \mathrm{L}G \times C$ and for $\mathbf{Gr}_{\mathcal{G}}(\underline{0}) \cong \mathrm{Fl}_G \times C$ are obvious.

Next, consider the ind-schemes $\mathcal{L}\mathcal{G}^{\mathrm{BD}}$, $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$, and $\mathbf{Gr}^{\mathrm{BD}}_{\mathcal{G}}$. It is immediate from Lemma 2.3.1 and from (2.3.3) that their special fibers are identified with those of $\mathcal{L}\mathcal{G}$, $\mathcal{L}^+\mathcal{G}$, and $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$, respectively. To describe their generic fibers, we use Lemma 2.1.6(2). This lemma implies that the generic fiber of, say, $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ is the product of the generic and special fibers of $\mathcal{L}^+\mathcal{G}$. Similar reasoning applies to $\mathcal{L}\mathcal{G}^{\mathrm{BD}}$ and $\mathbf{Gr}^{\mathrm{BD}}_{\mathcal{G}}$.

Ind-scheme Y		*Special fiber* $Y_{\underline{0}}$	*Generic fiber* $Y_{\mid C^\circ}$	
$\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{0})$	$\cong \mathrm{L}G \times C$	$\mathrm{L}G$	$\mathrm{L}G$	$\times C^\circ$
$\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathsf{y}}}(\underline{\mathsf{y}})$	$\cong \mathcal{L}\mathcal{G}$	$\mathrm{L}G$	$\mathrm{L}G$	$\times C^\circ$
	$\mathcal{L}^+\mathcal{G}$	I	L^+G	$\times C^\circ$
$\mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{\mathsf{y}}\cup\underline{0})$	$\cong \mathcal{L}\mathcal{G}^{\mathrm{BD}}$	$\mathrm{L}G$	$\mathrm{L}G \times \mathrm{L}G$	$\times C^\circ$
	$\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$	I	$\mathcal{L}^+\mathcal{G}_{\mid C^\circ} \underset{C^\circ}{\times} I \cong \mathrm{L}^+G \times I$	$\times C^\circ$
$\mathbf{Gr}_{\mathcal{G}}(\underline{0})$	$\cong \mathrm{Fl}_G \times C$	Fl_G	Fl_G	$\times C^\circ$
$\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})$	$\cong \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$	Fl_G	Gr_G	$\times C^\circ$
$\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}\cup\underline{0})$	$\cong \mathbf{Gr}^{\mathrm{BD}}_{\mathcal{G}}$	Fl_G	$\mathrm{Gr}_G \times \mathrm{Fl}_G$	$\times C^\circ$
$\mathbf{Gr}_{\mathcal{G}}(\underline{0},\dots,\underline{0})$		$\mathrm{L}G \overset{I}{\times} \cdots \overset{I}{\times} \mathrm{L}G \overset{I}{\times} \mathrm{Fl}_G$	$\mathrm{L}G \overset{I}{\times} \cdots \overset{I}{\times} \mathrm{L}G \overset{I}{\times} \mathrm{Fl}_G$	$\times C^\circ$
$\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}},\dots,\underline{\mathsf{y}})$		$\mathrm{L}G \overset{I}{\times} \cdots \overset{I}{\times} \mathrm{L}G \overset{I}{\times} \mathrm{Fl}_G$	$\mathrm{L}G \overset{\mathrm{L}^+G}{\times} \cdots \overset{\mathrm{L}^+G}{\times} \mathrm{L}G \overset{\mathrm{L}^+G}{\times} \mathrm{Gr}_G$	$\times C^\circ$
$\mathbf{Gr}_{\mathcal{G}}(\underline{S})$		$\mathrm{L}G \overset{I}{\times} \cdots \overset{I}{\times} \mathrm{L}G \overset{I}{\times} \mathrm{Fl}_G$	$\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}\cap\underline{S})_{\mid C^\circ} \underset{C^\circ}{\times} \mathbf{Gr}_{\mathcal{G}}(\underline{0}\cap\underline{S})_{\mid C^\circ}$	

TABLE 2.3.1. Special and generic fibers

It remains to describe the special and generic fibers of $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$ in general. The special fibers are given by the following lemma.

Lemma 2.3.18. — *There is a canonical isomorphism of ind-schemes*

$$\mathbf{Gr}_{\mathcal{G}}(S_1,\dots,S_n)_{\underline{0}} \cong \underbrace{\mathrm{L}G \times^I \cdots \times^I \mathrm{L}G \times^I \mathrm{Fl}_G}_{m\ \text{factors}},$$

where m is the number of labels S_i with $S_i \neq \varnothing$. Moreover, via this isomorphism, the convolution map

$$(\mu_{i,j})_{\underline{0}} : \mathbf{Gr}_{\mathcal{G}}(S_1,\dots,S_n)_{\underline{0}} \to \mathbf{Gr}_{\mathcal{G}}(S_1,\dots,S_{i-1},S_i\cup\cdots\cup S_j,S_{j+1},\dots,S_n)_{\underline{0}}$$

is identified with the usual convolution map for twisted products of Fl_G*.*

Proof. — In view of (2.3.7), we may assume that none of the S_i's are $\varnothing$, so that $m = n$. And since we are taking the fiber at 0, we may as well further assume that $S_1 = \cdots = S_n = \underline{0}$. From Lemma 2.2.4 and Proposition 2.2.5, the right-hand side represents the functor

$$R \mapsto \left\{(\mathcal{E}^1,\dots,\mathcal{E}^n,\beta_1,\dots,\beta_n) \;\middle|\; \begin{array}{c} \mathcal{E}^1,\dots,\mathcal{E}^n \text{ are principal } \mathcal{I}\text{-bundles, and} \\ \text{for } 1 \le i \le n,\ \beta_i : \mathcal{E}^i_{\mid \mathrm{Spec}(R((x)))} \xrightarrow{\sim} \mathcal{E}^{i-1}_{\mid \mathrm{Spec}(R((x)))} \text{ is an isomorphism} \end{array}\right\},$$

where, as usual, β_1 should be understood to be a trivialization. Since the restriction of $\mathcal{G}$ to $\mathrm{Spec}(\mathscr{O})$ is $\mathcal{I}$, this scheme coincides with $\mathbf{Gr}_{\mathcal{G}}(S_1,\dots,S_n) \times_C \{0\}$.

The final assertion is easy, and left to the reader. $\square$

Before studying the remaining generic fibers, we need some new notation. Note that there is an obvious "intersection" operation on the set of symbols $\varnothing, \underline{0}, \underline{\mathsf{y}}, \underline{\mathsf{y}}\cup\underline{0}$. If

$\underline{S} = (S_1, \dots, S_n)$, we write

$$\underline{\mathsf{y}} \cap \underline{S} = (\underline{\mathsf{y}} \cap S_0, \dots, \underline{\mathsf{y}} \cap S_n) \qquad \text{and} \qquad \underline{0} \cap \underline{S} = (\underline{0} \cap S_0, \dots, \underline{0} \cap S_n).$$

Proposition 2.3.19. — *Let $\underline{S} = (S_1, \dots, S_n)$ be a sequence of symbols among $\varnothing, \underline{0}, \underline{\mathsf{y}}, \underline{\mathsf{y}}\cup\underline{0}$.*

1. *There is a canonical isomorphism*

$$\nu : \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cap \underline{S})_{|C^\circ} \times_{C^\circ} \mathbf{Gr}_{\mathcal{G}}(\underline{0} \cap \underline{S})_{|C^\circ} \xrightarrow{\sim} \mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}.$$

Moreover, this isomorphism is compatible with convolution; that is, if we let $\underline{S}' = (S_1, \dots, S_{i-1}, S_i \cup \dots \cup S_j, S_{j+1}, \dots, S_n)$, then the following diagram commutes:

$$\begin{tikzcd} \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cap \underline{S})_{|C^\circ} \times_{C^\circ} \mathbf{Gr}_{\mathcal{G}}(\underline{0} \cap \underline{S})_{|C^\circ} \arrow[r, "\nu"] \arrow[d, "\mu_{i,j}"'] & \mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ} \arrow[d, "\mu_{i,j}"] \\ \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cap \underline{S}')_{|C^\circ} \times_{C^\circ} \mathbf{Gr}_{\mathcal{G}}(\underline{0} \cap \underline{S}')_{|C^\circ} \arrow[r, "\nu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{S}')_{|C^\circ}. \end{tikzcd} \tag{2.3.15}$$

2. *There is a canonical isomorphism*

$$\tilde{\nu} : \mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathsf{y}}}(\underline{\mathsf{y}} \cap \underline{S})_{|C^\circ} \times_{C^\circ} \mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{0} \cap \underline{S})_{|C^\circ} \xrightarrow{\sim} \mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{S})_{|C^\circ}.$$

Moreover, this isomorphism is compatible with the actions of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}_{|C^\circ} \cong \mathcal{L}^+\mathcal{G}_{|C^\circ} \times I$ on both sides.

Proof. — We establish the isomorphisms ν and $\tilde{\nu}$ simultaneously by induction on n. If $n = 1$, these claims are contained in the part of Table 2.3.1 that has already been established. Suppose now that $n > 1$, and set $\underline{S}' = (S_1, \dots, S_{n-1})$. By Corollary 2.3.14 and induction, we have

$$\begin{aligned} \mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ} &\cong \mathbf{Gr}^{(\infty)}_{\mathcal{G}}(\underline{S}')_{|C^\circ} \times_{C^\circ}^{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}_{|C^\circ}} \mathbf{Gr}_{\mathcal{G}}(S_n)_{|C^\circ} \\ &\cong \Big(\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{\mathsf{y}}}(\underline{\mathsf{y}} \cap \underline{S}')_{|C^\circ} \times_{C^\circ}^{\mathcal{L}^+\mathcal{G}_{|C^\circ}} \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cap S_n)_{|C^\circ}\Big) \\ &\qquad \times_{C^\circ} \Big(\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{0} \cap \underline{S}')_{|C^\circ} \times_{C^\circ}^{I \times C^\circ} \mathbf{Gr}_{\mathcal{G}}(\underline{0} \cap S_n)_{|C^\circ}\Big) \\ &\cong \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cap \underline{S})_{|C^\circ} \times_{C^\circ} \mathbf{Gr}_{\mathcal{G}}(\underline{0} \cap \underline{S})_{|C^\circ}. \end{aligned}$$

This is the desired isomorphism ν. The inductive step for $\tilde{\nu}$ is similar. The proof of compatibility with μ is routine and will be omitted. □

In view of Proposition 2.3.19 and (2.3.7), to finish the description of the generic fiber $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}$, we need only consider the case where $\underline{S}$ consists only of $\underline{0}$'s, or only of $\underline{\mathsf{y}}$'s. These cases are treated in the following lemma.

Lemma 2.3.20. — 1. *There is a canonical isomorphism of ind-schemes over C°*

$$\mathbf{Gr}_{\mathcal{G}}(\underline{0}, \dots, \underline{0})_{|C^\circ} \cong \big(\mathrm{L}G \times^I \cdots \times^I \mathrm{L}G \times^I \mathrm{Fl}_G\big) \times C^\circ,$$

where the number of factors in parentheses in the right-hand side is the number of symbols $\underline{0}$ in the left-hand side.

2. *There is a canonical isomorphism of ind-schemes over* C°

$$\mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}}, \dots, \underline{\mathrm{y}})_{|C^\circ} \cong \big(\mathrm{L}G \times^{\mathrm{L}^+G} \cdots \times^{\mathrm{L}^+G} \mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G\big) \times C^\circ,$$

where the number of factors in parentheses in the right-hand side is the number of symbols $\underline{\mathrm{y}}$ *in the left-hand side.*

Proof. — In each case, the proof is by induction on the number of factors, the base case being provided by previously established entries in Table 2.3.1. □

Finally, combining Proposition 2.3.19 and Lemma 2.3.20, for any $\underline{S}$ we obtain a canonical isomorphism

$$\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ} \cong \big(\underbrace{\mathrm{L}G \times^{\mathrm{L}^+G} \cdots \times^{\mathrm{L}^+G} \mathrm{Gr}_G}_{m \text{ factors}}\big) \times \big(\underbrace{\mathrm{L}G \times^{I} \cdots \times^{I} \mathrm{Fl}_G}_{n \text{ factors}}\big) \times C^\circ \tag{2.3.16}$$

where m is the number of symbols belonging to $\{\underline{\mathrm{y}}, \underline{\mathrm{y}}\cup\underline{0}\}$, and n is the number of symbols belonging to $\{\underline{0}, \underline{\mathrm{y}}\cup\underline{0}\}$. In subsequent sections, this isomorphism will typically be denoted by ν.

2.4. Nearby cycles on iterated global affine Grassmannians

2.4.1. Equivariant derived categories. — Given a sequence $\underline{S}$ and an integer $m \geq 1$, let j_m be such that the action of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ on $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m$ factors through $\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}$, in the notation of Proposition 2.3.11. Then the theory in Chapter 10 lets us define the equivariant derived category

$$D^{\mathrm{b}}_{\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m, \Bbbk).$$

This category is independent of the choice of j_m (up to canonical equivalence); indeed, for any $k \geq 0$, because the kernel of $\mathcal{L}^+_{j_{m+k}}\mathcal{G}^{\mathrm{BD}} \to \mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}$ is smooth with unipotent fibers (see Lemma 2.3.2), there is a canonical equivalence of categories

$$D^{\mathrm{b}}_{\mathcal{L}^+_{j_{m+k}}\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m, \Bbbk),$$

see Proposition 10.2.8. This justifies defining the "$\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-equivariant derived category" by

$$D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m, \Bbbk) := D^{\mathrm{b}}_{\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m, \Bbbk) \text{ for any sufficiently large } j_m,$$

even though $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-equivariance is not explicitly treated by the theory of Chapter 10.

Next, the closed embedding $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m \hookrightarrow \mathbf{Gr}_{\mathcal{G}}(\underline{S})_{m+1}$ gives rise to a fully faithful functor

$$D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{m+1}, \Bbbk).$$

Using these transition functors, we define

$$D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S}), \Bbbk) := \varinjlim_{m} D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m, \Bbbk).$$

In particular, by definition, any object of $D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S}), \Bbbk)$ is supported on some finite-type subscheme $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m$.

In the special case of $\mathbf{Gr}_{\mathcal{G}}(\underline{y}) = \mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$, the reasoning above can be carried out using $\mathcal{L}^+\mathcal{G}$ instead of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$. One then obtains the equivariant derived category

$$D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}}(\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}, \Bbbk).$$

2.4.2. $\mathbb{G}_{\mathrm{m}}$-actions. — Consider the natural action of $\mathbb{G}_{\mathrm{m}}$ on $C = \mathbb{A}^1_{\mathbb{C}}$. According to [**Zh1**, Lemma 5.4], there is an action

$$\mathbb{G}_{\mathrm{m}} \times \mathcal{G} \to \mathcal{G} \tag{2.4.1}$$

such that the structure map $\mathcal{G} \to C$ is $\mathbb{G}_{\mathrm{m}}$-equivariant. Over C°, $\mathbb{G}_{\mathrm{m}}$ acts on $\mathcal{G}_{|C^\circ} \cong G \times C^\circ$ by the natural action on the second factor. Thus, most of the work in constructing (2.4.1) goes into showing that the Iwahori group scheme $\mathcal{I}$ admits a suitable $\mathbb{G}_{\mathrm{m}}$-action, using Bruhat–Tits theory.

This action can also be constructed using the formalism of Néron blowups (see Remark 2.2.9) as follows. By compatibility of Néron blowups with base change (see [**MRR**, Theorem 3.2(6)]), the fiber product

$$\mathcal{G} \times_{\mathbb{A}^1} (\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1),$$

where the morphism $\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1 \to \mathbb{A}^1$ is the action morphism, is the Néron blowup of $(G \times \mathbb{A}^1) \times_{\mathbb{A}^1} (\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1)$ in $B \times (\mathbb{G}_{\mathrm{m}} \times \{0\})$ along $\mathbb{G}_{\mathrm{m}} \times \{0\}$. Now using the $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{A}^1$ we obtain an identification of $(G \times \mathbb{A}^1) \times_{\mathbb{A}^1} (\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1)$ with the similar fiber product where the morphism $\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1 \to \mathbb{A}^1$ is the projection. Again by compatibility of Néron blowups with base change, we deduce an isomorphism

$$\mathcal{G} \times_{\mathbb{A}^1} (\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1) \xrightarrow{\sim} \mathbb{G}_{\mathrm{m}} \times \mathcal{G}$$

as schemes over $\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1$. Composing the inverse isomorphism with the natural projection on $\mathcal{G}$ defines the desired action.

The action (2.4.1) gives rise to an action

$$\mathbb{G}_{\mathrm{m}} \times \mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \to \mathcal{L}^+\mathcal{G}^{\mathrm{BD}}.$$

Let us write this action down explicitly in the language of Lemma 2.3.1. Given $r \in \mathbb{G}_{\mathrm{m}}(R) = R^\times$, let $\rho_r : C_R \to C_R$ be the map given by multiplication by r. We also write ρ_r for the induced map on closed subschemes of C_R. If $\mathcal{E}$ is a $\mathcal{G}$-bundle on $\widehat{\Gamma_0 \cup \Gamma_y}$, then the pullback $\rho_r^*\mathcal{E}$ is a $\rho_r^*\mathcal{G}$-bundle on $\widehat{\Gamma_0 \cup \Gamma_{ry}}$. But the action map (2.4.1) gives rise to a canonical isomorphism $\rho_r^*\mathcal{G} \cong \mathcal{G}$, so $\rho_r^*\mathcal{E}$ can be regarded as a $\mathcal{G}$-bundle. The pullback along ρ_r of the trivial bundle is canonically identified with the trivial bundle, so the pullback of a trivialization is again a trivialization. To summarize, the action map $\mathbb{G}_{\mathrm{m}}(R) \times \mathcal{L}^+\mathcal{G}^{\mathrm{BD}}(R) \to \mathcal{L}^+\mathcal{G}^{\mathrm{BD}}(R)$ is given by

$$r \cdot (y, \mathcal{E}, \beta, \gamma) = (ry, \rho_r^*\mathcal{E}, \rho_r^*\beta, \rho_r^*\gamma).$$

By construction, the structure map $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \to C$ is $\mathbb{G}_{\mathrm{m}}$-equivariant.

The same construction gives rise to $\mathbb{G}_{\mathrm{m}}$-actions on many other (ind-)schemes over C, including $\mathcal{L}^+\mathcal{G}$, $\mathcal{L}\mathcal{G}^{\mathrm{BD}}$, $\mathcal{L}\mathcal{G}$, and $\mathbf{Gr}_{\mathcal{G}}(\underline{S})$ for any sequence $\underline{S}$. It can also be applied to the finite-type group schemes $\mathcal{L}^+_m\mathcal{G}^{\mathrm{BD}}$ and $\mathcal{L}^+_m\mathcal{G}$, and to the finite-type schemes $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m$.

Let m and j_m be such that the action of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ on $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m$ factors through $\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}$. Then Chapter 10 explains how to define the equivariant derived category

$D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \ltimes \mathcal{L}^+_{j_m} \mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m, \Bbbk)$. The procedure from §2.4.1 can be repeated with this additional $\mathbb{G}_{\mathrm{m}}$-action to define the category

$$D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \ltimes \mathcal{L}^+ \mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S}), \Bbbk).$$

In the special case of $\mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}}) = \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$, we similarly obtain the category

$$D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \ltimes \mathcal{L}^+ \mathcal{G}}(\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}, \Bbbk).$$

Taking the fiber at $0 \in C$, one obtains an action of $\mathbb{G}_{\mathrm{m}} \ltimes I$ on $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_0$, or in the case where $\underline{S} = (\underline{\mathrm{y}})$, an action of $\mathbb{G}_{\mathrm{m}} \ltimes I$ on Fl_G. The $\mathbb{G}_{\mathrm{m}}$-action in this case is commonly known as the "loop rotation" action. Following the discussion in Section 9.3, one may consider the categories of $\mathbb{G}_{\mathrm{m}}$-monodromic or unipotent $\mathbb{G}_{\mathrm{m}}$-monodromic I-equivariant complexes on Fl_G. In fact, it turns out that every I-equivariant complex is unipotent $\mathbb{G}_{\mathrm{m}}$-monodromic.

Lemma 2.4.1. — *Let $\mathbb{G}_{\mathrm{m}}$ act on Fl_G by loop rotation. For this action, we have*

$$D^{\mathrm{b}}_I(\mathrm{Fl}_G /\!\!/ \mathbb{G}_{\mathrm{m}}, \Bbbk) = D^{\mathrm{b}}_{I, \mathbb{G}_{\mathrm{m}}\text{-mon}}(\mathrm{Fl}_G, \Bbbk) = D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk).$$

Proof. — By definition, $D^{\mathrm{b}}_I(\mathrm{Fl}_G /\!\!/ \mathbb{G}_{\mathrm{m}}, \Bbbk)$ is a full triangulated subcategory of $D^{\mathrm{b}}_{I, \mathbb{G}_{\mathrm{m}}\text{-mon}}(\mathrm{Fl}_G, \Bbbk)$, which is in turn a full triangulated subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$. To prove the lemma, it is enough to show that $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ is generated by objects in $D^{\mathrm{b}}_I(\mathrm{Fl}_G /\!\!/ \mathbb{G}_{\mathrm{m}}, \Bbbk)$. Every I-equivariant perverse sheaf admits a filtration whose subquotients are intersection cohomology complexes associated to a constant sheaf on some I-orbit in Fl. (Recall that every I-equivariant local system on an orbit is constant, since each stabilizer is connected.) Every such orbit is stable under the loop rotation $\mathbb{G}_{\mathrm{m}}$-action, so such intersection cohomology complexes are $\mathbb{G}_{\mathrm{m}}$-equivariant. We conclude that every I-equivariant perverse sheaf lies in $D^{\mathrm{b}}_I(\mathrm{Fl}_G /\!\!/ \mathbb{G}_{\mathrm{m}}, \Bbbk)$, as desired. □

2.4.3. Twisted products and convolution. — Suppose $\underline{S} = (S_1, \dots, S_n)$, and let $\underline{S}' = (S_1, \dots, S_i)$ and $\underline{S}'' = (S_{i+1}, \dots, S_n)$. Let

$$\mathscr{F} \in D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S}'), \Bbbk) \qquad \text{and} \qquad \mathscr{G} \in D^{\mathrm{b}}_{\mathcal{L}^+ \mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S}''), \Bbbk).$$

We define

$$\mathscr{F} \overset{L}{\boxtimes}_C \mathscr{G} := p_1^* \mathscr{F} \overset{L}{\otimes} p_2^* \mathscr{G}[-1] \in D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}''), \Bbbk),$$

where

$$p_1 : \mathbf{Gr}_{\mathcal{G}}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'') \to \mathbf{Gr}_{\mathcal{G}}(\underline{S}'),$$
$$p_2 : \mathbf{Gr}_{\mathcal{G}}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'') \to \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')$$

are the projection maps.

Next, we wish to define an object $\mathscr{F} \widetilde{\boxtimes}_C \mathscr{G}$ in $D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S}), \Bbbk)$, called their *twisted external tensor product.* The definition involves the following diagram, where p and

q are the obvious maps:

$$\begin{aligned}(2.4.2)\quad \mathbf{Gr}_{\mathcal{G}}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'') &\xleftarrow{p} \mathbf{Gr}_{\mathcal{G}}^{(\infty)}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'') \\ &\xrightarrow{q} \mathbf{Gr}_{\mathcal{G}}^{(\infty)}(\underline{S}') \times_C^{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}} \mathbf{Gr}_{\mathcal{G}}(\underline{S}'') \xrightarrow[\sim]{\text{Cor. 2.3.14}} \mathbf{Gr}_{\mathcal{G}}(\underline{S}).\end{aligned}$$

Identify the last two spaces in this diagram. Informally, we would like to define $\mathscr{F} \mathbin{\widetilde{\boxtimes}_C} \mathscr{G}$ to be the object uniquely characterized by the property that there is an isomorphism

$$(2.4.3)\qquad p^*(\mathscr{F} \overset{L}{\boxtimes}_C \mathscr{G}) \cong q^*(\mathscr{F} \mathbin{\widetilde{\boxtimes}_C} \mathscr{G})$$

of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \times_C \mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-equivariant complexes on $\mathbf{Gr}_{\mathcal{G}}^{(\infty)}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')$. Unfortunately, this does not quite make sense: the latter scheme is not of finite type over C, so Chapter 10 does not treat equivariant sheaves on it.

The solution to this difficulty is to approximate $\mathbf{Gr}_{\mathcal{G}}^{(\infty)}(\underline{S}') \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')$ by a scheme of finite type. (The same difficulty and the same solution implicitly arose in §1.3.1.) Choose an integer $m \geq 0$ such that $\mathscr{F}$ is supported on $\mathbf{Gr}_{\mathcal{G}}(\underline{S}')_m$ and $\mathscr{G}$ is supported on $\mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_m$, and then choose an integer $j_m \geq 1$ such that the $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-action on $\mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_m$ factors through $\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}$. Define a functor $\mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}')$ by copying the definition of $\mathbf{Gr}_{\mathcal{G}}^{(\infty)}(\underline{S}')$ from §2.3.5, but requiring γ instead to be a trivialization of $\mathcal{E}^n$ over the j_m-th nilpotent thickening $(\Gamma_0 \cup \Gamma_y)^{(j_m)}$ of $\Gamma_0 \cup \Gamma_y$, rather than over its completion. It follows easily by the methods of §2.3.5 that

$$\mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}') \to \mathbf{Gr}_{\mathcal{G}}(\underline{S}')$$

is a principal $\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}$-bundle. Let $\mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}')_m$ be the preimage of $\mathbf{Gr}_{\mathcal{G}}(\underline{S}')_m$ under this map. Then $\mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}')_m \to \mathbf{Gr}_{\mathcal{G}}(\underline{S}')_m$ is also a principal $\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}$-bundle; in particular, $\mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}')_m$ is a scheme of finite type over C. We can now consider the diagram

$$\begin{aligned}(2.4.4)\quad \mathbf{Gr}_{\mathcal{G}}(\underline{S}')_m \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_m &\xleftarrow{p} \mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}')_m \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_m \\ &\xrightarrow{q} \mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}')_m \times_C^{\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}} \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_m \xrightarrow{\sim} \mathbf{Gr}_{\mathcal{G}}(\underline{S})_m.\end{aligned}$$

(Here, the last isomorphism holds by the same reasoning as in Corollary 2.3.14.) According to Proposition 10.2.6, the functor q^* gives rise to an equivalence of categories

$$\begin{aligned}q^* : D^{\mathrm{b}}_c\left(\mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}')_m \times_C^{\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}} \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_m, \Bbbk\right)& \\ \xrightarrow{\sim} D^{\mathrm{b}}_{\mathcal{L}^+_{j_m}\mathcal{G}^{\mathrm{BD}}}&\left(\mathbf{Gr}_{\mathcal{G}}^{(j_m)}(\underline{S}')_m \times_C \mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_m, \Bbbk\right).\end{aligned}$$

We now define $\mathscr{F} \mathbin{\widetilde{\boxtimes}_C} \mathscr{G} \in D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S}), \Bbbk)$ to be the unique object that is supported on $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_m$ and satisfying (2.4.3), but with p and q as in (2.4.4) rather than (2.4.2). A routine argument shows that this object is independent of the choice of m and j_m.

If $\mathscr{F}$ lives in $D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S}'), \Bbbk)$ instead, the same construction gives us

$$\mathscr{F} \mathbin{\widetilde{\boxtimes}}_C \mathscr{G} \in D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S}), \Bbbk).$$

Now suppose that $i = 1$ and $n = 2$, i.e., that $\underline{S}'$ and $\underline{S}''$ each consist of a single symbol, say $\underline{S}' = (S')$ and $\underline{S}'' = (S'')$. Then we have a convolution map

$$\mu = \mu_{1,2} : \mathbf{Gr}_{\mathcal{G}}(S', S'') \to \mathbf{Gr}_{\mathcal{G}}(S' \cup S''),$$

see (2.3.8). For $\mathscr{F}$ and $\mathscr{G}$ as above, we define their *convolution product* to be the object $\mathcal{F} \star_C \mathcal{G} \in D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(S' \cup S''), \Bbbk)$ given by

$$\mathscr{F} \star_C \mathscr{G} = \mu_*(\mathscr{F} \mathbin{\widetilde{\boxtimes}}_C \mathscr{G}).$$

As above, if $\mathscr{F}$ happens to belong to the $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-equivariant derived category, then $\mathscr{F} \star_C \mathscr{G}$ does as well.

If now $\underline{S}''' = (S''')$ is a third symbol, and if $\mathscr{H} \in D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}}(\mathbf{Gr}(S'''), \Bbbk)$, then there is a natural associativity isomorphism

$$\alpha = \alpha_{\mathscr{F},\mathscr{G},\mathscr{H}} : (\mathscr{F} \star_C \mathscr{G}) \star_C \mathscr{H} \xrightarrow{\sim} \mathscr{F} \star_C (\mathscr{G} \star_C \mathscr{H}).$$

This associativity map satisfies an appropriate version of the "pentagon axiom." Explicitly, α is constructed using the commutativity of the diagram

$$(2.4.5)\qquad \begin{tikzcd} & \mathbf{Gr}_{\mathcal{G}}(S', S'', S''') \arrow[dl, "\mu_{1,2}"'] \arrow[dr, "\mu_{2,3}"] \arrow[dd, "\mu_{1,3}"] & \\ \mathbf{Gr}_{\mathcal{G}}(S' \cup S'', S''') \arrow[dr, "\mu_{1,2}"'] & & \mathbf{Gr}_{\mathcal{G}}(S', S'' \cup S''') \arrow[dl, "\mu_{1,2}"] \\ & \mathbf{Gr}_{\mathcal{G}}(S' \cup S'' \cup S'''). & \end{tikzcd}$$

Finally, we remark that there are a few straightforward variants of the construction above that will be useful:

- one can work with $\mathbb{G}_{\mathrm{m}} \ltimes \mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-equivariant complexes instead of just $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-equivariant complexes;
- one can carry out these constructions over C° instead of C.

We will use obvious notations for these variants.

2.4.4. Nearby cycles. — Let $\underline{S} = (S_1, \dots, S_n)$, and consider the structure map $\mathbf{Gr}_{\mathcal{G}}(\underline{S}) \to C$. Using Lemma 2.3.18, we have a nearby cycles functor

$$\Psi = \Psi_{\underline{S}} : D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}, \Bbbk) \to D^{\mathrm{b}}_c(\mathrm{L}G \times^I \cdots \times^I \mathrm{L}G \times^I \mathrm{Fl}_G, \Bbbk),$$

where the number of factors on the right-hand side is as in Lemma 2.3.18. For any $\mathscr{F} \in D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}, \Bbbk)$, the object $\Psi_{\underline{S}}(\mathscr{F})$ is equipped with a natural "monodromy automorphism" (see §9.1.3):

$$\mathsf{m}_{\mathscr{F}} : \Psi_{\underline{S}}(\mathscr{F}) \to \Psi_{\underline{S}}(\mathscr{F}).$$

As explained in Section 10.3, this functor can be "upgraded" to the level of equivariant derived categories: since $(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{\underline{0}} \cong I$, we have functors

$$\Psi_{\underline{S}} : D^{\mathrm{b}}_{(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{L}G \times^I \cdots \times^I \mathrm{Fl}_G, \Bbbk),$$

$$\Psi_{\underline{S}} : D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \ltimes (\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}, \Bbbk) \to D^{\mathrm{b}}_{I,\mathbb{G}_{\mathrm{m}}\text{-mon}}(\mathrm{L}G \times^I \cdots \times^I \mathrm{Fl}_G, \Bbbk),$$

and $\mathsf{m}_{\mathscr{F}}$ can be considered as a morphism in the appropriate equivariant derived category. (In the second case, we use the notation from Section 9.3 for the $\mathbb{G}_{\mathrm{m}}$-action by loop rotation.) To simplify notation we will use the same notation for the equivariant and nonequivariant versions of these functors; it should always be clear which version is used.

Since nearby cycles commute with proper push-forward (see Proposition 9.1.4(1)), in view of Remark 2.3.12 we have a canonical isomorphism

$$(\mu_{i,j})_{\underline{0}*} \circ \Psi_{(S_1,\dots,S_n)} \cong \Psi_{(S_1,\dots,S_{i-1},S_i\cup\cdots\cup S_j,S_{j+1},\dots,S_n)} \circ (\mu_{i,j\,|C^\circ})_*. \tag{2.4.6}$$

Moreover, the monodromy automorphisms on the left- and right-hand sides are identified via this isomorphism:

$$(\mu_{i,j})_{\underline{0}*}\mathsf{m}_{\mathscr{F}} = \mathsf{m}_{(\mu_{i,j\,|C^\circ})_*\mathscr{F}}. \tag{2.4.7}$$

We will use these observations repeatedly in the sequel.

Lemma 2.4.2. — *Let $\underline{S}' = (S_1,\dots,S_i)$ and $\underline{S}'' = (S_{i+1},\dots,S_n)$. Let*

$$\mathscr{F} \in D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S}')_{|C^\circ},\Bbbk) \quad \textit{resp.} \quad D^{\mathrm{b}}_{(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S}')_{|C^\circ},\Bbbk)$$

and

$$\mathscr{G} \in D^{\mathrm{b}}_{(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_{|C^\circ},\Bbbk).$$

Then, setting $\underline{S} = (S_1,\dots,S_n)$, there is a natural isomorphism

$$\Psi_{\underline{S}}(\mathscr{F}\,\widetilde{\boxtimes}_{C^\circ}\,\mathscr{G}) \cong \Psi_{\underline{S}'}(\mathscr{F})\,\widetilde{\boxtimes}\,\Psi_{\underline{S}''}(\mathscr{G})$$

in $D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{\underline{0}})$, resp. in $D^{\mathrm{b}}_I(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{\underline{0}})$. Moreover, via this isomorphism, we have

$$\mathsf{m}_{\mathscr{F}\widetilde{\boxtimes}_{C^\circ}\mathscr{G}} = \mathsf{m}_{\mathscr{F}}\,\widetilde{\boxtimes}\,\mathsf{m}_{\mathscr{G}}.$$

Proof. — Let $m \geq 0$ and $j_m \geq 1$ be integers such that $\mathscr{F}\widetilde{\boxtimes}_{C^\circ}\mathscr{G}$ can be computed using (the restriction to C° of) the diagram (2.4.4). Since the nearby cycles construction commutes with external tensor product (see Proposition 9.1.6(3)), we have a natural isomorphism

$$\Psi_{\mathbf{Gr}^{(j_m)}_{\mathcal{G}}(\underline{S}')_m\times_C\mathbf{Gr}_{\mathcal{G}}(\underline{S}'')_m}\big(\widetilde{\mathscr{F}}\overset{L}{\boxtimes}_{\Bbbk}\mathscr{G}\big) \cong \Psi_{\mathbf{Gr}^{(j_m)}_{\mathcal{G}}(\underline{S}')_m}(\widetilde{\mathscr{F}})\overset{L}{\boxtimes}_{\Bbbk}\Psi_{\underline{S}''}(\mathscr{G}),$$

where $\widetilde{\mathscr{F}}$ is the pullback of $\mathscr{F}$ to $\mathbf{Gr}^{(j_m)}_{\mathcal{G}}(\underline{S}')_{m|C^\circ}$. Moreover, the monodromy automorphism of the left-hand side is the external tensor product of the monodromy automorphisms on the right-hand side. The result then follows from compatibility of the nearby cycles functor with smooth pullback (see Proposition 9.1.4(2)) and the fact that the maps p and q in (2.4.4) are smooth since they are principal bundles under smooth groups. □

Combining Lemma 2.4.2 with (2.4.6) and (2.4.7), we obtain the following result.

Corollary 2.4.3. — *If*

$$\mathscr{F} \in D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(S_1)_{|C^\circ},\Bbbk) \quad \textit{resp.} \quad D^{\mathrm{b}}_{(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(S_1)_{|C^\circ},\Bbbk)$$

and

$$\mathscr{G} \in D^{\mathrm{b}}_{(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(S_2)_{|C^\circ}, \Bbbk),$$

there is a natural isomorphism

$$\Psi_{(S_1 \cup S_2)}(\mathscr{F} \star_{C^\circ} \mathscr{G}) \cong \Psi_{S_1}(\mathscr{F}) \star^I \Psi_{S_2}(\mathscr{G})$$

in $D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(S_1 \cup S_2)_{\underline{0}})$, *resp. in* $D^{\mathrm{b}}_I(\mathbf{Gr}_{\mathcal{G}}(S_1 \cup S_2)_{\underline{0}})$. *Moreover, via this isomorphism, we have*

$$\mathsf{m}_{\mathscr{F} \star_{C^\circ} \mathscr{G}} = \mathsf{m}_{\mathscr{F}} \star^I \mathsf{m}_{\mathscr{G}}.$$

2.4.5. The functor Z. — Consider the special case of the construction from §2.4.4 when $\underline{S} = \underline{\mathsf{y}}$. In this case, since the action of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ factors through an action of $\mathcal{L}^+\mathcal{G}$, one can also consider $\Psi_{\underline{\mathsf{y}}}$ as a functor

$$\Psi_{\underline{\mathsf{y}}} : D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})_{|C^\circ}, \Bbbk) \to D^{\mathrm{b}}_I(\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})_{\underline{0}}, \Bbbk),$$

or in other words (using the identifications (2.2.7) and (2.2.8), together with Lemma 2.3.6), as a functor

$$\Psi_{\underline{\mathsf{y}}} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G \times C^\circ, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk).$$

We can now define the main player of this book, namely the functor

$$\mathsf{Z} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$$

defined by setting

$$\mathsf{Z}(\mathscr{A}) = \Psi_{\underline{\mathsf{y}}}(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]).$$

The functor Z is called the *central sheaf functor*. Perverse sheaves of the form $\mathsf{Z}(\mathscr{A})$, where $\mathscr{A} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, are called *central sheaves*. This terminology will be justified in Chapter 3.

Remark 2.4.4. — 1. The idea of constructing "central sheaves" on Fl_G using nearby cycles seems to be due to Beĭlinson (inspired by a similar construction in the framework of Shimura varieties suggested by Haines and Kottwitz), and was first realized concretely by Gaitsgory, see [**G1**]. However, the ind-scheme used by Gaitsgory is *not* the same as the one considered above: specifically, over a point of C°, the fiber of Gaitsgory's ind-scheme is $\mathrm{Gr}_G \times G/B$, while the fiber of $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ is just Gr_G (see (2.2.8)). The idea of using a nonconstant group scheme over C in this construction is suggested in [**He**, Remark 3], and used concretely in [**PZ**] and [**Zh1**]. It seems reasonable to expect that $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ is isomorphic to the closure of $\mathrm{Gr}_G \times C^\circ$ in Gaitsgory's ind-scheme, but no proof of this claim appears in the literature, as far as we know.

See Remark 3.2.4 below for an explanation of why we prefer working with $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$ rather than with Gaitsgory's ind-scheme.

2. Of course, it is not necessary to consider equivariant derived categories when defining Z: in the nonequivariant setting, the same construction yields a functor

$$\mathsf{Z} : D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk). \tag{2.4.8}$$

(Once again, for simplicity we will not distinguish the equivariant and nonequivariant versions in the notation.) Indeed, the papers [**G1, Zh1**] do not mention the equivariant version. This is not so problematic for these authors, since they are mainly interested in perverse sheaves, and one can still prove without considering $\mathcal{L}^+\mathcal{G}$-equivariant categories that Z sends L^+G-equivariant perverse sheaves to I-equivariant perverse sheaves: see [**G1**, Proposition 4] and [**Zh1**, Lemma 7.2]. However this creates some technical complications in some proofs, which we want to avoid. Considering the equivariant version will also allow us to state (and prove) properties of the complexes $\mathsf{Z}(\mathscr{A})$ similar to those obtained in [**G1, Zh1**], but now for *any* $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, and not only for perverse sheaves.

Lemma 2.4.5. — *The functor Z is t-exact for the perverse t-structures.*

Proof. — The functor $\mathscr{F} \mapsto \mathscr{F} \boxtimes^L_\Bbbk \underline{\Bbbk}_{C^\circ}[1]$ is t-exact (as a pullback under a smooth morphism), as is the functor $\Psi_{\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}}$ (see Theorem 9.1.3(2)). Therefore their composition, namely Z, is t-exact. □

As in §2.4.4, we have a monodromy operator

$$\mathsf{m}_{\mathscr{A}} : \mathsf{Z}(\mathscr{A}) \to \mathsf{Z}(\mathscr{A})$$

for any $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. The proof of the following claim is based on some constructions recalled in Section 9.3.

Proposition 2.4.6. —
1. *For any $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, the automorphism $\mathsf{m}_{\mathscr{A}}$ is unipotent.*
2. *For any $\mathscr{A}, \mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and any $f \in \mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)}(\mathsf{Z}(\mathscr{A}), \mathsf{Z}(\mathscr{B}))$ we have $f \circ \mathsf{m}_{\mathscr{A}} = \mathsf{m}_{\mathscr{B}} \circ f$.*

Proof. — (1) Using the constructions of §2.4.2 we can apply Proposition 10.3.1, which shows that the complex $\mathsf{Z}(\mathscr{A})$ is monodromic, and moreover that we have $\mathsf{m}_{\mathscr{A}} = \mu_{\mathsf{Z}(\mathscr{A})}(-1)$. Now by Lemma 2.4.1 and Corollary 9.3.5 the automorphism $\mu_{\mathsf{Z}(\mathscr{A})}(-1)$ is unipotent, and (1) follows.

(2) The claim follows from similar considerations, using the fact that monodromy in the sense of Section 9.3 commutes with all morphisms, see Proposition 9.3.2. □

Remark 2.4.7. —
1. An alternative proof of Proposition 2.4.6(1) in case $\mathscr{A}$ is perverse, based on different considerations, will be given in §4.6.2 below.
2. Using the notation of §9.2.1, Proposition 2.4.6(1) says in particular that for $\mathscr{A}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ we have $\Psi_{\underline{\mathsf{y}}}(\mathscr{A} \boxtimes^L_\Bbbk \underline{\Bbbk}_{C^\circ}[1]) = \Psi^{\mathrm{un}}_{\underline{\mathsf{y}}}(\mathscr{A} \boxtimes^L_\Bbbk \underline{\Bbbk}_{C^\circ}[1])$.
3. In this statement it is important to restrict to L^+G-equivariant complexes. We do not know if these properties hold for the functor (2.4.8).

2.5. Global affine Grassmannians and nearby cycles for G_C

All the work carried out in Sections 2.2, 2.3, and 2.4 can be repeated with the group scheme $\mathcal{G}$ replaced by the constant group scheme

$$G_C := G \times C.$$

Roughly, this has the effect of making the fiber at 0 look like all other fibers. In the present section we explain more precisely how this affects the previous constructions. The proofs of all the statements we make are similar to (and in general simpler than) their counterparts for $\mathcal{G}$, and will therefore be omitted.

2.5.1. Global group scheme and global affine Grassmannian. — First, as in §2.2.3, one can define global loop and arc groups $\mathcal{L}G_C$ and $\mathcal{L}^+G_C$ associated with G_C. In this case, using the additive structure on $C = \mathbb{A}^1$ and the fact that our group scheme is constant we see that we have canonical identifications

$$\mathcal{L}G_C \cong \mathrm{L}G \times C, \quad \mathcal{L}^+G_C \cong \mathrm{L}^+G \times C.$$

Then we have a central affine Grassmannian $\mathbf{Gr}^{\mathrm{Cen}}_{G_C}$, which identifies with the fppf sheafification of the quotient $\mathcal{L}G_C/\mathcal{L}^+G_C$, and an identification

$$\mathbf{Gr}^{\mathrm{Cen}}_{G_C} \cong \mathrm{Gr}_G \times C. \tag{2.5.1}$$

There is a natural proper morphism

$$\varpi : \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}} \to \mathbf{Gr}^{\mathrm{Cen}}_{G_C} \tag{2.5.2}$$

which sends a triple $(y, \mathcal{E}, \beta)$ to the triple $(y, \widetilde{\mathcal{E}}, \widetilde{\beta})$ where $\widetilde{\mathcal{E}}$ is the G_C-bundle induced from $\mathcal{E}$ via the canonical morphism $\mathcal{G} \to G_C$, and $\widetilde{\beta}$ is the trivialization of $\widetilde{\mathcal{E}}_{|C^\circ}$ induced by β. Here the induced map

$$\varpi_{|C^\circ} : (\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}})_{|C^\circ} \to (\mathbf{Gr}^{\mathrm{Cen}}_{G_C})_{|C^\circ}$$

is an isomorphism (in fact, the identity map under the identifications (2.2.8) and (2.5.1)). On the other hand,

$$\varpi_{\underline{0}} : (\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}})_{\underline{0}} \to (\mathbf{Gr}^{\mathrm{Cen}}_{G_C})_{\underline{0}}$$

can be identified with the map $\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G$ from (2.2.1).

2.5.2. Beĭlinson–Drinfeld group schemes and affine Grassmannian. — Next, as in §2.3.2 we have Beĭlinson–Drinfeld group ind-schemes $\mathcal{L}G^{\mathrm{BD}}_C$ and $\mathcal{L}^+G^{\mathrm{BD}}_C$ over C, and two natural morphisms

$$\mathcal{L}^+G^{\mathrm{BD}}_C \rightrightarrows \mathrm{L}^+G \times C$$

defined as in (2.3.1) and (2.3.2). We also have

$$(\mathcal{L}^+G^{\mathrm{BD}}_C)_{\underline{0}} \cong \mathrm{L}^+G, \quad (\mathcal{L}^+G^{\mathrm{BD}}_C)_{|C^\circ} \cong \mathrm{L}^+G \times \mathrm{L}^+G \times C^\circ.$$

As in §2.3.3 we have an ind-scheme $\mathbf{Gr}^{\mathrm{BD}}_{G_C}$ over C, which identifies with the fpqc sheafification of the quotient $\mathcal{L}G^{\mathrm{BD}}_C/\mathcal{L}^+G^{\mathrm{BD}}_C$, and a canonical morphism

$$\varpi : \mathbf{Gr}^{\mathrm{BD}}_{\mathcal{G}} \to \mathbf{Gr}^{\mathrm{BD}}_{G_C}.$$

In fact, $\mathbf{Gr}^{\mathrm{BD}}_{G_C}$ is the restriction to $\{0\} \times C$ of the ind-scheme Fus_G from §1.3.3. We have canonical identifications

$$(\mathbf{Gr}^{\mathrm{BD}}_{G_C})_{|C^\circ} \cong \mathrm{Gr}_G \times \mathrm{Gr}_G \times C^\circ, \qquad (\mathbf{Gr}^{\mathrm{BD}}_{G_C})_{\underline{0}} \cong \mathrm{Gr}_G \tag{2.5.3}$$

(compare with (1.3.3) or [**BR**, (1.7.3)]).

2.5.3. Iterated global affine Grassmannians. — Next, one can define iterated global affine Grassmannians $\mathbf{Gr}_{G_C}(S_1, \dots, S_n)$ associated with the group scheme G_C. In this setting we have

$$\mathbf{Gr}_{G_C}(\varnothing) \cong C, \quad \mathbf{Gr}_{G_C}(\underline{0}) \cong \mathrm{Gr}_G \times C,$$
$$\mathbf{Gr}_{G_C}(\underline{\mathsf{y}}) \cong \mathbf{Gr}^{\mathrm{Cen}}_{G_C} \cong \mathrm{Gr}_G \times C, \quad \mathbf{Gr}_{G_C}(\underline{\mathsf{y}} \cup \underline{0}) \cong \mathbf{Gr}^{\mathrm{BD}}_{G_C},$$

and there is a natural morphism of functors

$$\varpi = \varpi_{\underline{S}} : \mathbf{Gr}_{\mathcal{G}}(S_1, \dots, S_n) \to \mathbf{Gr}_{G_C}(S_1, \dots, S_n). \tag{2.5.4}$$

Variants of these spaces have already appeared in the "global version of the convolution diagram" from [**BR**, §1.7.4]; for instance, $\mathbf{Gr}_{G_C}(\underline{0}, \underline{\mathsf{y}})$ is the restriction to $\{0\} \times C$ of the ind-scheme on the top of [**BR**, p. 64]. Using appropriate principal bundles, as in §2.3.5 one shows that the functors $\mathbf{Gr}_{G_C}(S_1, \dots, S_n)$ are represented by ind-proper ind-schemes.

As in Lemma 2.3.18 we have

$$\mathbf{Gr}_{G_C}(S_1, \dots, S_n)_{\underline{0}} \cong \underbrace{\mathrm{L}G \times^{\mathrm{L}^+G} \cdots \times^{\mathrm{L}^+G} \mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G}_{m \text{ factors}},$$

where m is the number of labels S_i with $S_i \neq \varnothing$. And the analogue of (2.3.16) states that we have a canonical isomorphism

$$\mathbf{Gr}_{G_C}(\underline{S})_{|C^\circ} \cong \big(\underbrace{\mathrm{L}G \times^{\mathrm{L}^+G} \cdots \times^{\mathrm{L}^+G} \mathrm{Gr}_G}_{n \text{ factors}}\big) \times \big(\underbrace{\mathrm{L}G \times^{\mathrm{L}^+G} \cdots \times^{\mathrm{L}^+G} \mathrm{Gr}_G}_{n' \text{ factors}}\big) \times C^\circ$$

where n is the number of symbols belonging to $\{\underline{\mathsf{y}}, \underline{\mathsf{y}} \cup \underline{0}\}$, and n' is the number of symbols belonging to $\{\underline{0}, \underline{\mathsf{y}} \cup \underline{0}\}$.

2.5.4. Nearby cycles. — The constructions and results from Section 2.4 also make sense (and hold true) when $\mathcal{G}$ is replaced by G_C, except that all mentions of I-equivariance should be replaced by L^+G-equivariance. The counterpart in this setting of the functor $\Psi_{\underline{S}}$ will be denoted $\overline{\Psi}_{\underline{S}}$. In particular, as an analogue of Corollary 2.4.3, we have that for

$$\mathscr{F} \in D^{\mathrm{b}}_c(\mathbf{Gr}_{G_C}(S_1)_{|C^\circ}, \Bbbk) \qquad \text{and} \qquad \mathscr{G} \in D^{\mathrm{b}}_{(\mathcal{L}^+G^{\mathrm{BD}}_C)_{|C^\circ}}(\mathbf{Gr}_{G_C}(S_2)_{|C^\circ}, \Bbbk),$$

there is a natural isomorphism

$$\overline{\Psi}_{(S_1,S_2)}(\mathscr{F} \star_{C^\circ} \mathscr{G}) \cong \overline{\Psi}_{S_1}(\mathscr{F}) \star^{\mathrm{L}^+G} \overline{\Psi}_{S_2}(\mathscr{G}).$$

In analogy with Z, we can define the functor

$$\mathsf{Z}^{\mathrm{sph}} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \qquad \text{by} \qquad \mathsf{Z}^{\mathrm{sph}}(\mathscr{A}) := \overline{\Psi}_{\underline{\mathsf{y}}}(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}),$$

where we implicitly use the analogue of (2.2.8) for the group scheme G_C. This may appear to be a somewhat silly definition to make: by (2.5.1) we have a canonical isomorphism

$$\mathsf{Z}^{\mathrm{sph}}(\mathscr{A}) \cong \mathscr{A}. \tag{2.5.5}$$

Nevertheless, this is useful because it lets us highlight parallels with the situation considered in the preceding sections. Here again we also have a nonequivariant version

$$\mathsf{Z}^{\mathrm{sph}} : D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk),$$

and (2.5.5) also holds for this version. For $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ or $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$, the monodromy automorphism of $\mathsf{Z}^{\mathrm{sph}}(\mathscr{A})$ (see §9.1.3) will be denoted

$$\mathsf{m}^{\mathrm{sph}}_{\mathscr{A}} : \mathsf{Z}^{\mathrm{sph}}(\mathscr{A}) \to \mathsf{Z}^{\mathrm{sph}}(\mathscr{A}).$$

In fact, the reasoning for (2.5.5) also shows that $\mathsf{m}^{\mathrm{sph}}_{\mathscr{A}}$ is the identity morphism of $\mathsf{Z}^{\mathrm{sph}}(\mathscr{A})$.

Recall the proper map

$$\varpi = \varpi_{\underline{S}} : \mathbf{Gr}_{\mathcal{G}}(\underline{S}) \to \mathbf{Gr}_{G_C}(\underline{S})$$

from (2.5.4). Using the fact that nearby cycles commute with push-forward along a proper map (see Proposition 9.1.4(1)), we obtain a natural isomorphism

$$(\varpi_0)_* \Psi_{\underline{S}}(\mathscr{F}) \cong \overline{\Psi}_{\underline{S}}((\varpi_{|C^\circ})_* \mathscr{F})$$

in $D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)$ for any $\mathscr{F} \in D^{\mathrm{b}}_{(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}, \Bbbk)$, or in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$ for any $\mathscr{F} \in D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}, \Bbbk)$. In the special case $\underline{S} = (\mathsf{y})$, in view of (2.5.5) we deduce the following claim.

Lemma 2.5.1. — *For any $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, resp. for any $\mathscr{A}$ in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$, there are canonical isomorphisms*

$$\pi_* \mathsf{Z}(\mathscr{A}) \cong \mathsf{Z}^{\mathrm{sph}}(\mathscr{A}) \cong \mathscr{A}$$

in $D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$. Moreover, under these identifications we have

$$\pi_* \mathsf{m}_{\mathscr{A}} = \mathsf{m}^{\mathrm{sph}}_{\mathscr{A}}.$$

In particular, Lemma 2.5.1 implies that for any $\mathscr{A}$ in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$ we have a canonical isomorphism

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}). \tag{2.5.6}$$

(This isomorphism can also be obtained directly from compatibility of nearby cycles with proper pushforward—see Proposition 9.1.4(1)—applied to the structure morphism $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}} \to C$.)

CHAPTER 3

BRAIDING COMPATIBILITIES

In Chapter 2 we have explained the definition of the functor

$$\mathsf{Z} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk).$$

Both the domain and the codomain come with monoidal structures (given by convolution), and our goal in this chapter is to prove that this functor is compatible with these monoidal structures in various ways.

Recall in particular from Chapter 1 that when $\Bbbk$ is a field, the convolution product on $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ is t-exact for the perverse t-structure, and the resulting monoidal structure on the abelian category $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ is *symmetric*. Perhaps the most significant result in this chapter is a description of how the commutativity constraint on $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ interacts with the monoidal structure of $D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)$ under the restricted functor

$$\mathsf{Z} : \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk).$$

A precise statement will be given in Section 3.1, along with a discussion of what one can say when $\Bbbk$ is not a field. Along the way, we will see that general properties of nearby cycles functors yield additional properties of Z (in particular in terms of exactness of convolution and of total cohomology), which we record in this chapter for later use in Chapter 4.

The most important tool used in the proofs in this chapter is the theory of iterated global affine Grassmannians associated to the group scheme $\mathcal{G}$ (see Section 2.3). In Section 2.5, we saw a variant of this theory in which $\mathcal{G}$ is replaced by the constant group scheme G_C, and one can ask what happens when one carries out the arguments of this chapter using these G_C-versions instead. It turns out that this idea leads to new perspectives or alternative approaches to certain aspects of the Satake category that were discussed in Section 1.3: see Section 3.3 and §3.5.1 for details.

3.1. Overview of central functors

To explain the kinds of compatibilities we are interested in, we recall the notion of a *central functor* from [**Be2**, Definition 1]. Consider a monoidal category $(\mathbf{A}, \otimes^{\mathbf{A}})$, and a symmetric monoidal category $(\mathbf{B}, \otimes^{\mathbf{B}})$. Then a *central functor* from $\mathbf{B}$ to $\mathbf{A}$ is a monoidal functor

$$F : \mathbf{B} \to \mathbf{A}$$

together with an isomorphism

$$\sigma : F(-) \otimes^{\mathbf{A}} (-) \xrightarrow{\sim} (-) \otimes^{\mathbf{A}} F(-) \tag{3.1.1}$$

of functors from $\mathbf{B} \times \mathbf{A}$ to $\mathbf{A}$, which satisfy the following conditions:

1. For X, X' in $\mathbf{B}$, the isomorphism $\sigma_{X,F(X')}$ coincides with the composition
$$F(X) \otimes^{\mathbf{A}} F(X') \xrightarrow{\sim} F(X \otimes^{\mathbf{B}} X') \xrightarrow{\sim} F(X' \otimes^{\mathbf{B}} X) \xrightarrow{\sim} F(X') \otimes^{\mathbf{A}} F(X)$$
where the middle isomorphism is the image under F of the commutativity constraint of $\mathbf{B}$ (applied to (X, X')), and the first and third isomorphisms are induced by the monoidal structure on F.
2. For Y_1, Y_2 in $\mathbf{A}$ and X in $\mathbf{B}$, the composition
$$F(X) \otimes^{\mathbf{A}} Y_1 \otimes^{\mathbf{A}} Y_2 \xrightarrow[\sim]{\sigma_{X,Y_1} \otimes \mathrm{id}_{Y_2}} Y_1 \otimes^{\mathbf{A}} F(X) \otimes^{\mathbf{A}} Y_2 \xrightarrow[\sim]{\mathrm{id}_{Y_1} \otimes \sigma_{X,Y_2}} Y_1 \otimes^{\mathbf{A}} Y_2 \otimes^{\mathbf{A}} F(X)$$
coincides with $\sigma_{X,Y_1 \otimes^{\mathbf{A}} Y_2}$ (where we omit the associativity constraint of $\mathbf{A}$).
3. For Y in $\mathbf{A}$ and X_1, X_2 in $\mathbf{B}$, the composition
$$F(X_1 \otimes^{\mathbf{B}} X_2) \otimes^{\mathbf{A}} Y \xrightarrow{\sim} F(X_1) \otimes^{\mathbf{A}} F(X_2) \otimes^{\mathbf{A}} Y \xrightarrow[\sim]{\mathrm{id}_{F(X_1)} \otimes \sigma_{X_2,Y}} F(X_1) \otimes^{\mathbf{A}} Y \otimes^{\mathbf{A}} F(X_2)$$
$$\xrightarrow[\sim]{\sigma_{X_1,Y} \otimes \mathrm{id}_{F(X_2)}} Y \otimes^{\mathbf{A}} F(X_1) \otimes^{\mathbf{A}} F(X_2) \xrightarrow{\sim} Y \otimes^{\mathbf{A}} F(X_1 \otimes^{\mathbf{B}} X_2)$$
(where the first and last isomorphisms are provided by the monoidal structure on F) coincides with $\sigma_{X_1 \otimes^{\mathbf{B}} X_2, Y}$.

As explained in [**Be2**, Remark 1], these conditions have natural interpretations in terms of the Drinfeld center $\mathcal{Z}(\mathbf{A})$ of $\mathbf{A}$:

- the datum of a functor $F : \mathbf{B} \to \mathbf{A}$ together with an isomorphism (3.1.1) satisfying (2) is equivalent to the datum of a functor $\widetilde{F}$ from $\mathbf{B}$ to $\mathcal{Z}(\mathbf{A})$;
- the datum of a monoidal structure on F satisfying (3) is equivalent to the datum of a monoidal structure on $\widetilde{F}$;
- condition (1) means that $\widetilde{F}$ (with its monoidal structure considered above) intertwines the commutativity constraint of $\mathbf{B}$ with the natural braiding on $\mathcal{Z}(\mathbf{A})$.

The main results of the present chapter can be compactly stated as follows. (This statement combines parts of Theorems 3.2.3, 3.4.1 and 3.5.1 below.)

Theorem 3.1.1. — 1. *Assume that* $\Bbbk$ *is a field. Then the functor*

$$\mathsf{Z} : (\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G}) \to (D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk), \star^I)$$

admits a natural structure of central functor. That is, this functor factors through a monoidal functor from $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ *to the Drinfeld center* $\mathcal{Z}(D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk))$ *that intertwines the commutativity constraint on the category* $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ *with the braiding on* $\mathcal{Z}(D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk))$.

2. *For a general noetherian commutative ring* $\Bbbk$ *of finite global dimension, the functor* Z *factors through a monoidal functor from* $(D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G})$ *to* $\mathcal{Z}(D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk))$.

This result was initially proved by Gaitsgory in the setting of étale $\mathbb{Q}_\ell$-sheaves in [**G1**] (where the "centrality" and "monoidality" isomorphisms were constructed) and [**G2**] (where the relevant compatibilities are checked). Our proofs essentially follow those of Gaitsgory, although there are some new features here: notably, the systematic use of iterated global affine Grassmannians, and of equivariance with respect to global group schemes such as $\mathcal{L}^+\mathcal{G}$ or $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$. (These features help clarify the details of some arguments from [**G2**].)

Theorem 3.1.1(1) appears to suffer from two deficits: it requires $\Bbbk$ to be a field, and it restricts the domain of Z to just perverse sheaves. Part (2) drops these restrictions but has a weaker conclusion. Let us briefly discuss why Theorem 3.1.1 has this form.

First, when $\Bbbk$ is not a field, the monoidal structure $\star^{\mathrm{L}^+G}$ on $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}, \Bbbk)$ need not be t-exact, so part (1) would not make sense as stated. Of course, by passing to perverse cohomology, we do obtain a monoidal structure $\star_0^{\mathrm{L}^+G}$ on $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}, \Bbbk)$ for general $\Bbbk$, and one could ask whether a version of Theorem 3.1.1(1) holds in this setting. We will see later that Z does indeed intertwine $\star_0^{\mathrm{L}^+G}$ with a certain bifunctor $\star_0^I$ on $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$, and prove a version of property (1) above in this generality (see Theorem 3.5.1); however, the latter bifunctor does *not* make $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$ into a monoidal category (in particular, it does not seem to admit an associativity constraint).

Second, we will see in Theorem 3.3.2 that the commutativity constraint on $(\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star_0^{\mathrm{L}^+G})$ can indeed be extended to a commutativity constraint for $(D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G})$, raising the possibility that Z might define a central functor from $(D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G})$ to $(D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk), \star^I)$. Unfortunately, there is a technical obstacle to this: our proof of this property in the context of Theorem 3.1.1(1) uses the theory of "nearby cycles over a 2-dimensional base" (reviewed in Section 9.4), but this theory is only available (at least, from the point of view adopted here) for perverse sheaves.

3.2. Centrality isomorphism

In this section, we will construct the "centrality" isomorphism (3.1.1) for the functor Z, and prove that this isomorphism satisfies property (2) from the definition of a central functor. (As in §1.3.3, we will later need to adjust this isomorphism by a sign.)

3.2.1. Convolution with central sheaves. —

3.2.1.1. *Notation.* — Throughout this section, we will make frequent (and silent) use of various identifications from Table 2.3.1, especially to regard external tensor products ("$\boxtimes$") of sheaves as objects living on some $\mathbf{Gr}_{\mathcal{G}}(\underline{S})_{|C^\circ}$. In addition, for clarity, we will explicitly mention the associativity constraint in the category $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, denoted by

$$\alpha_{\mathscr{F},\mathscr{G},\mathscr{H}} : (\mathscr{F} \star^I \mathscr{G}) \star^I \mathscr{H} \xrightarrow{\sim} \mathscr{F} \star^I (\mathscr{G} \star^I \mathscr{H}).$$

In the proof, it will be convenient to make use of the functors

$$\tau_{\underline{0}} : D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}_{(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{0})_{|C^\circ}, \Bbbk),$$
$$\tau_{\underline{\mathsf{y}}} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_{(\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})_{|C^\circ}, \Bbbk)$$

both defined by the formula $\mathscr{F} \mapsto \mathscr{F} \boxtimes^L_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]$. We will sometimes regard $\tau_{\underline{0}}$, resp. $\tau_{\underline{\mathsf{y}}}$, as taking values in $D^{\mathrm{b}}_{I\times C^\circ}(\mathbf{Gr}_{\mathcal{G}}(\underline{0})_{|C^\circ}, \Bbbk)$, resp. in $D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}_{|C^\circ}}(\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})_{|C^\circ}, \Bbbk)$, instead; this should not cause any confusion. It is immediate from the definitions that for $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ we have

$$\mathsf{Z}(\mathscr{F}) = \Psi_{\underline{\mathsf{y}}}(\tau_{\underline{\mathsf{y}}}(\mathscr{F})). \tag{3.2.1}$$

On the other hand, it follows from the second isomorphism in Lemma 2.3.6 that for $\mathscr{F} \in D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ there is a canonical isomorphism

$$\mathscr{F} \cong \Psi_{\underline{0}}(\tau_{\underline{0}}(\mathscr{F})). \tag{3.2.2}$$

In practice, we will often omit the subscript and denote both $\tau_{\underline{0}}$ and $\tau_{\underline{\mathsf{y}}}$ simply by τ. No ambiguity should result from this.

3.2.1.2. *Convolution with central sheaves as nearby cycles.* — Let us consider the bifunctor

$$\mathsf{C} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \times D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$$

given by

$$\mathsf{C}(\mathscr{A}, \mathscr{F}) := \Psi_{\underline{\mathsf{y}}\cup\underline{0}}\big(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big).$$

This bifunctor has partially equivariant counterparts

$$D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \times D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk),$$
$$D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk) \times D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$$

which will also be denoted by C.

Proposition 3.2.1. — 1. *For $\mathscr{A}$ in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, and $\mathscr{F}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, there is a canonical isomorphism*

$$\mathsf{C}(\mathscr{A}, \mathscr{F}) \cong \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}$$

in $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$.

2. *For $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{F}$ in $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, there is a canonical isomorphism*

$$\mathsf{C}(\mathscr{A}, \mathscr{F}) \cong \mathscr{F} \star^I \mathsf{Z}(\mathscr{A})$$

in $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$.

Proof. — The proof will make use of the commutative diagram

$$\begin{tikzcd} & \mathrm{Gr}_G \times \mathrm{Fl}_G \times C^\circ \arrow[dl, "\nu"'] \arrow[d, "\nu"] \arrow[dr, "\nu"] & \\ \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{0})_{|C^\circ} \arrow[dr, "\mu"'] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cup \underline{0})_{|C^\circ} & \mathbf{Gr}_{\mathcal{G}}(\underline{0}, \underline{\mathsf{y}})_{|C^\circ} \arrow[dl, "\mu"] \end{tikzcd} \tag{3.2.3}$$

in which every map is an isomorphism. (See §2.3.4 for the definition of μ.) In the setting of (1), the left-hand part of this diagram gives us a natural isomorphism

$$\tau(\mathscr{A}) \star_{C^\circ} \tau(\mathscr{F}) \cong \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]$$

(where we identify all the spaces in (3.2.3) without mention). Using Corollary 2.4.3 along with (3.2.1) and (3.2.2), we deduce the desired isomorphism

$$\mathsf{C}(\mathscr{A}, \mathscr{F}) \cong \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F},$$

either in $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$ or in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$.

The proof of (2) is similar, using now the natural isomorphism

$$\tau(\mathscr{F}) \star_{C^\circ} \tau(\mathscr{A}) \cong \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]$$

provided by the right-hand part of (3.2.3). □

Remark 3.2.2. — As in the case of Gr_G (see §1.3.1), for $\mathscr{F}, \mathscr{G}$ in $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$ we set

$$\mathscr{F} \star^I_0 \mathscr{G} := {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \star^I \mathscr{G}).$$

(Note that it is *not* clear, and probably false, that one can endow the bifunctor $\star^I_0$ with an associativity constraint.) Taking the perverse degree-0 part in Proposition 3.2.1, in the setting of (1), if $\mathscr{A}$ and $\mathscr{F}$ are perverse we obtain a canonical isomorphism

$$\Psi_{\underline{\mathsf{y}} \cup \underline{0}}\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big) \cong \mathsf{Z}(\mathscr{A}) \star^I_0 \mathscr{F},$$

and in the setting of (2), if $\mathscr{A}$ and $\mathscr{F}$ are perverse we obtain a canonical isomorphism

$$\Psi_{\underline{\mathsf{y}} \cup \underline{0}}\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big) \cong \mathscr{F} \star^I_0 \mathsf{Z}(\mathscr{A}).$$

3.2.1.3. *Centrality isomorphism and compatibilities.* — The following theorem is the main application of Proposition 3.2.1, and the main result of the present section.

Theorem 3.2.3. — *For $\mathscr{A} \in D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{F} \in D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, there is a natural isomorphism*

$$\widetilde{\sigma}_{\mathscr{A}, \mathscr{F}} : \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F} \xrightarrow{\sim} \mathscr{F} \star^I \mathsf{Z}(\mathscr{A})$$

in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$. *Moreover, for* $\mathscr{A} \in D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ *and* $\mathscr{F}_1, \mathscr{F}_2 \in D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$*, the following diagram commutes:*

(3.2.4)
$$\begin{tikzcd}
(\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}_1) \star^I \mathscr{F}_2 \arrow[r, "\alpha_{\mathsf{Z}(\mathscr{A}),\mathscr{F}_1,\mathscr{F}_2}"] \arrow[d, "\widetilde{\sigma}_{\mathscr{A},\mathscr{F}_1} \star^I \mathrm{id}_{\mathscr{F}_2}"'] & \mathsf{Z}(\mathscr{A}) \star^I (\mathscr{F}_1 \star^I \mathscr{F}_2) \arrow[ddd, "\widetilde{\sigma}_{\mathscr{A},\mathscr{F}_1 \star^I \mathscr{F}_2}"] \\
(\mathscr{F}_1 \star^I \mathsf{Z}(\mathscr{A})) \star^I \mathscr{F}_2 \arrow[d, "\alpha_{\mathscr{F}_1,\mathsf{Z}(\mathscr{A}),\mathscr{F}_2}"'] & \\
\mathscr{F}_1 \star^I (\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}_2) \arrow[d, "\mathrm{id}_{\mathscr{F}_1} \star^I \widetilde{\sigma}_{\mathscr{A},\mathscr{F}_2}"'] & \\
\mathscr{F}_1 \star^I (\mathscr{F}_2 \star^I \mathsf{Z}(\mathscr{A})) \arrow[r, "\alpha^{-1}_{\mathscr{F}_1,\mathscr{F}_2,\mathsf{Z}(\mathscr{A})}"] & (\mathscr{F}_1 \star^I \mathscr{F}_2) \star^I \mathsf{Z}(\mathscr{A}).
\end{tikzcd}$$

Proof. — The isomorphism $\widetilde{\sigma}_{\mathscr{A},\mathscr{F}}$ is obtained as the composition of the isomorphisms

$$\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F} \xrightarrow{\sim} \mathsf{C}(\mathscr{A}, \mathscr{F}) \xrightarrow{\sim} \mathscr{F} \star^I \mathsf{Z}(\mathscr{A})$$

provided by Proposition 3.2.1.

For the second part of the theorem, consider the following two diagrams:

(3.2.5)
$$\begin{tikzcd}
& \mathrm{Gr}_G \times (\mathrm{L}G \times^I \mathrm{Fl}_G) \times C^\circ \arrow[r, "\nu"] \arrow[dl, "\nu"'] \arrow[d, "\nu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{0},\underline{0},\underline{\mathsf{y}})_{|C^\circ} \arrow[d, "\mu"] \arrow[dr, "\mu"] & \\
\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}},\underline{0},\underline{0})_{|C^\circ} \arrow[dr, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{0},\underline{\mathsf{y}},\underline{0})_{|C^\circ} \arrow[r, "\mu"] \arrow[d, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{0},\underline{\mathsf{y}}\cup\underline{0})_{|C^\circ} \arrow[d, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{0},\underline{\mathsf{y}})_{|C^\circ} \arrow[dl, "\mu"] \\
& \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}\cup\underline{0},\underline{0})_{|C^\circ} \arrow[r, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}\cup\underline{0})_{|C^\circ}, &
\end{tikzcd}$$

(3.2.6)
$$\begin{tikzcd}
& \mathrm{Gr}_G \times (\mathrm{L}G \times^I \mathrm{Fl}_G) \times C^\circ \arrow[r, "\nu"] \arrow[dl, "\nu"'] \arrow[dr, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{0},\underline{0},\underline{\mathsf{y}})_{|C^\circ} \arrow[dr, "\mu"] & \\
\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}},\underline{0},\underline{0})_{|C^\circ} \arrow[r, "\mu"] \arrow[dr, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}},\underline{0})_{|C^\circ} \arrow[dr, "\mu"] & \mathrm{Gr}_G \times \mathrm{Fl}_G \times C^\circ \arrow[l, "\nu"'] \arrow[r, "\nu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{0},\underline{\mathsf{y}})_{|C^\circ} \arrow[dl, "\mu"] \\
& \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}\cup\underline{0},\underline{0})_{|C^\circ} \arrow[r, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}\cup\underline{0})_{|C^\circ}, &
\end{tikzcd}$$

in which once again μ is as in §2.3.4 and ν is as in §2.3.7. Both diagrams are commutative; indeed, they are both assembled from copies of the commutative squares (2.3.15), (2.4.5), and (3.2.3).

By tracing through (3.2.5) and using associativity, we obtain the sequence of maps along the left-hand side and bottom of the following diagram; by tracing through (3.2.6) and using associativity, we obtain the maps along the top and right-hand side:

$$\begin{tikzcd}
(\tau(\mathscr{A}) \star_{C^\circ} \tau(\mathscr{F}_1)) \star_{C^\circ} \tau(\mathscr{F}_2) \arrow[r, "\alpha"] \arrow[d, "\tilde{\sigma}\star_{C^\circ}\mathrm{id}"'] & \tau(\mathscr{A}) \star_{C^\circ} (\tau(\mathscr{F}_1) \star_{C^\circ} \tau(\mathscr{F}_2)) \arrow[d, "\wr"] \\
(\tau(\mathscr{F}_1) \star_{C^\circ} \tau(\mathscr{A})) \star_{C^\circ} \tau(\mathscr{F}_2) \arrow[d, "\alpha"'] & \tau(\mathscr{A}) \star_{C^\circ} \tau(\mathscr{F}_1 \star^I \mathscr{F}_2) \arrow[d, "\tilde{\sigma}"] \\
\tau(\mathscr{F}_1) \star_{C^\circ} (\tau(\mathscr{A}) \star_{C^\circ} \tau(\mathscr{F}_2)) \arrow[d, "\mathrm{id}\star_{C^\circ}\tilde{\sigma}"'] & \tau(\mathscr{F}_1 \star^I \mathscr{F}_2) \star_{C^\circ} \tau(\mathscr{A}) \arrow[d, "\wr"] \\
\tau(\mathscr{F}_1) \star_{C^\circ} (\tau(\mathscr{F}_2) \star_{C^\circ} \tau(\mathscr{A})) \arrow[r, "\alpha^{-1}"] & (\tau(\mathscr{F}_1) \star_{C^\circ} \tau(\mathscr{F}_2)) \star_{C^\circ} \tau(\mathscr{A}).
\end{tikzcd}$$

This diagram commutes because (3.2.5) and (3.2.6) agree along their outermost edges. We apply $\Psi_{\underline{y}\cup\underline{0}}$ to this diagram to obtain (3.2.4), which is therefore commutative. □

Remark 3.2.4. — The proof of Theorem 3.2.3 is precisely the point where it is more convenient to work with the ind-scheme $\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}$ rather than the ind-scheme considered in [**G1**] in the definition of the functor Z. In fact, in [**G1**, Proposition 6] the isomorphisms of Proposition 3.2.1 are proved only in the case $\mathscr{A}$ and $\mathscr{F}$ are perverse sheaves, which makes it impossible to state the commutativity of (3.2.4) in a reasonable way (since $\mathscr{F}_1 \star^I \mathscr{F}_2$ might not be perverse even if $\mathscr{F}_1$ and $\mathscr{F}_2$ are). This issue was mentioned (and solved along the lines above) in [**Zh1**].

3.2.2. Some consequences. —

3.2.2.1. *Convolution exactness.* — The following corollary (to Proposition 3.2.1) will play an important role in Chapter 4.

Corollary 3.2.5. — *For $\mathscr{A} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{F} \in \mathsf{Perv}(\mathrm{Fl}_G, \Bbbk)$, the complex $\mathscr{F} \star^I \mathsf{Z}(\mathscr{A})$ is concentrated in nonpositive perverse degrees. If $\mathscr{A} \boxtimes^L_\Bbbk \mathscr{F} \in D^{\mathrm{b}}_c(\mathrm{Gr}_G \times \mathrm{Fl}_G, \Bbbk)$ is perverse, then $\mathscr{F} \star^I \mathsf{Z}(\mathscr{A})$ is perverse.*

Similarly, for $\mathscr{A} \in \mathsf{Perv}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{F} \in \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$, the complex $\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}$ is concentrated in nonpositive perverse degrees. If $\mathscr{A} \boxtimes^L_\Bbbk \mathscr{F} \in D^{\mathrm{b}}_c(\mathrm{Gr}_G \times \mathrm{Fl}_G, \Bbbk)$ is perverse, then $\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}$ is perverse.

Proof. — The claims follow from Proposition 3.2.1 and the t-exactness of nearby cycles (see Theorem 9.1.3(2)). □

Remark 3.2.6. — In case $\Bbbk$ is a field, the assumption that $\mathscr{A} \boxtimes^L_\Bbbk \mathscr{F}$ is perverse is always satisfied. Hence, in this case, Corollary 3.2.5 says that central sheaves are "convolution exact" in the sense that convolution with such an object (on the right, and on the left for I-equivariant objects) is an exact functor with respect to the perverse t-structure.

3.2.2.2. *Total cohomology.* — Proposition 3.2.1 also has the following consequence for total cohomology.

Corollary 3.2.7. — 1. *For $\mathscr{A}$ in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{F}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, there are canonical isomorphisms*

$$R\Gamma(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}) \cong R\Gamma(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F}) \cong R\Gamma(\mathrm{Gr}_G, \mathscr{A}) \overset{L}{\otimes}_{\Bbbk} R\Gamma(\mathrm{Fl}_G, \mathscr{F})$$

in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$. In particular, if $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A})$ or $\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F})$ is $\Bbbk$-flat, there are canonical isomorphisms of graded $\Bbbk$-modules

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F}) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\Bbbk} \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}). \tag{3.2.7}$$

2. *For $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{F}$ in $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$, there are canonical isomorphisms*

$$R\Gamma(\mathrm{Fl}_G, \mathscr{F} \star^I \mathsf{Z}(\mathscr{A})) \cong R\Gamma(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F}) \cong R\Gamma(\mathrm{Gr}_G, \mathscr{A}) \overset{L}{\otimes}_{\Bbbk} R\Gamma(\mathrm{Fl}_G, \mathscr{F})$$

in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$. In particular, if $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A})$ or $\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F})$ is $\Bbbk$-flat, there are canonical isomorphisms of graded $\Bbbk$-modules

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F} \star^I \mathsf{Z}(\mathscr{A})) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F}) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\Bbbk} \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}). \tag{3.2.8}$$

Proof. — By compatibility of nearby cycles with proper pushforward (see Proposition 9.1.4(1)) applied to the structure morphism $\mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}} \cup \underline{0}) \to C$, in both settings we obtain a canonical isomorphism

$$R\Gamma(\mathrm{Fl}_G, \mathsf{C}(\mathscr{A}, \mathscr{F})) \cong R\Gamma(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F}).$$

By a general form of the Künneth formula (see [**Ac3**, Proposition 2.9.2]), the right-hand side identifies with $R\Gamma(\mathrm{Gr}_G, \mathscr{A}) \otimes^L_{\Bbbk} R\Gamma(\mathrm{Fl}_G, \mathscr{F})$. In each case the first claim follows, in view of Proposition 3.2.1.

The second claim in each part of the proposition is a consequence of the following well-known property of derived tensor product (cf. [**Ac3**, Lemma A.6.17]): for $M, N \in D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$, there is a natural map

$$\mathsf{H}^\bullet(M) \otimes_{\Bbbk} \mathsf{H}^\bullet(N) \to \mathsf{H}^\bullet(M \overset{L}{\otimes}_{\Bbbk} N)$$

that is an isomorphism if $\mathsf{H}^\bullet(M)$ is $\Bbbk$-flat (or if $\mathsf{H}^\bullet(N)$ is $\Bbbk$-flat). ☐

3.2.2.3. *Interpretation in terms of equivariant cohomology.* — Recall that there exists a canonical algebra homomorphism

$$\mathsf{H}^\bullet_I(\mathrm{pt}; \Bbbk) \to \mathsf{H}^\bullet(\mathrm{Fl}_G; \Bbbk) \tag{3.2.9}$$

constructed as follows. Denote by I_{u} the preimage of U under the projection $\mathrm{L}^+G \to G$. (This subgroup is sometimes called the "pro-unipotent radical of I.") We denote by $\mathrm{Fl}^{(1)}_G$ the fppf quotient $(\mathrm{L}G/I_{\mathrm{u}})_{\mathrm{fppf}}$. As for Fl_G, this functor is represented by a separated ind-scheme of ind-finite type, and we have a canonical isomorphism

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \Bbbk) \cong \mathsf{H}^\bullet_T(\mathrm{Fl}^{(1)}_G, \Bbbk),$$

where T acts on $\mathrm{Fl}^{(1)}_G$ via the action induced by the action on $\mathrm{L}G$ given by $t \cdot \gamma = \gamma t^{-1}$. This provides an algebra morphism $\mathsf{H}^\bullet_T(\mathrm{pt}; \Bbbk) \to \mathsf{H}^\bullet(\mathrm{Fl}_G, \Bbbk)$. Now it is well known that the composition $I \to B \to B/U \cong T$ induces an isomorphism $\mathsf{H}^\bullet_T(\mathrm{pt}; \Bbbk) \xrightarrow{\sim} \mathsf{H}^\bullet_I(\mathrm{pt}; \Bbbk)$.

We deduce the desired morphism (3.2.9). Using this morphism, for any complex $\mathscr{F}$ in $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$ we obtain a canonical action of $\mathsf{H}^\bullet_I(\mathrm{pt}; \Bbbk)$ on $\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F})$.

Consider, as in Corollary 3.2.7(1), a complex $\mathscr{A}$ in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$ and a complex $\mathscr{F}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$. Then, as in the proof of this corollary, there exists a canonical morphism of graded $\Bbbk$-modules

$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \to \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \overset{L}{\boxtimes}_\Bbbk \mathscr{F}). \tag{3.2.10}$$

On the other hand, we also have a canonical morphism of graded $\Bbbk$-modules

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_{\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)} \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \to \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}) \tag{3.2.11}$$

which sends $f \otimes g$, where $f \in \mathsf{H}^p(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}))$ is regarded as a morphism $\underline{\Bbbk} \to \mathsf{Z}(\mathscr{A})[p]$ and $g \in \mathsf{H}^q_I(\mathrm{Fl}_G, \mathscr{F})$ is regarded as a morphism $\underline{\Bbbk} \to \mathscr{F}[q]$ in the *I-equivariant* derived category, to $f \mathbin{\widetilde{\boxtimes}} g : \underline{\Bbbk} = \underline{\Bbbk} \mathbin{\widetilde{\boxtimes}} \underline{\Bbbk} \to \mathsf{Z}(\mathscr{A}) \mathbin{\widetilde{\boxtimes}} \mathscr{F}[p+q]$, regarded as an element in

$$\mathsf{H}^{p+q}(\mathrm{L}G \times^I \mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \mathbin{\widetilde{\boxtimes}} \mathscr{F}) \cong \mathsf{H}^{p+q}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}).$$

Lemma 3.2.8. — *The following diagram commutes, where the right-hand vertical arrow is induced by the isomorphism of Corollary 3.2.7*(1)*, and in the left column the top vertical arrow is induced by the forgetful map from equivariant cohomology to cohomology and the bottom vertical arrow is the natural one:*

$$\begin{tikzcd}
\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \arrow[r, "(3.2.10)"] & \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \boxtimes^L_\Bbbk \mathscr{F}) \arrow[ddd, "\wr"] \\
\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \arrow[u] \arrow[d, "\wr"', "(2.5.6)"] & \\
\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_\Bbbk \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \arrow[d] & \\
\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_{\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)} \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \arrow[r, "(3.2.11)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}).
\end{tikzcd}$$

Proof. — The isomorphism of Corollary 3.2.7(1) is obtained by remarking that there exists a canonical isomorphism

$$\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{\mathsf{y}}) \times^I \mathrm{Fl}_G \xrightarrow{\sim} \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{0}).$$

From this construction and the description of (2.5.6) in terms of nearby cycles (see the last sentence in §2.5.4) it is clear that we have a commutative diagram

$$\begin{tikzcd}
\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \arrow[r] \arrow[d, "(2.5.6)"', "\wr"] & \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \boxtimes^L_\Bbbk \mathscr{F}) \arrow[d, "\wr"'] \\
\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_\Bbbk \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \arrow[r] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}).
\end{tikzcd}$$

Here the lower arrow factors as a composition

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_{\Bbbk} \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \to \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_{\mathsf{H}^\bullet_I(\mathrm{pt},\Bbbk)} \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \xrightarrow{(3.2.11)} \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F})$$

where the first map is as above. On the other hand, since the restriction of the map $\mathbf{Gr}^{(\infty)}_{\mathcal{G},\underline{0}}(\underline{\mathrm{y}}) \to \mathbf{Gr}_{\mathcal{G}}(\underline{\mathrm{y}})$ to C° is the trivial projection (see Example 2.3.17), the upper arrow factors as a composition

$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\Bbbk} \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \to \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\Bbbk} \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \xrightarrow{(3.2.10)} \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{F}),$$

which implies our claim. □

3.3. Variants for G_C

The constructions of Section 3.2 have obvious counterparts in the setting where the group scheme $\mathcal{G}$ is replaced by the constant group scheme G_C, which we explain in this section. These analogues provide different perspectives on the convolution product in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ (explained in §3.3.1) and on the tensor structure of the functor F (see §§3.3.3–3.3.4).

3.3.1. Convolution as nearby cycles. — We consider the bifunctor

$$\mathsf{C}_{\mathrm{Gr}} : D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \times D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$$

given by

$$\mathsf{C}_{\mathrm{Gr}}(\mathscr{A}, \mathscr{B}) := \overline{\Psi}_{\underline{\mathrm{y}} \cup \underline{0}}\big(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big),$$

where we are using the identifications in (2.5.3). This bifunctor has partially equivariant counterparts

$$D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \times D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk),$$
$$D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk) \times D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk)$$

which will also be denoted by C_{Gr}.

The proof of the following claim is similar to that of Proposition 3.2.1, and will therefore be omitted.

Proposition 3.3.1. — 1. *For $\mathscr{A}$ in $D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, and $\mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, there is a canonical isomorphism*

$$\mathsf{C}_{\mathrm{Gr}}(\mathscr{A}, \mathscr{B}) \cong \mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}$$

in $D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$.

2. *For $\mathscr{A}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{B}$ in $D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, there is a canonical isomorphism*
$$\mathsf{C}_{\mathrm{Gr}}(\mathscr{A}, \mathscr{B}) \cong \mathscr{B} \star^{\mathrm{L}^+G} \mathscr{A}$$
in $D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$.

Applying Proposition 3.3.1(1) in the case where $\mathscr{A}$ and $\mathscr{B}$ are perverse sheaves, and taking the 0-th perverse cohomology, we deduce a canonical isomorphism

$$\overline{\Psi}_{\underline{y} \cup \underline{0}}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]) \cong \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}. \tag{3.3.1}$$

See §3.3.2 below for the comparison with the "fusion product" construction of convolution recalled in §1.3.3.

As for Theorem 3.2.3, one deduces the following theorem.

Theorem 3.3.2. — *For $\mathscr{A}, \mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, there is a natural isomorphism*
$$\widetilde{\sigma}^{\mathrm{sph}}_{\mathscr{A},\mathscr{B}} : \mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B} \xrightarrow{\sim} \mathscr{B} \star^{\mathrm{L}^+G} \mathscr{A}.$$
These isomorphisms are compatible with the associativity constraint for $\star^{\mathrm{L}^+G}$, in the sense that (together with this associativity constraint) they endow the pair $(D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk), \star^{\mathrm{L}^+G})$ with the structure of a symmetric monoidal category.

The commutativity constraint considered in this theorem is closely related to the constraint of §1.3.3; see Corollary 3.5.5 below for a precise statement.

As for Corollary 3.2.5, we obtain the following corollary, which generalizes some results from §1.3.1 (see Remark 1.3.3).

Corollary 3.3.3. — *For $\mathscr{A} \in \mathsf{Perv}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, the complex $\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}$ is concentrated in nonpositive perverse degrees. If $\mathscr{A} \boxtimes^L_{\Bbbk} \mathscr{B} \in D^{\mathrm{b}}_{c}(\mathrm{Gr}_G \times \mathrm{Gr}_G, \Bbbk)$ is perverse, then $\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}$ is perverse.*

We also obtain the following counterpart of Corollary 3.2.7.

Corollary 3.3.4. — *For $\mathscr{A}$ in $D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, there exists a canonical isomorphism*
$$R\Gamma(\mathrm{Gr}_G, \mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \cong R\Gamma(\mathrm{Gr}_G, \mathscr{A}) \overset{L}{\otimes}_{\Bbbk} R\Gamma(\mathrm{Gr}_G, \mathscr{B})$$
in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$. In particular, if $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A})$ or $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{B})$ is $\Bbbk$-flat there exists a canonical isomorphism of graded $\Bbbk$-vector spaces
$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\Bbbk} \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{B}).$$

Finally we state a variant of Corollary 3.3.4, which will be considered more closely in §3.3.3 below.

Corollary 3.3.5. — *For $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, there exists a canonical isomorphism*
$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\Bbbk} \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{B}).$$

Proof. — Starting from (3.3.1), arguments similar to those encountered in the proof of Corollary 3.2.7 (i.e. relying on the compatibility of nearby cycles with proper push-forward) provide a canonical isomorphism

$$\mathsf{H}^{\bullet}(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \cong \mathsf{H}^{\bullet}(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B})). \tag{3.3.2}$$

The claim follows, in view of (1.3.7). □

3.3.2. Comparison with the fusion product. — Proposition 3.3.1 provides a description of convolution of L^+G-equivariant complexes on Gr_G in terms of nearby cycles. On the other hand, the proof of the geometric Satake equivalence involves a description of convolution in terms of a "fusion product," see §1.3.3. Let us now explain the relationship between these two descriptions.

The fusion product involves the fusion space Fus_G, which comes with a structure map $\mathrm{Fus}_G \to C^2$. Recall from §2.5.2 that $\mathbf{Gr}^{\mathrm{BD}}_{G_C}$ is identified with the restriction of Fus_G to $\{0\} \times C \subset C^2$. Let i, j, i', j', a, and b be the embeddings shown in the following diagram:

$$\begin{tikzcd}
\mathrm{Gr}_G \arrow[r, "i'"] \arrow[d, "b"'] & \mathbf{Gr}^{\mathrm{BD}}_{G_C} \arrow[d, "a"] & \mathrm{Gr}_G \times \mathrm{Gr}_G \times C^{\circ} \arrow[l, "j'"'] \arrow[d] \\
\mathrm{Gr}_G \times C \arrow[r, "i"] & \mathrm{Fus}_G & \mathrm{Gr}_G \times \mathrm{Gr}_G \times (C^2 \smallsetminus \Delta C) \arrow[l, "j"']
\end{tikzcd}$$

The maps i and j were previously introduced in §1.3.3, and the maps i' and j' come from (2.5.3).

Lemma 3.3.6. — *For any $\mathscr{C}$ in $\mathsf{Perv}(\mathrm{Gr}_G \times \mathrm{Gr}_G, \Bbbk)$, there exists a canonical isomorphism*

$$a^* j_{!*}\big(\mathscr{C} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^2 \smallsetminus \Delta C}[2]\big) \cong j'_{!*}\big(\mathscr{C} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^{\circ}}[1]\big)[1].$$

Proof. — Using the additive structure on $C = \mathbb{A}^1$ we obtain an isomorphism $C \times (\{0\} \times C) \to C \times C$ defined by $(x,(0,y)) \mapsto (x+y, y)$, which identifies $\{0\} \times (\{0\} \times C)$ in the left-hand side with ΔC in the right-hand side. Using this isomorphism (and the fact that the group scheme G_C is constant), given points $x, y \in C$ and a principal G-bundle over $\widehat{\Gamma_0 \cup \Gamma_y}$ with a trivialization away from $\Gamma_0 \cup \Gamma_y$ one obtains a G-bundle over $\widehat{\Gamma_{x+y} \cup \Gamma_y}$ with a trivialization away from $\Gamma_{x+y} \cup \Gamma_y$, which provides an isomorphism of ind-schemes

$$C \times \mathbf{Gr}^{\mathrm{BD}}_{G_C} \xrightarrow{\sim} \mathrm{Fus}_G.$$

Under this identification, the perverse sheaf $j_{!*}(\mathscr{C} \boxtimes^L_{\Bbbk} \underline{\Bbbk}_{C^2 \smallsetminus \Delta C}[2])$ on the right-hand side corresponds to the perverse sheaf

$$\big(\underline{\Bbbk}_C[1]\big) \overset{L}{\boxtimes}_{\Bbbk} j'_{!*}(\mathscr{C} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^{\circ}}[1])$$

on the left-hand side. The claim follows. □

Proposition 3.3.7. — *For any* $\mathscr{A}, \mathscr{B}$ *in* $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, *there exists a canonical isomorphism*

$$b^* i^* j_{!*}\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^2 \smallsetminus \Delta C}[2]\big) \cong \overline{\Psi}_{\underline{\mathsf{y}}\cup\underline{0}}\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big)[2].$$

Moreover, this isomorphism is compatible with those in (1.3.4) *and* (3.3.1) *in the obvious way.*

Proof. — We have $i \circ b = a \circ i'$, so that the left-hand side identifies with

$$(i')^* a^* j_{!*}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^2 \smallsetminus \Delta C}[2]).$$

In view of Lemma 3.3.6, our problem reduces to that of constructing an isomorphism

$$(i')^* j'_{!*}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1])[-1] \cong \overline{\Psi}_{\underline{\mathsf{y}}\cup\underline{0}}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]).$$

Since the left-hand side is known to be perverse (see (1.3.4)) this isomorphism will follow from Lemma 9.1.9 provided we prove that the monodromy automorphism of the right-hand side is trivial. However, using the compatibility of nearby cycles with proper pushforward applied to the structure morphism $\mathbf{Gr}_{G_C}(\underline{\mathsf{y}}\cup\underline{0}) \to C$ (as in the proof of Corollary 3.3.5) and the fact that the monodromy is trivial for a constant family, we obtain that this monodromy automorphism is sent to the identity by the functor $\mathsf{H}^\bullet(\mathrm{Gr}_G, -)$. Since the latter functor is faithful on perverse sheaves (see Proposition 1.3.4), this concludes the proof.

The compatibility between the various isomorphisms is left to the reader. □

The isomorphism of Proposition 3.3.7 can be described more explicitly as follows. By definition (see §9.1.1) we have

$$\overline{\Psi}_{\underline{\mathsf{y}}\cup\underline{0}}\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big) = (i')^* j'_*\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \exp_* \underline{\Bbbk}_{\mathbb{C}}[1]\big)[-1],$$

where $\exp : \mathbb{C} \to \mathbb{C}^\times$ is the exponential map. Now we have canonical morphisms

$$j'_{!*}\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big) \to j'_*\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big)$$

and $\underline{\Bbbk}_{C^\circ} \to \exp_* \underline{\Bbbk}_{\mathbb{C}}$, which induce an isomorphism

$$(i')^* j'_{!*}\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big)[-1] \xrightarrow{\sim} (i')^* j'_*\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \exp_* \underline{\Bbbk}_{\mathbb{C}}[1]\big)[-1].$$

3.3.3. Monoidal structure on total cohomology via nearby cycles. — Corollary 3.3.5 provides a monoidal structure on the functor

$$\mathsf{F} = \mathsf{H}^\bullet(\mathrm{Gr}_G, -) : \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to \mathsf{Mof}_{\Bbbk}.$$

On the other hand, another monoidal structure on this functor is involved in the proof of the geometric Satake equivalence, see §1.3.4. We will now prove that these structures in fact coincide.

Proposition 3.3.8. — *For any* $\mathscr{A}, \mathscr{B}$ *in* $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, *the isomorphisms*

$$\mathsf{F}(\mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \xrightarrow{\sim} \mathsf{F}(\mathscr{A}) \otimes_{\Bbbk} \mathsf{F}(\mathscr{B})$$

constructed in §1.3.6 and in Corollary 3.3.5 coincide.

Proof. — In the construction of both isomorphisms, one produces an isomorphism

$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B})), \tag{3.3.3}$$

and then uses (1.3.7). To conclude, we therefore only have to prove that the two versions of (3.3.3) coincide.

Set

$$\mathscr{C} := j'_{!*}\big({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]\big),$$

and denote by $f' : \mathbf{Gr}_{G_C}(\underline{\mathrm{y}}\cup\underline{0}) \to C$ the structure morphism. Then, taking into account Lemma 3.3.6, the construction of §1.3.4 is based on the fact that the complex $f'_*\mathscr{C}$ is constant in the sense considered in Example 9.1.2; then, identifying the stalks of these sheaves at 0 and $1 \in \mathbb{C}^\times = C^\circ$ (see (9.1.3)) one obtains an isomorphism

$$R\Gamma(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \xrightarrow{\sim} R\Gamma(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B})),$$

from which one deduces (3.3.3) by taking cohomology. On the other hand, the construction in Corollary 3.3.5 is based on applying the nearby cycles functor associated with $\mathrm{id} : C \to C$ to

$$R\Gamma(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B})) \overset{L}{\otimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1] = (f'_{|C^\circ})_*({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B}) \overset{L}{\otimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]),$$

and observing that the complex so-obtained identifies with $R\Gamma(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \boxtimes^L_{\Bbbk} \mathscr{B}))$ on the one hand (see (9.1.4)), and with $R\Gamma(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B})$ on the other hand (by (3.3.1) and compatibility of nearby cycles with proper pushforward). Once again, taking cohomology we deduce the isomorphism (3.3.3).

Once these constructions are recalled, the claim follows from the commutativity of the diagram in Example 9.1.2 (applied to the constant complex $f'_*\mathscr{C}$). □

3.3.4. Monoidal structure on total cohomology via equivariant cohomology. — We conclude this section by giving a third description of the monoidal structure on the functor F, in terms of equivariant cohomology, valid under the assumption that all the torsion primes of G are invertible in $\Bbbk$. (For a brief reminder on how to compute the torsion primes, see [**JMW**, §2.6]. For a more detailled account, and more details, see [**To**].) This construction of the monoidal structure is considered in particular in [**Zh4**, §5.2] (in the case when $\Bbbk$ is a field of characteristic 0) and (in a "mixed characteristic setting," and again for characteristic-0 coefficients) in [**Zh3**].

Under our assumption on $\Bbbk$, it is known that the equivariant cohomology $\mathsf{H}^\bullet_G(\mathrm{pt};\Bbbk)$ is concentrated in even degrees, and free (of finite rank in each degree) over $\Bbbk$. A construction similar to that of (3.2.9) yields a canonical algebra morphism

$$\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \to \mathsf{H}^\bullet(\mathrm{Gr}_G;\Bbbk). \tag{3.3.4}$$

From this morphism we deduce, for any $\mathscr{A}$ in $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$, a canonical action of $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$ on $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A})$. Note also that the projection $\mathrm{L}^+G \to G$ induces an algebra isomorphism

$$\mathsf{H}^\bullet_G(\mathrm{pt};\Bbbk) \xrightarrow{\sim} \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk).$$

Lemma 3.3.9. — *Under our assumption above:*

1. *the morphism* (3.3.4) *vanishes on* $\bigoplus_{i>0} \mathsf{H}^i_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$*;*
2. *for any* $\mathscr{A}$ *in* $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk)$*, there exists a (noncanonical) isomorphism of graded* $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$*-modules*
$$\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G,\mathscr{A}) \cong \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Gr}_G,\mathscr{A}).$$
As a consequence, the forgetful functor induces an isomorphism of graded $\Bbbk$*-modules*
$$\Bbbk \otimes_{\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)} \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G,\mathscr{A}) \xrightarrow{\sim} \mathsf{H}^\bullet(\mathrm{Gr}_G,\mathscr{A}).$$

In part (2) of this lemma, we regard $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G,\mathscr{A})$ as an $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$-module in the usual way for equivariant cohomology (cf. [**Ac3**, §6.7]).

Proof. — (1) There are two ways to construct a ring homomorphism $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \to \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G;\Bbbk)$. The first is the "usual" way that comes from the general theory of equivariant cohomology. The second, which we will call the "special" way, arises by repeating the construction of (3.3.4) (following (3.2.9)) with $\mathsf{H}^\bullet(\mathrm{Gr}_G,\mathscr{A})$ replaced by $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G,\mathscr{A})$. Now consider the compositions
$$\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \underset{\text{special}}{\overset{\text{usual}}{\rightrightarrows}} \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G;\Bbbk) \xrightarrow{\text{forget}} \mathsf{H}^\bullet(\mathrm{Gr}_G;\Bbbk).$$
In the usual case, this composition factors through the forgetful map $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \to \mathsf{H}^\bullet(\mathrm{pt};\Bbbk) = \Bbbk$: thus, it vanishes on $\bigoplus_{i>0} \mathsf{H}^i_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$. In the special case, this composition is equal to the map (3.3.4).

Thus, to finish the proof, it is enough to show that the usual and special maps $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \to \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G;\Bbbk)$ coincide. This claim is essentially established in [**MR2**, proof of Lemma 3.6], following an argument of Ginzburg. (A different proof of this claim is given in [**Zh4**, Lemma 5.2.5].) We will not repeat the argument here, but will only briefly explain how to translate from our setting into that of [**MR2**]. Consider the ring homomorphism
$$\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \otimes_\Bbbk \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \to \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G;\Bbbk) \tag{3.3.5}$$
given by having the left-hand, resp. right-hand, copy of $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$ act by the usual, resp. special, action. Since $\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$ is free (and hence flat) over $\Bbbk$, the Künneth formula (in the form stated in, say, [**Ac3**, Proposition 6.7.5]) lets us identify the domain of (3.3.5) with $\mathsf{H}^\bullet_{\mathrm{L}^+G\times\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$. In view of this observation, the argument explained in the second half of the proof of [**MR2**, Lemma 3.6] (starting with equation (3.9)) shows that (3.3.5) factors through the multiplication map
$$\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \otimes_\Bbbk \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \to \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk),$$
and thus the usual and special actions coincide.

(2) As usual one can assume that $\mathscr{A}$ is supported on a single connected component of Gr_G. Recall that there exists a (Leray–Serre) convergent spectral sequence
$$E_2^{p,q} = \mathsf{H}^p_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk) \otimes_\Bbbk \mathsf{H}^q(\mathrm{Gr}_G,\mathscr{A}) \ \Rightarrow\ \mathsf{H}^{p+q}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\mathscr{A}).$$
Our assumption guarantees that $\mathsf{H}^p_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)$ vanishes unless p is even. On the other hand, since $\mathscr{A}$ is supported on a single connected component of Gr_G, all the coweights λ such that $\mathsf{F}_\lambda(\mathscr{A}) \neq 0$ belong to the same $\mathbb{Z}\mathfrak{R}$-coset. Therefore the parity of the

integer $\langle\lambda, 2\rho\rangle$ is the same for all these λ's, which implies that $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A})$ is concentrated in degrees of constant parity. These observations imply that the spectral sequence above degenerates at the E_2-term, from which we deduce our claims. □

As for (3.2.10)–(3.2.11), for $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ we have canonical morphisms of graded $\Bbbk$-modules

$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{B}) \to \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \overset{L}{\boxtimes}_\Bbbk \mathscr{B})) \tag{3.3.6}$$

and

$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)} \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G, \mathscr{B}) \to \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}). \tag{3.3.7}$$

The following lemma is analogous to Lemma 3.2.8, and its proof is similar. (In particular, this proof does not require our running assumption on $\Bbbk$.)

Lemma 3.3.10. — *For any $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ the following diagram commutes, where in the left column the upper arrow is induced by the forgetful map from equivariant cohomology to cohomology and the lower arrow is the natural one:*

$$\begin{tikzcd}
\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{B}) \arrow[r, "(3.3.6)"] & \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \boxtimes^L_\Bbbk \mathscr{B})) \arrow[dd, "\wr"', "(3.3.2)"] \\
\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G, \mathscr{B}) \arrow[u] \arrow[d] & \\
\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)} \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G, \mathscr{B}) \arrow[r, "(3.3.7)"] & \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}).
\end{tikzcd}$$

Now we combine Lemma 3.3.9 and Lemma 3.3.10. Under our assumptions on $\Bbbk$, for $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, Lemma 3.3.9 implies that we have a canonical isomorphism

$$\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_{\mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{pt};\Bbbk)} \mathsf{H}^\bullet_{\mathrm{L}^+G}(\mathrm{Gr}_G, \mathscr{B}) \cong \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{B}).$$

(We insist that this isomorphism uses the construction in the second claim of part (2) of Lemma 3.3.9, which *is* canonical, and not the *non*canonical isomorphism in the first claim of this statement.) Moreover, the upper map in the left-hand column of the diagram of Lemma 3.3.10 coincides with the composition of the lower map in this diagram with this isomorphism. This shows that the composition of the upper arrow with the right vertical arrow (i.e., the monoidal structure on F from Proposition 3.3.8) coincides with the lower horizontal arrow. In other words, the latter arrow provides a third equivalent description of the monoidal structure on F.

3.4. The central functor is monoidal

We now drop the restrictions on $\Bbbk$ from §3.3.4.

3.4.1. Monoidality isomorphism. — Our main goal in this section is to construct the "monoidality" isomorphism for Z, and to prove the analogue of property (3) of a central functor, along the following lines.

Theorem 3.4.1. — *For $\mathscr{A} \in D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, resp. $\mathscr{A} \in D^{\mathrm{b}}_{c}(\mathrm{Gr}_G, \Bbbk)$, and $\mathscr{B} \in D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, there is a natural isomorphism*

$$\phi_{\mathscr{A},\mathscr{B}} : \mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B}) \xrightarrow{\sim} \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B})$$

in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, resp. in $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$. Moreover, for $\mathscr{A}_1, \mathscr{A}_2 \in D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and $\mathscr{F} \in D^{\mathrm{b}}_I(\mathrm{Gr}, \Bbbk)$, the following diagram commutes:

$$\begin{tikzcd}
\big(\mathsf{Z}(\mathscr{A}_1) \star^I \mathsf{Z}(\mathscr{A}_2)\big) \star^I \mathscr{F} \arrow[r, "\phi_{\mathscr{A}_1,\mathscr{A}_2} \star^I \mathrm{id}_{\mathscr{F}}"] \arrow[d, "\alpha_{\mathsf{Z}(\mathscr{A}_1),\mathsf{Z}(\mathscr{A}_2),\mathscr{F}}"'] & \mathsf{Z}(\mathscr{A}_1 \star^{\mathrm{L}^+G} \mathscr{A}_2) \star^I \mathscr{F} \arrow[d, "\widetilde{\sigma}_{\mathscr{A}_1 \star^{\mathrm{L}^+G} \mathscr{A}_2, \mathscr{F}}"] \\
\mathsf{Z}(\mathscr{A}_1) \star^I \big(\mathsf{Z}(\mathscr{A}_2) \star^I \mathscr{F}\big) \arrow[d, "\mathrm{id}_{\mathsf{Z}(\mathscr{A}_1)} \star^I \widetilde{\sigma}_{\mathscr{A}_2,\mathscr{F}}"'] & \mathscr{F} \star^I \mathsf{Z}(\mathscr{A}_1 \star^{\mathrm{L}^+G} \mathscr{A}_2) \arrow[d, "\mathrm{id}_{\mathscr{F}} \star^I \phi^{-1}_{\mathscr{A}_1,\mathscr{A}_2}"] \\
\mathsf{Z}(\mathscr{A}_1) \star^I \big(\mathscr{F} \star^I \mathsf{Z}(\mathscr{A}_2)\big) \arrow[d, "\alpha^{-1}_{\mathsf{Z}(\mathscr{A}_1),\mathscr{F},\mathsf{Z}(\mathscr{A}_2)}"'] & \mathscr{F} \star^I \big(\mathsf{Z}(\mathscr{A}_1) \star^I \mathsf{Z}(\mathscr{A}_2)\big) \arrow[d, "\alpha^{-1}_{\mathscr{F},\mathsf{Z}(\mathscr{A}_1),\mathsf{Z}(\mathscr{A}_2)}"] \\
\big(\mathsf{Z}(\mathscr{A}_1) \star^I \mathscr{F}\big) \star^I \mathsf{Z}(\mathscr{A}_2) \arrow[r, "\widetilde{\sigma}_{\mathscr{A}_1,\mathscr{F}} \star^I \mathrm{id}_{\mathsf{Z}(\mathscr{A}_2)}"] & \big(\mathscr{F} \star^I \mathsf{Z}(\mathscr{A}_1)\big) \star^I \mathsf{Z}(\mathscr{A}_2).
\end{tikzcd} \tag{3.4.1}$$

Proof. — The proof is very similar to that of Theorem 3.2.3. Let $\eta : (\mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G) \times C^\circ \to \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})_{|C^\circ}$ be the unique map making the following diagram commute:

$$\begin{tikzcd}
 & (\mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G) \times C^\circ \arrow[dl, "\nu"'] \arrow[d, "\eta"] \arrow[dr, "\mu"] & \\
\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{\mathsf{y}})_{|C^\circ} \arrow[dr, "\mu"'] & & \mathrm{Gr}_G \times C^\circ \arrow[dl, "\nu"] \\
 & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})_{|C^\circ} &
\end{tikzcd} \tag{3.4.2}$$

(see (2.3.16)). Comparing the two sides of (3.4.2), we obtain natural isomorphisms

$$\tau(\mathscr{A}) \star_{C^\circ} \tau(\mathscr{B}) \cong \eta_*(\mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^\circ}[1]) \cong \tau(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}).$$

We define $\tilde{\phi} : \tau(\mathscr{A}) \star_{C^\circ} \tau(\mathscr{B}) \xrightarrow{\sim} \tau(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B})$ to be the composition of these isomorphisms. By Corollary 2.4.3 along with (3.2.1) and (3.2.2), we obtain an isomorphism

$$\phi_{\mathscr{A},\mathscr{B}} : \mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B}) \xrightarrow{\sim} \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}).$$

The rest of theorem involves the study of the following two diagrams:

$$\begin{tikzcd}
& (\mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G) \times \mathrm{Fl}_G \times C^\circ \arrow[r, "\nu"] \arrow[dl, "\nu"'] \arrow[d, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{0}, \underline{\mathsf{y}}, \underline{\mathsf{y}})_{|C^\circ} \arrow[d, "\mu"] \arrow[dr, "\mu"] & \\
\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{\mathsf{y}}, \underline{0})_{|C^\circ} \arrow[dr, "\mu"'] & \mathrm{Gr}_G \times \mathrm{Fl}_G \times C^\circ \arrow[r, "\nu"] \arrow[d, "\nu"'] & \mathbf{Gr}_{\mathcal{G}}(\underline{0}, \underline{\mathsf{y}})_{|C^\circ} \arrow[d, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cup \underline{0}, \underline{\mathsf{y}})_{|C^\circ} \arrow[dl, "\mu"] \\
& \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{0})_{|C^\circ} \arrow[r, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cup \underline{0})_{|C^\circ}, &
\end{tikzcd}$$

$$\begin{tikzcd}
& (\mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G) \times \mathrm{Fl}_G \times C^\circ \arrow[r, "\nu"] \arrow[dl, "\nu"'] \arrow[dr, "\nu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{0}, \underline{\mathsf{y}}, \underline{\mathsf{y}})_{|C^\circ} \arrow[dr, "\mu"] & \\
\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{\mathsf{y}}, \underline{0})_{|C^\circ} \arrow[r, "\mu"] \arrow[dr, "\mu"'] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{\mathsf{y}} \cup \underline{0})_{|C^\circ} \arrow[dr, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{0}, \underline{\mathsf{y}})_{|C^\circ} \arrow[l, "\mu"'] \arrow[r, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cup \underline{0}, \underline{\mathsf{y}})_{|C^\circ} \arrow[dl, "\mu"] \\
& \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{0})_{|C^\circ} \arrow[r, "\mu"] & \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cup \underline{0})_{|C^\circ}. &
\end{tikzcd}$$

Both diagrams are assembled from copies of (2.3.15), (2.4.5), and (3.4.2), and they coincide along their outermost edges. From this point on, the proof of Theorem 3.2.3 can be repeated essentially verbatim. We omit further details. $\square$

The isomorphism from Theorem 3.4.1 is compatible with monodromy in the sense of the following claim.

Proposition 3.4.2. — *For any $\mathscr{A}, \mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B}) \arrow[r, "\phi_{\mathscr{A},\mathscr{B}}"] \arrow[d, "\mathsf{m}_{\mathscr{A}} \star^I \mathsf{m}_{\mathscr{B}}"'] & \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \arrow[d, "\mathsf{m}_{\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}}"] \\
\mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B}) \arrow[r, "\phi_{\mathscr{A},\mathscr{B}}"] & \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B})
\end{tikzcd}$$

Proof. — After tracing through the construction of ϕ in the proof of Theorem 3.4.1, this claim is easily deduced from the last assertion in Corollary 2.4.3. $\square$

Recall the bifunctor $\star^I_0$ introduced in Remark 3.2.2. Theorem 3.4.1 and the t-exactness of Z (see Lemma 2.4.5) ensure that for $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ we have a bifunctorial isomorphism

$$\phi^0_{\mathscr{A},\mathscr{B}} : \mathsf{Z}(\mathscr{A}) \star^I_0 \mathsf{Z}(\mathscr{B}) \xrightarrow{\sim} \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G}_0 \mathscr{B}). \tag{3.4.3}$$

3.4.2. Projection to Gr_G. — The isomorphism of Theorem 3.4.1 has no meaningful analogue in the setting where $\mathcal{G}$ is replaced by G_C, since forgetting Z and replacing I by L^+G we obtain a tautology. This suggests that this isomorphism "disappears" when one applies the functor π_*. In this subsection, for later use we make this idea precise.

Consider the map

$$\pi \mathbin{\widetilde{\times}} \pi : \mathrm{L}G \times^I \mathrm{Fl}_G \to \mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G.$$

For $\mathscr{A}, \mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, using the fact that $\pi_* \mathsf{Z}(\mathscr{A}) \cong \mathscr{A}$ and $\pi_* \mathsf{Z}(\mathscr{B}) \cong \mathscr{B}$ (see Lemma 2.5.1), it is not difficult to check that we have a canonical isomorphism

$$(\pi \mathbin{\widetilde{\times}} \pi)_* \big(\mathsf{Z}(\mathscr{A}) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathscr{B})\big) \cong \mathscr{A} \mathbin{\widetilde{\boxtimes}} \mathscr{B}. \tag{3.4.4}$$

Recall the maps m and m' from (1.3.1) and (2.2.2), respectively. Applying the functor m_*, and using the fact that $m \circ (\pi \mathbin{\widetilde{\times}} \pi) = \pi \circ m'$, we deduce an isomorphism

$$\pi_* \big(\mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B})\big) \xrightarrow{\sim} \mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}. \tag{3.4.5}$$

Lemma 3.4.3. — *The following diagram commutes:*

$$\begin{tikzcd}
\pi_* \big(\mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B})\big) \arrow[rr, "\pi_* \phi_{\mathscr{A}, \mathscr{B}}"] \arrow[dr, "(3.4.5)"'] & & \pi_* \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \arrow[dl, "\textit{Lemma } 2.5.1"] \\
& \mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}. &
\end{tikzcd}$$

Proof. — Consider the following two commutative diagrams:

$$\begin{tikzcd}
& (\mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G) \times C^\circ \arrow[dl, "\nu"'] \arrow[d, "\nu"] \arrow[dr, "\mu"] & \\
\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{\mathsf{y}})_{|C^\circ} \arrow[r, "\varpi"] \arrow[dr, "\mu"'] & \mathbf{Gr}_{G_C}(\underline{\mathsf{y}}, \underline{\mathsf{y}})_{|C^\circ} \arrow[dr, "\mu"] & \mathrm{Gr}_G \times C^\circ \arrow[d, "\nu"] \\
& \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})_{|C^\circ} \arrow[r, "\varpi"'] & \mathbf{Gr}_{G_C}(\underline{\mathsf{y}})_{|C^\circ},
\end{tikzcd} \tag{3.4.6}$$

$$\begin{tikzcd}
& (\mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G) \times C^\circ \arrow[dl, "\nu"'] \arrow[dr, "\mu"] & \\
\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}, \underline{\mathsf{y}})_{|C^\circ} \arrow[dr, "\mu"'] & & \mathrm{Gr}_G \times C^\circ \arrow[dl, "\nu"'] \arrow[d, "\nu"] \\
& \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}})_{|C^\circ} \arrow[r, "\varpi"'] & \mathbf{Gr}_{G_C}(\underline{\mathsf{y}})_{|C^\circ}.
\end{tikzcd} \tag{3.4.7}$$

Arguing as in the proof of Theorem 3.2.3, using these diagrams we obtain a commutative diagram

$$\begin{tikzcd}
\pi_* \big(\mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B})\big) \arrow[r, "(3.4.5)", "\sim"'] \arrow[d, "\pi_* \phi_{\mathscr{A}, \mathscr{B}}"', "\wr"] & \pi_* \mathsf{Z}(\mathscr{A}) \star^{\mathrm{L}^+G} \pi_* \mathsf{Z}(\mathscr{B}) \arrow[d, "\text{Lemma 2.5.1}", "\wr"'] \\
\pi_* \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \arrow[d, "\text{Lemma 2.5.1}"', "\wr"] & \mathsf{Z}^{\mathrm{sph}}(\mathscr{A}) \star^{\mathrm{L}^+G} \mathsf{Z}^{\mathrm{sph}}(\mathscr{B}) \arrow[d, "(2.5.5)", "\wr"'] \\
\mathsf{Z}^{\mathrm{sph}}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \arrow[r, "(2.5.5)", "\sim"'] & \mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}
\end{tikzcd}$$

where the maps along the left-hand side and bottom come from (3.4.7), and those on the top and right-hand side come from (3.4.6). $\square$

The isomorphism (3.4.5) also explains the relationship between the isomorphisms $\widetilde{\sigma}_{-,-}$ and $\widetilde{\sigma}^{\mathrm{sph}}_{-,-}$, as follows.

Lemma 3.4.4. — *For any $\mathscr{A},\mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk)$ the following diagram commutes:*

$$\begin{tikzcd}
\pi_*\big(\mathsf{Z}(\mathscr{A})\star^I \mathsf{Z}(\mathscr{B})\big) \arrow[r, "\pi_*\widetilde{\sigma}_{\mathscr{A},\mathsf{Z}(\mathscr{B})}"] \arrow[d, "(3.4.5)"'] & \pi_*\big(\mathsf{Z}(\mathscr{B})\star^I \mathsf{Z}(\mathscr{A})\big) \arrow[d, "(3.4.5)"] \\
\mathscr{A}\star^{\mathrm{L}^+G}\mathscr{B} \arrow[r, "\widetilde{\sigma}^{\mathrm{sph}}_{\mathscr{A},\mathscr{B}}"] & \mathscr{B}\star^{\mathrm{L}^+G}\mathscr{A}.
\end{tikzcd}$$

Proof. — The claim follows from the compatibility of nearby cycles with proper pushforward, see Proposition 9.1.4. □

3.5. Compatibility of commutativity

The goal of this section is to establish a compatibility result between the natural transformation $\widetilde{\sigma}_{\mathscr{A},\mathscr{F}}$ of Theorem 3.2.3 and the intrinsic commutativity constraint of the symmetric monoidal category $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk)$, which specializes to property (1) in Section 3.1 in case $\Bbbk$ is a field.

3.5.1. Statement. — Recall (see §1.3.3) that the morphism of bifunctors $\sigma^{\mathrm{Fus}}_{-,-}$ is not the commutativity constraint for the monoidal category $(\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk),\star_0^{\mathrm{L}^+G})$; they differ by a sign modification. To take this into account, we will similarly modify the definition of $\widetilde{\sigma}_{\mathscr{A},\mathscr{F}}$. By additivity again, we only have to explain this modification in the case where $\mathscr{A}$ and $\mathscr{F}$ are supported on a single connected component of Gr_G and Fl_G respectively. In this case, we set

$$\sigma_{\mathscr{A},\mathscr{F}} = -\widetilde{\sigma}_{\mathscr{A},\mathscr{F}}$$

in case the connected component supporting $\mathscr{A}$ and the image in Gr_G of the connected component supporting $\mathscr{F}$ are both odd, and

$$\sigma_{\mathscr{A},\mathscr{F}} = \widetilde{\sigma}_{\mathscr{A},\mathscr{F}}$$

otherwise.

We are now ready to state the main result of this section. In the following statement, the maps $\phi^0_{\mathscr{A},\mathscr{B}}$ and $\phi^0_{\mathscr{B},\mathscr{A}}$ are defined as in (3.4.3).

Theorem 3.5.1. — *For any $\mathscr{A},\mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk)$ the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{Z}(\mathscr{A})\star^I_0 \mathsf{Z}(\mathscr{B}) \arrow[r, "\phi^0_{\mathscr{A},\mathscr{B}}"] \arrow[d, "{}^{\mathrm{p}}\mathscr{H}^0(\sigma_{\mathscr{A},\mathsf{Z}(\mathscr{B})})"'] & \mathsf{Z}(\mathscr{A}\star_0^{\mathrm{L}^+G}\mathscr{B}) \arrow[d, "\mathsf{Z}(\sigma^{\mathrm{Com}}_{\mathscr{A},\mathscr{B}})"] \\
\mathsf{Z}(\mathscr{B})\star^I_0 \mathsf{Z}(\mathscr{A}) \arrow[r, "\phi^0_{\mathscr{B},\mathscr{A}}"] & \mathsf{Z}(\mathscr{B}\star_0^{\mathrm{L}^+G}\mathscr{A}).
\end{tikzcd}$$

The proof will be given in §3.5.8, following some preliminaries.

3.5.2. Affine Grassmannians over a 2-dimensional base. — The proof of Theorem 3.5.1 will require the use of "nearby cycles over a 2-dimensional base." A general theory of nearby cycles over an arbitrary base exists, and has seen progress recently (see [**I2**]), but this is not the theory we will use. Instead, in Section 9.4 (following Gaitsgory [**G2**]) we explain how such a functor can be constructed (and sometimes computed) in the situation at hand by generalizing Beĭlinson's construction of (unipotent) nearby cycles. (The latter construction is reviewed in Section 9.2.)

In the present subsection we introduce the geometric data that will be used in this construction, namely certain analogues of the iterated affine Grassmannians introduced in Section 2.3, but which now are ind-schemes over $C^2 = \mathbb{A}^2$.

Let $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}})$ be the ind-scheme over C^2 representing the functor given by

$$R \mapsto \left\{ (x, y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{c} x, y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma_x \cup \Gamma_y}, \\ \text{and } \beta : \mathcal{E}_{|\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_x} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_x} \text{ a trivialization} \end{array} \right\}.$$

(Here, following the same conventions as in §3.5.2 and Chapter 2, in the right-hand side we denote by Γ_x, resp. Γ_y, the graph of x, resp. y, and by $\widehat{\Gamma_x \cup \Gamma_y}$ the completion of C_R along the closed subscheme $\Gamma_x \cup \Gamma_y$. Also, we tacitly consider the *equivalence classes* of quadruples $(x, y, \mathcal{E}, \beta)$ for the obvious equivalence relation.) By restricting $\mathcal{E}$ from $\widehat{\Gamma_x \cup \Gamma_y}$ to $\widehat{\Gamma}_x$ and β from $\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_x$ to $\widehat{\Gamma}^{\circ}_x$, we get a map

$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}) \to \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}} \times C.$$

Reasoning similar to that in Remark 2.2.14 shows that this is an isomorphism, which justifies the claim that this functor is indeed represented by an (ind-projective) ind-scheme over C^2.

Similar considerations allow to define the ind-scheme $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{y}})$ over C^2 representing the functor given by

$$R \mapsto \left\{ (x, y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{c} x, y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma_x \cup \Gamma_y}, \\ \text{and } \beta : \mathcal{E}_{|\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_y} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_y} \text{ a trivialization} \end{array} \right\}.$$

Next, let $\underline{\mathcal{L}^+\mathcal{G}}^{\mathrm{BD}}$ be the affine group scheme over C^2 defined by

$$R \mapsto \{(x, y, \gamma) \mid x, y \in C(R),\ \gamma \in \mathcal{G}(\widehat{\Gamma_x \cup \Gamma_y})\}.$$

(The representability of this functor can be obtained as an application of [**HR1**, Lemma 3.2], as for Lemma 2.3.1. Moreover, as in this case, this group scheme is an inverse limit of smooth group schemes of finite type over C^2.) We also let $\underline{\mathbf{Gr}}^{(\infty)}_{\mathcal{G}}(\underline{\mathsf{x}})$ be the ind-scheme over C^2 given by

$$\underline{\mathbf{Gr}}^{(\infty)}_{\mathcal{G}}(\underline{\mathsf{x}})(R) = \{(x, y, \mathcal{E}, \beta, \gamma) \mid$$
$$\left. \begin{array}{c} x, y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma_x \cup \Gamma_y}, \\ \beta : \mathcal{E}_{|\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_x} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_x} \text{ and } \gamma : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^0_{\widehat{\Gamma_x \cup \Gamma_y}} \text{ trivializations} \end{array} \right\}.$$

There is a natural right action of $\underline{\mathcal{L}^+\mathcal{G}}^{\mathrm{BD}}$ on this ind-scheme (by twisting γ), and as in Lemma 2.3.9 the map

$$\underline{\mathbf{Gr}}_{\mathcal{G}}^{(\infty)}(\underline{\mathsf{x}}) \to \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}})$$

given by forgetting γ is an étale locally trivial principal $\underline{\mathcal{L}^+\mathcal{G}}^{\mathrm{BD}}$-bundle.

Let $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}, \underline{\mathsf{y}})$ be the functor from $\mathbb{C}$-algebras to sets defined by

$$R \mapsto \{(x, y, \mathcal{E}^1, \mathcal{E}^2, \beta_1, \beta_2) \mid \\ \left. \begin{array}{c} x, y \in C(R),\ \mathcal{E}^1, \mathcal{E}^2 \text{ principal } \mathcal{G}\text{-bundles over } \widehat{\Gamma_x \cup \Gamma_y}, \\ \text{and } \beta_1 : \mathcal{E}^1_{|\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_x} \to \mathcal{E}^0_{\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_x},\ \beta_2 : \mathcal{E}^2_{|\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_y} \to \mathcal{E}^1_{|\widehat{\Gamma_x \cup \Gamma_y} \smallsetminus \Gamma_y} \text{ isom.} \end{array} \right\}.$$

In analogy with Proposition 2.3.11, one can show that

$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}, \underline{\mathsf{y}}) \cong \underline{\mathbf{Gr}}_{\mathcal{G}}^{(\infty)}(\underline{\mathsf{x}}) \times_{C^2}^{\underline{\mathcal{L}^+\mathcal{G}}^{\mathrm{BD}}} \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{y}}),$$

and thereby conclude that $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}, \underline{\mathsf{y}})$ is represented by an ind-proper ind-scheme over C^2.

Finally, given $x, y \in C(R)$, recall the scheme $(\widehat{\Gamma_x \cup \Gamma_y})^\circ = \widehat{\Gamma_x \cup \Gamma_y} \smallsetminus (\Gamma_x \cup \Gamma_y)$ from §1.3.3. We define the functor $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})$ by

$$R \mapsto \left\{ (x, y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{c} x, y \in C(R),\ \mathcal{E} \text{ a principal } \mathcal{G}\text{-bundle over } \widehat{\Gamma_x \cup \Gamma_y}, \\ \text{and } \beta : \mathcal{E}_{|(\widehat{\Gamma_x \cup \Gamma_y})^\circ} \xrightarrow{\sim} \mathcal{E}^0_{(\widehat{\Gamma_x \cup \Gamma_y})^\circ} \text{ an isomorphism} \end{array} \right\}.$$

As in the proof of Proposition 2.3.4, using [**HR1**, Corollary 3.10] (now for the noetherian ring $\mathcal{O} = \mathbb{C}[x, y]$, the $\mathcal{O}$-curve $X = C^3$, and the divisor given by the union $\{(x, y, x) : x, y \in C\} \cup \{(x, y, y) : x, y \in C\}$) one obtains that $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})$ is represented by an ind-scheme over C^2. There is an obvious proper map

$$\mu : \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}, \underline{\mathsf{y}}) \to \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})$$

given by $(x, y, \mathcal{E}^1, \mathcal{E}^2, \beta_1, \beta_2) \mapsto (x, y, \mathcal{E}^2, \beta_{1|(\widehat{\Gamma_x \cup \Gamma_y})^\circ} \circ \beta_{2|(\widehat{\Gamma_x \cup \Gamma_y})^\circ})$.

3.5.3. Convolution. — We are now ready to combine these ingredients. Using the natural maps

$$\begin{aligned} (3.5.1) \quad \mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}} \times \mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}} &\cong \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}) \times_{C^2} \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{y}}) \xleftarrow{p} \underline{\mathbf{Gr}}_{\mathcal{G}}^{(\infty)}(\underline{\mathsf{x}}) \times_{C^2} \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{y}}) \\ &\xrightarrow{q} \underline{\mathbf{Gr}}_{\mathcal{G}}^{(\infty)}(\underline{\mathsf{x}}) \times_{C^2}^{\underline{\mathcal{L}^+\mathcal{G}}^{\mathrm{BD}}} \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{y}}) \cong \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}, \underline{\mathsf{y}}) \xrightarrow{\mu} \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}}) \end{aligned}$$

we can define a twisted external tensor product and a convolution product over C^2, denoted by

$$\widetilde{\boxtimes}_{C^2} : D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}, \Bbbk) \times D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}}(\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}, \Bbbk) \to D^{\mathrm{b}}_c(\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}, \underline{\mathsf{y}}), \Bbbk),$$
$$\star_{C^2} : D^{\mathrm{b}}_c(\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}, \Bbbk) \times D^{\mathrm{b}}_{\mathcal{L}^+\mathcal{G}}(\mathbf{Gr}_{\mathcal{G}}^{\mathrm{Cen}}, \Bbbk) \to D^{\mathrm{b}}_c(\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}}), \Bbbk)$$

respectively, by the same procedure as in §1.3.1 or in §2.4.3. Of course, they are related by

$$\mu_*(\mathscr{F} \,\widetilde{\boxtimes}_{C^2}\, \mathscr{G}) = \mathscr{F} \star_{C^2} \mathscr{G}.$$

These bifunctors have obvious analogues over $(C^\circ)^2$ instead of C^2, which will be denoted $\widetilde{\boxtimes}_{(C^\circ)^2}$ and $\star_{(C^\circ)^2}$ respectively.

3.5.4. Relation with the fusion product. — The ind-schemes introduced above in §3.5.2 have obvious analogues in the setting where the group scheme $\mathcal{G}$ is replaced by the constant group scheme G_C, which have in fact already appeared in the construction of the fusion product from §1.3.3 (see also [**BR**, §1.7] for more details). Namely, the ind-scheme $\underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}},\underline{\mathsf{y}})$ coincides with the one denoted $\mathrm{Gr}_{G,X}\widetilde{\times}\mathrm{Gr}_{G,X}$ in [**BR**, §1.7.4][(1)]; the ind-scheme $\underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})$ coincides with the one denoted Gr_{G,X^2} in [**BR**, §1.7.3] (and denoted Fus_G in §1.3.3); and the analogues of the constructions from §3.5.3 in this setting were considered in [**BR**, §1.7.5]. We will use the same notation for these variants.

We have a canonical isomorphism

$$\underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|C^2\smallsetminus\Delta C} \xrightarrow{\sim} \big(\underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}}) \times_{C^2} \underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{y}})\big)_{|C^2\smallsetminus\Delta C}$$

which sends the datum of $(\mathcal{E}^1,\mathcal{E}^2,\beta_1,\beta_2)$ to $((\mathcal{E}^1,\beta_1),(\tilde{\mathcal{E}}^2,\tilde{\beta}_2))$ where $\tilde{\mathcal{E}}^2$ is obtained by gluing the trivial bundle away from Γ_y with $\mathcal{E}^2$ away from Γ_x using the composition of β_2 and β_1 as gluing datum. In turn, we have a natural identification

$$\big(\underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}}) \times_{C^2} \underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{y}})\big)_{|C^2\smallsetminus\Delta C} \xrightarrow{\sim} \mathrm{Gr}_G \times \mathrm{Gr}_G \times (C^2 \smallsetminus \Delta C)$$

obtained by moving x and y to the origin, under which the restriction of $\tau(\mathscr{A})\,\widetilde{\boxtimes}_{C^2}\,\tau(\mathscr{B})$ identifies with $\mathscr{A} \boxtimes^L \mathscr{B} \boxtimes^L \underline{\Bbbk}[2]$, for any $\mathscr{A},\mathscr{B}$ in $D^{\mathrm{b}}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk)$.

The natural morphism

$$\underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}},\underline{\mathsf{y}}) \to \underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})$$

restricts to an isomorphism over $C^2 \smallsetminus \Delta C$ and, from this point of view, [**BR**, Lemma 1.7.10] states that for $\mathscr{A},\mathscr{B}$ perverse sheaves such that $\mathscr{A} \boxtimes^L_{\Bbbk} \mathscr{B}$ is perverse we have

$$\tau(\mathscr{A}) \star_{C^2} \tau(\mathscr{B}) \cong j_{!*}(\mathscr{A} \overset{L}{\boxtimes} \mathscr{B} \overset{L}{\boxtimes} \underline{\Bbbk}[2])[-1],$$

where we use the notation of §1.3.3. Restricting to the diagonal we obtain the isomorphism

$$\tau(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \cong i^* j_{!*}(\mathscr{A} \overset{L}{\boxtimes} \mathscr{B} \overset{L}{\boxtimes} \underline{\Bbbk}[2])$$

from (1.3.4). Now, exchanging the points x and y defines an automorphism

$$\mathrm{swap} : \underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}}) \xrightarrow{\sim} \underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})$$

which stabilizes the preimage of $C^2 \smallsetminus \Delta C$, and restricts on this preimage to the automorphism of $\mathrm{Gr}_G \times \mathrm{Gr}_G \times (C^2 \smallsetminus \Delta C)$ sending $(a,b,(x,y))$ to $(b,a,(y,x))$. The commutativity constraint $\sigma^{\mathrm{Fus}}_{\mathscr{A},\mathscr{B}}$ is then obtained using the fact that $i = \mathrm{swap} \circ i$, so that $i^* \cong i^* \circ \mathrm{swap}^*$.

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(1) In [**BR**], the curve $\mathbb{A}^1$ was denoted X rather than C.

3.5.5. Restriction to subvarieties of C^2. — The ind-schemes defined above can be restricted to various subsets of C^2, such as the open subset $(C^\circ)^2$, or the diagonal copy ΔC of C inside C^2. We also consider $\Delta C^\circ := (C^\circ)^2 \cap \Delta C$, and the open subset

$$C^{2\dagger} := (C^\circ)^2 \smallsetminus \Delta C^\circ.$$

The following observations are essentially immediate from the definitions.

1. Over $(C^\circ)^2$, since the restrictions of $\mathcal{G}$ and G_C coincide we have canonical identifications

$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|(C^\circ)^2} \cong \underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|(C^\circ)^2},$$
$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|(C^\circ)^2} \cong \underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|(C^\circ)^2} = (\mathrm{Fus}_G)_{|(C^\circ)^2}.$$

2. Over $\{0\}\times C$ we have

$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|\{0\}\times C} \cong \mathbf{Gr}_{\mathcal{G}}(\underline{0},\underline{\mathsf{y}}), \quad \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|\{0\}\times C} \cong \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}\cup\underline{0}).$$

3. Similarly, over $C\times\{0\}$ we have

$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|C\times\{0\}} \cong \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}},\underline{0}), \quad \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|C\times\{0\}} \cong \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}}\cup\underline{0}).$$

4. Over ΔC, we have

$$(\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}) \times_{C^2} \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{y}}))_{|\Delta C} \cong \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}} \times_C \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}},$$
$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|\Delta C} \cong \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}},\underline{\mathsf{y}}),$$
$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|\Delta C} \cong \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}.$$

5. Over $C^{2\dagger}$, the morphism

$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|C^{2\dagger}} \to \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|C^{2\dagger}}$$

   is an isomorphism, and these ind-schemes identify canonically with $\mathrm{Gr}_G \times \mathrm{Gr}_G \times C^{2\dagger}$.

3.5.6. Iterated cleanness. — Below we will make use of the "2-dimensional nearby cycles" construction from Section 9.4, applied to the ind-schemes $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})$ and $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})$. For this, it will be convenient to let $i^{\Delta\times}$ denote either of the inclusion maps

$$\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|\Delta C^\circ} \hookrightarrow \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|(C^\circ)^2}, \quad \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|\Delta C^\circ} \hookrightarrow \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|(C^\circ)^2}.$$

If $\mathscr{F}$ is a perverse sheaf that lives on $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|(C^\circ)^2}$ or $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|(C^\circ)^2}$, one can ask whether it is *iterated-clean* in the sense of Definition 9.4.6.

Lemma 3.5.2. — *Let $\mathscr{A},\mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk)$ be such that $\mathscr{A} \boxtimes^L_{\Bbbk} \mathscr{B}$ is perverse. Then $\tau(\mathscr{A})\, \widetilde{\boxtimes}_{(C^\circ)^2}\, \tau(\mathscr{B})$ and $\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B})$ are iterated-clean.*

Proof. — Consider the perverse sheaves $\tau(\mathscr{A}),\tau(\mathscr{B})$ in $\mathsf{Perv}_{\mathcal{L}^+\mathcal{G}_{|C^\circ}}(\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}{}_{|C^\circ},\Bbbk)$. Our assumption implies that the object

$$\tau(\mathscr{A}) \overset{L}{\boxtimes}_{\Bbbk} \tau(\mathscr{B}) \cong \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{(C^\circ)^2}[2]$$

is perverse. It is iterated-clean by Lemma 9.4.15. Since the maps p and q involved in the definition of convolution are smooth (and surjective), and the map μ is proper, applying Lemma 9.4.7 repeatedly yields the result. $\square$

For the rest of this subsection and the next one we fix $\mathscr{A}, \mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ such that $\mathscr{A} \boxtimes^L_{\Bbbk} \mathscr{B}$ is perverse.

Lemma 3.5.3. — *Let $\mathscr{A}, \mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ be such that $\mathscr{A} \boxtimes^L_{\Bbbk} \mathscr{B}$ is perverse.*

1. *Let $h : \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})_{|C^{2\dagger}} \hookrightarrow \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})_{(C^\circ)^2}$ be the inclusion map. Then there is a natural isomorphism*
$$\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B}) \cong h_{!*}(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B} \overset{L}{\boxtimes}_{\Bbbk} \underline{\Bbbk}_{C^{2\dagger}}[2]).$$
2. *There is a natural isomorphism*
$$(i^{\Delta \times})^* \big(\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B})\big)[-1] \cong \tau(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}).$$

In particular, these complexes are perverse.

Proof. — These statements are clear from the study of the scheme $\underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})$ in §3.5.4, since $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})$ and $\underline{\mathbf{Gr}}_{G_C}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})$ coincide over $(C^\circ)^2$ (see §3.5.5). $\square$

3.5.7. Study of nearby cycles over C^2. — We continue with our $\mathscr{A}, \mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ such that $\mathscr{A} \boxtimes^L_{\Bbbk} \mathscr{B}$ is perverse. We first want to apply the formalism of Section 9.4 to the ind-scheme $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}, \underline{\mathsf{y}})$ over C^2 and the perverse sheaf $\tau(\mathscr{A}) \widetilde{\boxtimes}_{(C^\circ)^2} \tau(\mathscr{B})$. By Lemma 3.5.2 this object is iterated clean. In order to apply Proposition 9.4.8 (which guarantees in particular that the 2-dimensional nearby cycles construction is well defined), we need to check two further conditions, stating that some nearby cycles are unipotent. By Lemma 9.4.16 and Remark 2.4.7(2) these assumptions are satisfied by the object $\tau(\mathscr{A}) \boxtimes^L_{\Bbbk} \tau(\mathscr{B})$ on $\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}} \times \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}$. Using Remark 9.4.14 we deduce that they also hold for $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}, \underline{\mathsf{y}})$ and $\tau(\mathscr{A}) \widetilde{\boxtimes}_{(C^\circ)^2} \tau(\mathscr{B})$. Therefore, applying Proposition 9.4.8 one can consider the objects

$$\begin{aligned}
\mathsf{F}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A}, \mathscr{B}) &= \Upsilon_{\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})}\big(\tau(\mathscr{A}) \widetilde{\boxtimes}_{(C^\circ)^2} \tau(\mathscr{B})\big),\\
\mathsf{F}^{(2)}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A}, \mathscr{B}) &= \Psi^{(2)}_{\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})}\big(\tau(\mathscr{A}) \widetilde{\boxtimes}_{(C^\circ)^2} \tau(\mathscr{B})\big),\\
\mathsf{F}^{\Delta}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A}, \mathscr{B}) &= \Psi^{\mathrm{un}}_{\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})_{|\Delta C}}\big((i^{\Delta\times})^*(\tau(\mathscr{A}) \widetilde{\boxtimes}_{(C^\circ)^2} \tau(\mathscr{B}))[-1]\big).
\end{aligned}$$

Now, consider the ind-scheme $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}} \cup \underline{\mathsf{y}})$ over C^2 and the perverse sheaf $\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B})$ (see Lemma 3.5.3). Using again Remark 9.4.14 we obtain that the conditions of Proposition 9.4.8 are again satisfied in this context, so that one can consider the objects

$$\begin{aligned}
\mathsf{F}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A}, \mathscr{B}) &= \Upsilon_{\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}\big(\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B})\big),\\
\mathsf{F}^{(2)}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A}, \mathscr{B}) &= \Psi^{(2)}_{\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}\big(\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B})\big),\\
\mathsf{F}^{\Delta}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A}, \mathscr{B}) &= \Psi^{\mathrm{un}}_{\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})_{|\Delta C}}\big((i^{\Delta\times})^*(\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B}))[-1]\big).
\end{aligned}$$

Lemma 3.5.4. — *Let $\mathscr{A}, \mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ be such that $\mathscr{A} \boxtimes^L_\Bbbk \mathscr{B}$ is perverse. We then have natural isomorphisms*

$$\mathsf{F}^{(2)}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \xleftarrow{\sim} \mathsf{F}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \xrightarrow{\sim} \mathsf{F}^{\Delta}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}), \tag{3.5.2}$$

and likewise for the $(\underline{x} \cup \underline{y})$ versions. Moreover, there is a commutative diagram

$$\begin{tikzcd} \mu_* \mathsf{F}^{(2)}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \arrow[d, "\wr"'] & \mu_* \mathsf{F}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mu_* \mathsf{F}^{\Delta}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \arrow[d, "\wr"] \\ \mathsf{F}^{(2)}_{(\underline{x}\cup\underline{y})}(\mathscr{A}, \mathscr{B}) & \mathsf{F}_{(\underline{x}\cup\underline{y})}(\mathscr{A}, \mathscr{B}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathsf{F}^{\Delta}_{(\underline{x}\cup\underline{y})}(\mathscr{A}, \mathscr{B}) \end{tikzcd} \tag{3.5.3}$$

where the horizontal maps are induced by (3.5.2) *and its variant for $(\underline{x} \cup \underline{y})$.*

Proof. — As explained above Proposition 9.4.8 applies in both settings, which justifies the isomorphisms

$$\mathsf{F}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \xrightarrow{\sim} \mathsf{F}^{(2)}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}), \quad \mathsf{F}_{(\underline{x}\cup\underline{y})}(\mathscr{A}, \mathscr{B}) \xrightarrow{\sim} \mathsf{F}^{(2)}_{(\underline{x}\cup\underline{y})}(\mathscr{A}, \mathscr{B}).$$

The isomorphism in the left column of (3.5.3) and the commutativity of the left part of this diagram follow from Corollary 9.4.9 and its proof.

By Lemma 9.4.12, we also have natural maps

$$\mathsf{F}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \to \mathsf{F}^{\Delta}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}), \quad \mathsf{F}_{(\underline{x}\cup\underline{y})}(\mathscr{A}, \mathscr{B}) \to \mathsf{F}^{\Delta}_{(\underline{x}\cup\underline{y})}(\mathscr{A}, \mathscr{B}). \tag{3.5.4}$$

To check that these maps are isomorphisms, we first work on $\mathbf{Gr}_{\mathcal{G}}(\underline{x}) \times_{C^2} \mathbf{Gr}_{\mathcal{G}}(\underline{y})$. In this setting, as above we have a morphism

$$\Upsilon_{\mathbf{Gr}_{\mathcal{G}}(\underline{x}) \times_{C^2} \mathbf{Gr}_{\mathcal{G}}(\underline{y})}(\tau(\mathscr{A}) \overset{L}{\boxtimes}_\Bbbk \tau(\mathscr{B})) \to \Psi^{\mathrm{un}}_{\mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}} \times_C \mathbf{Gr}^{\mathrm{Cen}}_{\mathcal{G}}}(\tau(\mathscr{A}) \overset{L}{\boxtimes}_\Bbbk \tau(\mathscr{B})), \tag{3.5.5}$$

which now is known to be an isomorphism thanks to Proposition 9.4.18. The first map in (3.5.4) is related to that in (3.5.5) by smooth pullback along the maps p and q from (3.5.1), so it is an isomorphism by Proposition 9.4.13(2).

For the second map in (3.5.4), we use the proper map μ from (3.5.1). The fact that our map is an isomorphism follows from Proposition 9.4.13(1). Similarly, the isomorphism in the right column of (3.5.3), and the commutativity of the right part of this diagram, follow from the proof of this proposition. □

3.5.8. Proof of Theorem 3.5.1. — We can finally give the proof of Theorem 3.5.1. So, we fix $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$.

Assume for now that $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A})$ is flat over $\Bbbk$. This implies that $\mathscr{A} \boxtimes^L_\Bbbk \mathscr{B}$ is a perverse sheaf (see Remark 1.3.10), so that the discussion above can be applied. Using the compatibility of (iterated) nearby cycles with smooth pullback along the maps p and q in (3.5.1) and the considerations in §9.4.6, we obtain an isomorphism

$$\mathsf{F}^{(2)}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \cong \mathsf{Z}(\mathscr{A}) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathscr{B}). \tag{3.5.6}$$

On the other hand, Lemma 2.4.2 implies that we have

$$\mathsf{F}^{\Delta}_{(\underline{x},\underline{y})}(\mathscr{A}, \mathscr{B}) \cong \mathsf{Z}(\mathscr{A}) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathscr{B}).$$

In fact, unwrapping the constructions we see that the isomorphism

$$\mathsf{F}^{(2)}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \xrightarrow{\sim} \mathsf{F}^{\Delta}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B})$$

obtained by composing the two identifications in (3.5.2) is the identity.

For the complexes constructed using the $(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})$ variant, we remark that from the definition (see the discussion following (9.4.1)), if $Z \subset (C^\circ)^2$ is a closed subset that remains closed in $C \times C^\circ$, then the output of the functor $\Psi^{(2)}$ depends only on the restriction of the given perverse sheaf to $(C^\circ)^2 \smallsetminus Z$. In particular, this holds for $Z = \Delta C^\circ$, in which case $(C^\circ)^2 \smallsetminus Z = C^{2\dagger}$. Over $C^{2\dagger}$, the ind-schemes $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}},\underline{\mathsf{y}})$, $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})$ and $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}) \times_{C^2} \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{y}})$ are all canonically identified (see §3.5.5), and from this we obtain a canonical isomorphism

$$\mathsf{F}^{(2)}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \cong \mathsf{C}(\mathscr{B},\mathsf{Z}(\mathscr{A})),$$

where C is defined as in §3.2.1. On the other hand, from the definitions we see that

$$\mathsf{F}^{\Delta}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \cong \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}). \tag{3.5.7}$$

Combining these identifications, we conclude that the commutative diagram (3.5.3) can be rewritten as

$$\begin{tikzcd}
\mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B}) \arrow[d, "\wr"'] & \mu_*\mathsf{F}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B}) \arrow[d, "\wr"] \\
\mathsf{C}(\mathscr{B},\mathsf{Z}(\mathscr{A})) & \mathsf{F}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}).
\arrow[from=1-1, to=1-3, bend left, "\mathrm{id}"]
\end{tikzcd} \tag{3.5.8}$$

Moreover, here the right vertical isomorphism coincides with the isomorphism $\phi_{\mathscr{A},\mathscr{B}}$ from Theorem 3.4.1, and the left vertical isomorphism coincides with that of Proposition 3.2.1(2).

In the construction of $\mathsf{F}^{(2)}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B})$ and $\mathsf{F}^{(2)}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B})$, we have made a choice of first applying nearby cycles in the direction of $\{0\}\times C \subset C^2$, and then in the direction of $\{(0,0)\} \subset C\times\{0\}$. But we could have also first taken nearby cycles in the direction of $C \times \{0\} \subset C^2$, and then in the direction of $\{(0,0)\} \subset C \times \{0\}$. If we denote by

$$\mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}), \qquad \mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B})$$

the objects obtained in this way, then we have canonical isomorphisms

$$\mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \cong \mathsf{Z}(\mathscr{A}) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathscr{B}), \quad \mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \cong \mathsf{C}(\mathscr{A},\mathsf{Z}(\mathscr{B})). \tag{3.5.9}$$

We also have a canonical isomorphism

$$\mu_*\mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \cong \mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}),$$

which now coincides under the identifications (3.5.9) with the isomorphism of Proposition 3.2.1(1), and these complexes are related to $\mathsf{F}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B})$ and $\mathsf{F}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B})$ by canonical isomorphisms.

For the next step, we observe as in §3.5.4 that we have an automorphism of the ind-scheme $\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})$ obtained by exchanging the points x and y, which we will also denote by $\mathrm{swap} : \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}}) \to \underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})$. This automorphism preserves the preimage of $\{(0,0)\}$, and induces the identity on this preimage. As in the construction of $\sigma^{\mathrm{Fus}}_{\mathscr{A},\mathscr{B}}$, we therefore obtain a canonical isomorphism

$$\Upsilon_{\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}\big(\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B})\big) \xrightarrow{\sim} \Upsilon_{\underline{\mathbf{Gr}}_{\mathcal{G}}(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}\big(\mathrm{swap}^*(\tau(\mathscr{A}) \star_{(C^\circ)^2} \tau(\mathscr{B}))\big),$$

or in other words (in view of Lemma 3.5.3(1)) a canonical isomorphism

$$\mathsf{F}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \xrightarrow{\sim} \mathsf{F}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{B},\mathscr{A}). \tag{3.5.10}$$

More generally swap restricts to the identity on the preimage of ΔC, so that we similarly obtain an isomorphism

$$\mathsf{F}^{\Delta}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \xrightarrow{\sim} \mathsf{F}^{\Delta}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{B},\mathscr{A}) \tag{3.5.11}$$

which, by construction (see §3.5.4), identifies under the isomorphism (3.5.7) and its version for the pair $(\mathscr{B},\mathscr{A})$ with $\mathsf{Z}(\sigma^{\mathrm{Fus}}_{\mathscr{A},\mathscr{B}})$.

The automorphism swap does not stabilize the preimage of $\{0\}\times C$, so it does not "commute" with $\mathsf{F}^{(2)}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}$ in the sense above. Instead, we obtain a canonical isomorphism

$$\mathsf{F}^{(2)}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \xrightarrow{\sim} \mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{B},\mathscr{A}). \tag{3.5.12}$$

As explained above, both of these complexes identify with $\mathsf{C}(\mathscr{B},\mathsf{Z}(\mathscr{A}))$, and in fact the isomorphism so constructed is just the identity map of this complex. But this identification can be combined with the isomorphisms

$$\mu_*\mathsf{F}^{(2)}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \xrightarrow{\sim} \mathsf{F}^{(2)}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}), \quad \mu_*\mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{B},\mathscr{A}) \xrightarrow{\sim} \mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}}\cup\underline{\mathsf{y}})}(\mathscr{B},\mathscr{A})$$

to provide a canonical isomorphism

$$\mu_*\mathsf{F}^{(2)}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{A},\mathscr{B}) \xrightarrow{\sim} \mu_*\mathsf{F}^{(2)\prime}_{(\underline{\mathsf{x}},\underline{\mathsf{y}})}(\mathscr{B},\mathscr{A}),$$

which via the identifications (3.5.6) and (3.5.9) coincides with $\widetilde{\sigma}_{\mathscr{A},\mathsf{Z}(\mathscr{B})}$.

Of course the isomorphisms (3.5.10), (3.5.11) and (3.5.12) are compatible in the obvious way, and from the identifications above we deduce the commutativity of a diagram similar to that of Theorem 3.5.1, but for the isomorphisms $\widetilde{\sigma}_{\mathscr{A},\mathsf{Z}(\mathscr{B})}$ and $\sigma^{\mathrm{Fus}}_{\mathscr{A},\mathscr{B}}$. Twisting by the appropriate signs, this proves the theorem in the special case where $\mathsf{H}^\bullet(\mathrm{Gr}_G,\mathscr{A})$ is flat over $\Bbbk$.

For the general case, recall that any $\mathscr{A} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr},\Bbbk)$ can be written as a quotient of a perverse sheaf $\mathscr{A}'$ such that $\mathsf{H}^\bullet(\mathrm{Gr}_G,\mathscr{A}')$ is flat (see [**BR**, Proof of Lemma 1.10.10]). Consider the commutative diagram

$$\begin{array}{ccc}
\mathsf{Z}(\mathscr{A}') \star^I \mathsf{Z}(\mathscr{B}) & \longrightarrow & \mathsf{Z}(\mathscr{A}) \star^I_0 \mathsf{Z}(\mathscr{B}) \\
{\scriptstyle \phi_{\mathscr{B},\mathscr{A}'}\sigma_{\mathscr{A}',\mathsf{Z}(\mathscr{B})} - \mathsf{Z}(\sigma^{\mathrm{Com}}_{\mathscr{A}',\mathscr{B}})\phi_{\mathscr{A}',\mathscr{B}}}\big\downarrow & & \big\downarrow{\scriptstyle \phi^0_{\mathscr{B},\mathscr{A}}\,{}^{\mathrm{p}}\mathscr{H}^0(\sigma_{\mathscr{A},\mathsf{Z}(\mathscr{B})}) - \mathsf{Z}(\sigma^{\mathrm{Com}}_{\mathscr{A},\mathscr{B}})\phi^0_{\mathscr{A},\mathscr{B}}} \\
\mathsf{Z}(\mathscr{B} \star^{\mathrm{L}^+G} \mathscr{A}') & \longrightarrow & \mathsf{Z}(\mathscr{B} \star^{\mathrm{L}^+G}_0 \mathscr{A}),
\end{array}$$

where the horizontal maps are induced by the quotient morphism $\mathscr{A}' \to \mathscr{A}$. The left-hand vertical map is 0 by the special case of the theorem that is already proved. The upper horizontal arrow is surjective by Corollary 3.2.5. It follows that the right-hand vertical map is 0 as well, which finishes the proof of the theorem.

3.5.9. Alternative description of the commutativity constraint of the Satake category. — We conclude this chapter with some consequences of Theorem 3.5.1.

First, as a corollary of Theorem 3.5.1 and Lemma 3.4.3 we get a commutative diagram

$$\begin{tikzcd} {}^{\mathrm{p}}\mathscr{H}^0(\pi_*(\mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B}))) \arrow[r, "\sim", "(3.4.5)"'] \arrow[d, "{}^{\mathrm{p}}\mathscr{H}^0(\pi_*\sigma_{\mathscr{A},\mathsf{Z}(\mathscr{B})})"'] & \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B} \arrow[d, "\sigma^{\mathrm{Com}}_{\mathscr{A},\mathscr{B}}"] \\ {}^{\mathrm{p}}\mathscr{H}^0(\pi_*(\mathsf{Z}(\mathscr{B}) \star^I \mathsf{Z}(\mathscr{A}))) \arrow[r, "\sim", "(3.4.5)"'] & \mathscr{B} \star_0^{\mathrm{L}^+G} \mathscr{A} \end{tikzcd}$$

for all $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. Informally, this diagram says that "π_* sends $\sigma_{\mathscr{A},\mathsf{Z}(\mathscr{B})}$ to $\sigma^{\mathrm{Com}}_{\mathscr{A},\mathscr{B}}$."

If we further take Lemma 3.4.4 into account we obtain the following corollary, which can be used to provide an alternative description of the commutativity constraint $\sigma^{\mathrm{Com}}_{-,-}$.

Corollary 3.5.5. — *For $\mathscr{A}, \mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ we have*

$$ {}^{\mathrm{p}}\mathscr{H}^0(\widetilde{\sigma}^{\mathrm{sph}}_{\mathscr{A},\mathscr{B}}) = \sigma^{\mathrm{Fus}}_{\mathscr{A},\mathscr{B}} : \mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B} \xrightarrow{\sim} \mathscr{B} \star_0^{\mathrm{L}^+G} \mathscr{A}. $$

CHAPTER 4

CENTRAL SHEAVES, WAKIMOTO SHEAVES, AND THE SATAKE EQUIVALENCE

In this chapter we establish a number of properties of central sheaves due to Arkhipov–Bezrukavnikov [**AB**]. In particular we explain the construction (due to Mirković) of "Wakimoto sheaves," which are certain I-equivariant perverse sheaves on Fl_G parametrized by $\mathbf{X}^\vee$ which "categorify" the Bernstein elements in the affine Hecke algebra (in a sense that will be made precise in Chapter 5). We then show that for each $\mathscr{A}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ the central sheaf $\mathsf{Z}(\mathscr{A})$ admits a canonical filtration whose subquotients are Wakimoto sheaves, with multiplicities governed by weight spaces of $\mathcal{S}(\mathscr{A})$. The proof of this fact uses in a crucial way the results on centrality and convolution-exactness proved in Chapter 3.

We then explore some consequences of this property, and use it in particular to provide yet another equivalent description of the fiber functor F. These properties are essential for applications of central sheaves to representation theory, and in particular for the construction of the Arkhipov–Bezrukavnikov equivalence explained in Chapter 6.

In the constructions from [**AB**], a distinguished role is played by the dominant weights. Here, following Zhu [**Zh1**], we generalize these constructions in a way that allows this role to be played by any choice of Weyl chamber. The other main difference from [**AB**] is that we allow the coefficients to be an arbitrary noetherian commutative ring of finite global dimension, instead of a field of characteristic 0.

4.1. Standard sheaves, costandard sheaves, and their convolutions

4.1.1. Affine Weyl group combinatorics. — The (extended) affine Weyl group is the semi-direct product

$$W := W_{\mathrm{f}} \ltimes \mathbf{X}^\vee.$$

To avoid any possible confusion, for $\lambda \in \mathbf{X}^\vee$ we will denote by $\mathsf{t}(\lambda)$ the element $(e, \lambda) \in W$ (where $e \in W_{\mathrm{f}}$ is the unit element). The group W parametrizes the I-orbits on Fl_G; more precisely if $w = \mathsf{t}(\lambda)v$ with $\lambda \in \mathbf{X}^\vee$ and $v \in W_{\mathrm{f}}$, we set

$$\mathrm{Fl}_{G,w} := I \cdot x^\lambda \dot{v} I/I \subset \mathrm{Fl}_G,$$

where $\dot{v}$ is any choice of lift of v in $N_G(T)$. Then we have

$$(\mathrm{Fl}_G)_{\mathrm{red}} = \bigsqcup_{w \in W} \mathrm{Fl}_{G,w},$$

see [**PR**, Proposition 8.1]. If we denote by $\ell(w)$ the dimension of $\mathrm{Fl}_{G,w}$, then a formula due to Iwahori–Matsumoto [**IM**] states that if $w = x\mathrm{t}(\lambda)$ with $x \in W_{\mathrm{f}}$ and $\lambda \in \mathbf{X}^\vee$, we have

$$\ell(w) = \sum_{\substack{\alpha \in \mathfrak{R}_+ \\ x(\alpha) \in \mathfrak{R}_+}} |\langle \lambda, \alpha \rangle| + \sum_{\substack{\alpha \in \mathfrak{R}_+ \\ x(\alpha) \in -\mathfrak{R}_+}} |\langle \lambda, \alpha \rangle + 1|. \tag{4.1.1}$$

This function ℓ defines a structure of a "quasi-Coxeter group" on W. In more detail, recall that $\mathfrak{R}^\vee \subset \mathbf{X}^\vee$ denotes the set of coroots, and set

$$W_{\mathrm{Cox}} := W_{\mathrm{f}} \ltimes \mathbb{Z}\mathfrak{R}^\vee \subset W.$$

If we also set $S := \{w \in W_{\mathrm{Cox}} \mid \ell(w) = 1\}$, then the pair (W_{Cox}, S) is a Coxeter system, with length function the restriction of ℓ. Furthermore, if we set $\Omega := \{w \in W \mid \ell(w) = 0\}$, then conjugation by any element of Ω stabilizes S, and hence acts on W_{Cox} as a Coxeter group automorphism. Finally, multiplication in W induces a group isomorphism

$$\Omega \ltimes W_{\mathrm{Cox}} \xrightarrow{\sim} W.$$

Remark 4.1.1. — One can easily check that the composition of natural projections

$$W = W_{\mathrm{f}} \ltimes \mathbf{X}^\vee \to \mathbf{X}^\vee \to \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$$

factors through a group isomorphism $W/W_{\mathrm{Cox}} \xrightarrow{\sim} \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$. In particular we have a group isomorphism $\Omega \xrightarrow{\sim} \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$, showing that Ω is a finitely generated abelian group.

This bijection can be made explicit as follows. Recall the subset $\mathbf{X}^\vee_{\mathrm{min}} \subset \mathbf{X}^\vee$ of minuscule coweights, see §1.2.1.6. Then it is known that we have bijections

$$\Omega \xleftarrow{\sim} \mathbf{X}^\vee_{\mathrm{min}} \xrightarrow{\sim} \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$$

where the right arrow is the composition of natural maps $\mathbf{X}^\vee_{\mathrm{min}} \hookrightarrow \mathbf{X}^\vee \twoheadrightarrow \mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$ and the left arrow sends λ to $\mathrm{t}(\lambda)v$, where $v \in W_{\mathrm{f}}$ is the unique element of minimal length such that $v(\lambda) \in -\mathbf{X}^\vee_+$.

The Bruhat order $\leq_{\mathrm{Bru}}$ on W_{Cox} can be extended in a canonical way to W by setting, for $w, w' \in W_{\mathrm{Cox}}$ and $\omega, \omega' \in \Omega$,

$$w\omega \leq_{\mathrm{Bru}} w'\omega' \quad \text{iff } \omega = \omega' \text{ and } w \leq_{\mathrm{Bru}} w'.$$

We then have, for any $w, w' \in W$,

$$\mathrm{Fl}_{G,w} \subset \overline{\mathrm{Fl}_{G,w'}} \quad \text{iff} \quad w \leq_{\mathrm{Bru}} w'. \tag{4.1.2}$$

We denote by $\mathbf{X}^\vee_+ \subset \mathbf{X}^\vee$ the subset consisting of dominant coweights. If λ is dominant, then $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \mathfrak{R}_+$, so (4.1.1) implies that

$$\ell(\mathrm{t}(\lambda)) = \langle \lambda, 2\rho \rangle \quad \text{for } \lambda \in \mathbf{X}^\vee_+. \tag{4.1.3}$$

In particular, it follows that

$$\ell(\mathsf{t}(\lambda+\mu)) = \ell(\mathsf{t}(\lambda)) + \ell(\mathsf{t}(\mu)) \quad \text{for } \lambda, \mu \in \mathbf{X}^\vee_+. \tag{4.1.4}$$

Similarly, using (4.1.1) one sees that

$$\ell(\mathsf{t}(x\lambda)) = \ell(\mathsf{t}(\lambda)) \quad \text{for } \lambda \in \mathbf{X}^\vee \text{ and } x \in W_{\mathrm{f}}. \tag{4.1.5}$$

It is also well known that for $\lambda, \mu \in \mathbf{X}^\vee_+$ we have

$$\mathsf{t}(\lambda) \leq_{\mathrm{Bru}} \mathsf{t}(\mu) \quad \text{iff} \quad \mu - \lambda \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+. \tag{4.1.6}$$

This can e.g. by justified as follows. Observe first that $\mathsf{t}(\lambda) \leq_{\mathrm{Bru}} \mathsf{t}(\mu)$ iff $\mathsf{t}(-\lambda) \leq_{\mathrm{Bru}} \mathsf{t}(-\mu)$. Now using (4.1.1) one can check that for any $\nu \in -\mathbf{X}^\vee_+$, the element $\mathsf{t}(\nu)$ is minimal in the coset $\mathsf{t}(\nu)W_{\mathrm{f}}$. Using (4.1.2), one can see that the restriction of $\leq_{\mathrm{Bru}}$ to the elements $w \in W$ which are minimal in wW_{f} describes the closure order on I-orbits in Gr_G. Since for $\nu \in -\mathbf{X}^\vee_+$ the I-orbit attached to $\mathsf{t}(\nu)$ is dense in the L^+G-orbit $\mathrm{Gr}_G^{w_\circ(\nu)}$ attached to $w_\circ(\nu)$, using the description of closures of L^+G-orbits in Gr_G (see (1.2.2)), we conclude that $\mathsf{t}(\lambda) \leq_{\mathrm{Bru}} \mathsf{t}(\mu)$ iff $w_\circ(-\mu) - w_\circ(-\lambda) \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+$, which is equivalent to the condition that $\mu - \lambda \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+$.

Recall that the *(closed) Weyl chambers* are the closures of the connected components of the complement in $\mathbf{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ of the hyperplanes determined by the elements of $\mathfrak{R}$. In this section we fix such a chamber, and denote by Λ, resp. Λ°, its intersection with $\mathbf{X}^\vee$, resp. the intersection of its interior with $\mathbf{X}^\vee$. Then there exists a unique element $x_\Lambda \in W_{\mathrm{f}}$ such that $\Lambda = x_\Lambda(\mathbf{X}^\vee_+)$, and we set $\mathfrak{R}^\Lambda := x_\Lambda(\mathfrak{R}_+)$. The subset $\mathfrak{R}^\Lambda$ is a positive system in $\mathfrak{R}$, and Λ consists of the weights which are dominant with respect to $\mathfrak{R}^\Lambda$. Attached to $\mathfrak{R}^\Lambda$ we have a partial order $\preceq_\Lambda$ on $\mathbf{X}^\vee$, determined by

$$\lambda \preceq_\Lambda \mu \quad \text{iff} \quad \mu - \lambda \in \mathbb{Z}_{\geq 0}\mathfrak{R}^\Lambda.$$

The following statement generalizes (4.1.4) and (4.1.6) to the setting where $\mathbf{X}^\vee_+$ is replaced by Λ.

Lemma 4.1.2. — *For any $\lambda, \mu \in \Lambda$, we have:*
1. *$\ell(\mathsf{t}(\lambda+\mu)) = \ell(\mathsf{t}(\lambda)) + \ell(\mathsf{t}(\mu))$;*
2. *$\mathsf{t}(\lambda) \leq_{\mathrm{Bru}} \mathsf{t}(\mu)$ iff $\lambda \preceq_\Lambda \mu$.*

Proof. — The formula in part (1) is immediate from (4.1.4) and (4.1.5). For part (2), we start from the fact that $\mathsf{t}(x_\Lambda^{-1}(\lambda)) \leq_{\mathrm{Bru}} \mathsf{t}(x_\Lambda^{-1}(\mu))$ iff $x_\Lambda^{-1}(\mu-\lambda) \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+$ (see (4.1.6)), which is equivalent to the condition that $\lambda \preceq_\Lambda \mu$. Now since $x_\Lambda^{-1}(\lambda)$ and $x_\Lambda^{-1}(\mu)$ are dominant, using (4.1.1) we see that

$$\ell(x_\Lambda \mathsf{t}(x_\Lambda^{-1}(\lambda))) = \ell(x_\Lambda) + \ell(\mathsf{t}(x_\Lambda^{-1}(\lambda))), \quad \ell(x_\Lambda \mathsf{t}(x_\Lambda^{-1}(\mu))) = \ell(x_\Lambda) + \ell(\mathsf{t}(x_\Lambda^{-1}(\mu))),$$

which in view of (4.1.5) can also be interpreted as

$$\ell(\mathsf{t}(\lambda)x_\Lambda) = \ell(\mathsf{t}(\lambda)) + \ell(x_\Lambda), \quad \ell(\mathsf{t}(\mu)x_\Lambda) = \ell(\mathsf{t}(\mu)) + \ell(x_\Lambda).$$

Now for $x, y, z \in W$ such that $\ell(xz) = \ell(x) + \ell(z)$ and $\ell(yz) = \ell(y) + \ell(z)$ we have

$$x \leq_{\mathrm{Bru}} y \quad \Leftrightarrow \quad xz \leq_{\mathrm{Bru}} yz.$$

(To establish this claim, it is enough to prove it when $\ell(z) \in \{0,1\}$. If $\ell(z) = 0$ then the equivalence is clear from definitions, and if $\ell(z) = 1$ it holds by, e.g., [**Hu2**, Proposition 5.9].) Similarly, for $x, y, z \in W$ such that $\ell(zx) = \ell(x) + \ell(z)$ and $\ell(zy) = \ell(y) + \ell(z)$ we have

$$x \leq_{\mathrm{Bru}} y \quad \Leftrightarrow \quad zx \leq_{\mathrm{Bru}} zy.$$

We deduce in our setting that

$$\mathsf{t}(x_\Lambda^{-1}(\lambda)) \leq_{\mathrm{Bru}} \mathsf{t}(x_\Lambda^{-1}(\mu)) \text{ iff } x_\Lambda \mathsf{t}(x_\Lambda^{-1}(\lambda)) \leq_{\mathrm{Bru}} x_\lambda \mathsf{t}(x_\Lambda^{-1}(\mu)) \text{ iff } \mathsf{t}(\lambda) \leq_{\mathrm{Bru}} \mathsf{t}(\mu),$$

which finishes the proof. □

4.1.2. Standard and costandard objects on Fl_G**.** — For $w \in W$, we denote by $j_w : \mathrm{Fl}_{G,w} \to \mathrm{Fl}_G$ the embedding. Then for M in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$ we set

$$\Delta_w^I(M) := (j_w)_! \underline{M}_{\mathrm{Fl}_{G,w}}[\ell(w)], \qquad \nabla_w^I(M) := (j_w)_* \underline{M}_{\mathrm{Fl}_{G,w}}[\ell(w)],$$

where $\underline{M}_{\mathrm{Fl}_{G,w}}$ means the pullback of M (considered as a constructible complex on a point) under the map $\mathrm{Fl}_{G,w} \to \mathrm{pt}$. Note that since the latter map is smooth and since the embedding $\mathrm{Fl}_{G,w} \to \mathrm{Fl}_G$ is affine (because Fl_G is separated and $\mathrm{Fl}_{G,w}$ is affine), the complexes $\Delta_w^I(M)$ and $\nabla_w^I(M)$ are perverse sheaves if M is concentrated in degree 0 (see [**Ac3**, Corollary 3.5.9]). In other words, the functors

$$\Delta_w^I(-) : D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk} \to D_I^{\mathrm{b}}(\mathrm{Fl}_G, \Bbbk)$$

and

$$\nabla_w^I(-) : D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk} \to D_I^{\mathrm{b}}(\mathrm{Fl}_G, \Bbbk)$$

are t-exact. If $\Bbbk$ is a field, we will also denote by $\mathscr{IC}_w^I$ the image of the canonical morphism $\Delta_w^I(\Bbbk) \to \nabla_w^I(\Bbbk)$; then $\mathscr{IC}_w^I$ is a simple perverse sheaf, and any simple I-equivariant perverse sheaf on Fl_G is isomorphic to some $\mathscr{IC}_w^I$.

The following claim will not be used in this chapter, but it will be required in later chapters.

Lemma 4.1.3. — *Assume that $\Bbbk$ is a field. Let $w \in W$, and let $\omega \in \Omega$ be the unique element such that $w \in \omega W_{\mathrm{Cox}}$. Then the socle (resp. top) of the perverse sheaf $\Delta_w^I(\Bbbk)$ (resp. $\nabla_w^I(\Bbbk)$) is $\mathscr{IC}_\omega^I$, and moreover the multiplicity of this simple object (as a composition factor) is* 1.

Proof. — A proof of an analogous claim for ordinary flag varieties appears in [**BBM**, §2.1] under the assumption that $\Bbbk$ has characteristic 0. In fact, the arguments given there apply for any field of coefficients, and also for affine flag varieties. □

The following (standard) properties will be very useful to us in various places.

Lemma 4.1.4. — 1. *For any two elements $w_1, w_2 \in W$ such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ and any M_1, M_2 in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$, there exists a canonical isomorphism*

$$\vartheta_{w_1,w_2}^{\Delta,M_1,M_2} : \Delta_{w_1}^I(M_1) \star^I \Delta_{w_2}^I(M_2) \xrightarrow{\sim} \Delta_{w_1 w_2}^I\big(M_1 \overset{L}{\otimes}_{\Bbbk} M_2\big).$$

Moreover, if w_1, w_2, w_3 satisfy $\ell(w_1w_2w_3) = \ell(w_1)+\ell(w_2)+\ell(w_3)$, the following diagram commutes:

$$\begin{tikzcd}
\Delta^I_{w_1}(M_1) \star^I \Delta^I_{w_2}(M_2) \star^I \Delta^I_{w_3}(M_3) \arrow[r, "\vartheta^{\Delta,M_1,M_2}_{w_1,w_2} \star^I \mathrm{id}", "\sim"'] \arrow[d, "\mathrm{id} \star^I \vartheta^{\Delta,M_2,M_3}_{w_2,w_3}"', "\wr"] & \Delta^I_{w_1w_2}\big(M_1 \otimes^L_{\Bbbk} M_2\big) \star^I \Delta^I_{w_3} \arrow[d, "\vartheta^{\Delta,M_1\otimes^L_{\Bbbk}M_2,M_3}_{w_1w_2,w_3}", "\wr"'] \\
\Delta^I_{w_1}(M_1) \star^I \Delta^I_{w_2w_3}\big(M_2 \otimes^L_{\Bbbk} M_3\big) \arrow[r, "\vartheta^{\Delta,M_1,M_2\otimes^L_{\Bbbk}M_3}_{w_1,w_2w_3}", "\sim"'] & \Delta^I_{w_1w_2w_3}\big(M_1 \otimes^L_{\Bbbk} M_2 \otimes^L_{\Bbbk} M_3\big).
\end{tikzcd}$$

2. *For any two elements $w_1, w_2 \in W$ such that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ and any M_1, M_2 in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$, there exists a canonical isomorphism*

$$\vartheta^{\nabla,M_1,M_2}_{w_1,w_2} : \nabla^I_{w_1}(M_1) \star^I \nabla^I_{w_2}(M_2) \xrightarrow{\sim} \nabla^I_{w_1w_2}\big(M_1 \overset{L}{\otimes}_{\Bbbk} M_2\big).$$

Moreover, if w_1, w_2, w_3 satisfy $\ell(w_1w_2w_3) = \ell(w_1)+\ell(w_2)+\ell(w_3)$, the following diagram commutes:

$$\begin{tikzcd}
\nabla^I_{w_1}(M_1) \star^I \nabla^I_{w_2}(M_2) \star^I \nabla^I_{w_3}(M_3) \arrow[r, "\vartheta^{\nabla,M_1,M_2}_{w_1,w_2} \star^I \mathrm{id}", "\sim"'] \arrow[d, "\mathrm{id} \star^I \vartheta^{\nabla,M_2,M_3}_{w_2,w_3}"', "\wr"] & \nabla^I_{w_1w_2}\big(M_1 \otimes^L_{\Bbbk} M_2\big) \star^I \nabla^I_{w_3} \arrow[d, "\vartheta^{\nabla,M_1\otimes^L_{\Bbbk}M_2,M_3}_{w_1w_2,w_3}", "\wr"'] \\
\nabla^I_{w_1}(M_1) \star^I \nabla^I_{w_2w_3}\big(M_2 \otimes^L_{\Bbbk} M_3\big) \arrow[r, "\vartheta^{\nabla,M_1,M_2\otimes^L_{\Bbbk}M_3}_{w_1,w_2w_3}", "\sim"'] & \nabla^I_{w_1w_2w_3}\big(M_1 \otimes^L_{\Bbbk} M_2 \otimes^L_{\Bbbk} M_3\big).
\end{tikzcd}$$

3. *For any $w \in W$ there exist canonical isomorphisms*

$$\Delta^I_w(\Bbbk) \star^I \nabla^I_{w^{-1}}(\Bbbk) \cong \Delta^I_e(\Bbbk) \cong \nabla^I_{w^{-1}}(\Bbbk) \star^I \Delta^I_w(\Bbbk).$$

In particular, the objects $\Delta^I_w(\Bbbk)$ and $\nabla^I_w(\Bbbk)$ are invertible in the monoidal category $(D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk), \star^I)$.

Note that in parts (1) and (2), the condition $\ell(w_1w_2w_3) = \ell(w_1) + \ell(w_2) + \ell(w_3)$ implies that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ and $\ell(w_2w_3) = \ell(w_2) + \ell(w_3)$.

Proof. — (1) It is well known from the (affine analogue of the) Bruhat decomposition that if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ then multiplication induces an isomorphism

$$q^{-1}(\mathrm{Fl}_{G,w_1}) \times^I \mathrm{Fl}_{G,w_2} \xrightarrow{\sim} \mathrm{Fl}_{G,w_1w_2},$$

where $q : \mathrm{L}G \to \mathrm{Fl}_G$ is the quotient map. Now it is easily seen that

$$\Delta^I_{w_1}(M_1) \star^I \Delta^I_{w_2}(M_2) \cong m'_! \circ (j_{w_1,w_2})_!\big(\underline{M_1}_{\mathrm{Fl}_{G,w_1}} \mathbin{\widetilde{\boxtimes}} \underline{M_2}_{\mathrm{Fl}_{G,w_2}}\big)$$

where $j_{w_1,w_2} : q^{-1}(\mathrm{Fl}_{G,w_1}) \times^I \mathrm{Fl}_{G,w_2} \to \mathrm{L}G \times^I \mathrm{Fl}_G$ is the embedding, and m' is the multiplication morphism for Fl_G (see (2.2.2)). We also have

$$\underline{M_1}_{\mathrm{Fl}_{G,w_1}} \mathbin{\widetilde{\boxtimes}} \underline{M_2}_{\mathrm{Fl}_{G,w_2}} \cong \underline{M_1 \overset{L}{\otimes}_{\Bbbk} M_2}_{q^{-1}(\mathrm{Fl}_{G,w_1}) \times^I \mathrm{Fl}_{G,w_2}}.$$

We deduce the isomorphism $\vartheta^{\Delta,M_1,M_2}_{w_1,w_2}$.

To prove the commutativity of the diagram in the setting where $\ell(w_1w_2w_3) = \ell(w_1) + \ell(w_2) + \ell(w_3)$ it suffices to consider the restrictions of all the objects involved to the stratum $\mathrm{Fl}_{G,w_1w_2w_3}$; then the claim is obvious.

(2) The proof is similar to that of (1).

(3)[1] We start with general considerations on stalks of complexes $\Delta^I_w(\Bbbk) \star^I \nabla^I_y(\Bbbk)$ for $w, y \in W$. Let $z \in W$, and consider the stalk

$$(\Delta^I_w(\Bbbk) \star^I \nabla^I_y(\Bbbk))_{zI} \tag{4.1.7}$$

of this complex at the point $zI \in \mathrm{Fl}_G$. By the definition of convolution and the base change theorem, we have

$$\left(\Delta^I_w(\Bbbk) \star^I \nabla^I_y(\Bbbk)\right)_{zI} \cong R\Gamma_c\left((m')^{-1}(zI), (\Delta^I_w(\Bbbk) \mathbin{\widetilde{\boxtimes}} \nabla^I_y(\Bbbk))_{|(m')^{-1}(zI)}\right).$$

Next, there exists an isomorphism

$$\mathrm{Fl}_G \xrightarrow{\sim} (m')^{-1}(zI)$$

which is given by $gI \mapsto [g, g^{-1}zI]$. Under this identification, we have

$$(\Delta^I_w(\Bbbk) \mathbin{\widetilde{\boxtimes}} \nabla^I_y(\Bbbk))_{|(m')^{-1}(zI)} \cong \Delta^I_w(\Bbbk) \overset{L}{\otimes}_{\Bbbk} (j_{z,y})_* \underline{\Bbbk}[\ell(y)],$$

where $j_{z,y}$ is the embedding

$$\{gI \in \mathrm{Fl}_G \mid zI \in gIyI\} \hookrightarrow \mathrm{Fl}_G \tag{4.1.8}$$

and $\underline{\Bbbk}$ is the constant sheaf on the left-hand side. (This can be checked by taking pullbacks to $\mathrm{L}G$.) Using the projection formula (see e.g. [**Ac3**, Theorem 1.4.9]), we deduce an isomorphism

$$(\Delta^I_w(\Bbbk) \star^I \nabla^I_y(\Bbbk))_{zI} \cong R\Gamma_c(\mathrm{Fl}_{G,w}, (j_w)^*(j_{z,y})_*\underline{\Bbbk})[\ell(y) + \ell(w)].$$

Now, assume that $y = w^{-1}$ and $z = e$. Then the left-hand side in (4.1.8) is $\mathrm{Fl}_{G,w}$, so that $(j_w)^*(j_{e,w^{-1}})_*\underline{\Bbbk} = \underline{\Bbbk}_{\mathrm{Fl}_{G,w}}$, and we obtain an isomorphism

$$(\Delta^I_w(\Bbbk) \star^I \nabla^I_{w^{-1}}(\Bbbk))_I \cong R\Gamma_c(\mathrm{Fl}_{G,w}; \Bbbk)[2\ell(w)].$$

Since $\mathrm{Fl}_{G,w}$ is an affine space of dimension $\ell(w)$, the right-hand side identifies canonically with $\Bbbk$.

Next, assume that $w = y = z \in S$, and denote this element by s. In this case, the left-hand side in (4.1.8) is

$$(\mathrm{Fl}_{G,s} \smallsetminus \{sI\}) \sqcup \mathrm{Fl}_{G,e},$$

hence $(j_s)^*(j_{s,s})_*\underline{\Bbbk}$ is the $*$-pushforward of the constant sheaf under the embedding $\mathrm{Fl}_{G,s} \smallsetminus \{sI\} \hookrightarrow \mathrm{Fl}_{G,s}$. The orbit $\mathrm{Fl}_{G,s}$ identifies with $\mathbb{A}^1$, in such a way that $\{sI\}$ corresponds to the origin 0. It is well known that

$$\mathsf{H}^\bullet_c(\mathbb{A}^1, j_*\underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}) = 0$$

where $j : \mathbb{A}^1 \smallsetminus \{0\} \to \mathbb{A}^1$ the embedding, see for instance [**Ac3**, Lemma B.2.6 or Lemma B.6.1]; so the stalk (4.1.7) vanishes. To summarize, we have shown that

$$(\Delta^I_w(\Bbbk) \star^I \nabla^I_y(\Bbbk))_{zI} = \begin{cases} \Bbbk & \text{if } y = w^{-1} \text{ and } z = e; \\ 0 & \text{if } w = y = z \in S. \end{cases}$$

[1] This part of the proof, which improves on a weaker claim from an earlier version of this book, was explained to us by G. Dhillon.

Similar considerations show that

$$(\nabla^I_w(\Bbbk) \star^I \Delta^I_y(\Bbbk))_{zI} = \begin{cases} \Bbbk & \text{if } y = w^{-1} \text{ and } z = e; \\ 0 & \text{if } w = y = z \in S. \end{cases}$$

In case $s \in S$, since the complexes $\Delta^I_s(\Bbbk) \star^I \nabla^I_s(\Bbbk)$ and $\nabla^I_s(\Bbbk) \star^I \Delta^I_s(\Bbbk)$ are I-equivariant and supported on $\overline{\mathrm{Fl}_{G,s}}$, these computations provide canonical isomorphisms

$$\Delta^I_s(\Bbbk) \star^I \nabla^I_s(\Bbbk) \cong \Delta^I_e \cong \nabla^I_s(\Bbbk) \star^I \Delta^I_s(\Bbbk).$$

It follows that $\Delta^I_s(\Bbbk)$ is invertible for the product $\star^I$, with inverse $\nabla^I_s(\Bbbk)$.

If $w \in \Omega$, since $\mathrm{Fl}_{G,w}$ is reduced to a point, and closed, it is clear that $\Delta^I_w(\Bbbk)$ is also invertible, with inverse $\nabla^I_{w^{-1}}(\Bbbk)$. For a general $w \in W$, using (1) and (2), and chosing a reduced decomposition of w, we deduce that $\Delta^I_w(\Bbbk)$ is an invertible object in $(D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk), \star^I)$, and that its inverse identifies (noncanonically at this stage) with $\nabla^I_{w^{-1}}(\Bbbk)$. Above we have constructed a canonical identification of $(\Delta^I_w(\Bbbk) \star^I \nabla^I_{w^{-1}}(\Bbbk))_I$ with $\Bbbk$; this provides the desired *canonical* identification of the inverse with $\nabla^I_{w^{-1}}(\Bbbk)$. □

Remark 4.1.5. — Combining the isomorphisms proved in Lemma 4.1.4 one can obtain more relations between the standard and costandard objects; for instance one can check that if $\ell(wv) = \ell(w) - \ell(v)$ then we have

$$\Delta^I_w(M) \star^I \nabla^I_v(N) \cong \Delta^I_{wv}(M \overset{L}{\otimes}_{\Bbbk} N)$$

for M, N in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$.

4.1.3. Convolutions of standard and costandard objects. — In this subsection we will study some objects obtained by convolving a standard and a costandard object. For this we will need the following observation.

Lemma 4.1.6. — *For any $w \in W$ and any closed finite union of I-orbits X, the morphisms*

$$q^{-1}(\mathrm{Fl}_{G,w}) \times^I X \to \mathrm{Fl}_G \quad \textit{and} \quad q^{-1}(X) \times^I \mathrm{Fl}_{G,w} \to \mathrm{Fl}_G$$

induced by the morphism m' from (2.2.2) *are affine.*

Proof. — Write $w = x\mathsf{t}(\lambda)$ with $x \in W_{\mathrm{f}}$ and $\lambda \in \mathbf{X}^\vee$, and choose a lift $\dot{x}$ of x in $N_G(T)$. Then if we write I_w for the stabilizer in I of the point $\dot{x}z^\lambda I \in \mathrm{Fl}_G$, the map $g \mapsto g\dot{x}z^\lambda I$ induces an isomorphism $I/I_w \xrightarrow{\sim} \mathrm{Fl}_{G,w}$. Let Y be a closed finite union of I-orbits containing $\dot{x}z^\lambda \cdot X$. Then we can identify $q^{-1}(\mathrm{Fl}_{G,w}) \times^I X$ with $I \times^{I_w} \dot{x}z^\lambda \cdot X$, and the map $q^{-1}(\mathrm{Fl}_{G,w}) \times^I X \to \mathrm{Fl}_G$ with the composition

$$I \times^{I_w} \dot{x}z^\lambda \cdot X \hookrightarrow I \times^{I_w} Y \xrightarrow{\sim} I/I_w \times Y \to Y \hookrightarrow \mathrm{Fl}_G$$

where the second map is given by $[g, y] \mapsto (gI_w, g \cdot y)$, and the third one is projection on the second factor. Here the first and last maps are closed immersions (and hence, in particular, affine), and the third one is projection along the affine space $I/I_w \cong \mathrm{Fl}_{G,w}$. Thus, our map $q^{-1}(\mathrm{Fl}_{G,w}) \times^I X \to \mathrm{Fl}_G$ is indeed affine.

The proof for the map $q^{-1}(X) \times^I \mathrm{Fl}_{G,w} \to \mathrm{Fl}_G$ is similar, using instead a closed finite union of orbits $Y' \subset \mathrm{Fl}_G$ such that $q^{-1}(X) \cdot \dot{x} z^\lambda \subset q^{-1}(Y')$. □

This lemma will allow us to prove the following claim.

Lemma 4.1.7. — *For any $w, y \in W$ and any M, N in $\mathsf{Mof}_\Bbbk$ such that the complex $M \otimes^L_\Bbbk N$ is concentrated in degree 0, the objects*

$$\Delta^I_w(M) \star^I \nabla^I_y(N) \quad \textit{and} \quad \nabla^I_w(M) \star^I \Delta^I_y(N)$$

are perverse sheaves.

Proof. — We treat the case of the object $\Delta^I_w(M) \star^I \nabla^I_y(N)$; the case of $\nabla^I_w(M) \star^I \Delta^I_y(N)$ is similar.

First, we remark that the object $\Delta^I_w(M) \star^I \nabla^I_y(N)$ can be obtained from the complex $\underline{M}_{\mathrm{Fl}_{G,w}} \mathbin{\widetilde{\boxtimes}} \underline{N}_{\mathrm{Fl}_{G,y}} \cong \underline{M \otimes^L_\Bbbk N}_{q^{-1}(\mathrm{Fl}_{G,w}) \times^I \mathrm{Fl}_{G,y}}$ by first taking the $*$-pushforward under the embedding

$$q^{-1}(\mathrm{Fl}_{G,w}) \times^I \mathrm{Fl}_{G,y} \hookrightarrow q^{-1}(\mathrm{Fl}_{G,w}) \times^I \overline{\mathrm{Fl}_{G,y}},$$

and then the !-pushforward under the morphism

$$q^{-1}(\mathrm{Fl}_{G,w}) \times^I \overline{\mathrm{Fl}_{G,y}} \to \mathrm{Fl}_G$$

induced by m'. Here the first map is an affine embedding, so the associated $*$-pushforward functor is t-exact for the perverse t-structure (see [**Ac3**, Corollary 3.5.9]), and the second map is affine by Lemma 4.1.6, so that the associated !-pushforward functor is left t-exact (see [**Ac3**, Theorem 3.5.8]). This shows that our object is concentrated in nonnegative perverse degrees. Similar arguments using the composition

$$q^{-1}(\mathrm{Fl}_{G,w}) \times^I \mathrm{Fl}_{G,y} \hookrightarrow q^{-1}(\overline{\mathrm{Fl}_{G,w}}) \times^I \mathrm{Fl}_{G,y} \to \mathrm{Fl}_G$$

show that this complex is also concentrated in nonpositive degrees, and hence that it is perverse. □

Remark 4.1.8. —
1. It is clear from the proof of Lemma 4.1.7 that the objects $\Delta^I_w(M) \star^I \nabla^I_y(N)$ and $\nabla^I_w(M) \star^I \Delta^I_y(N)$ do not depend on M and N individually, but only on the derived tensor product $M \otimes^L_\Bbbk N$ (up to canonical isomorphism). For this reason one can always assume that $N = \Bbbk$, which we will do below.
2. Lemma 4.1.7 has the following slight extension, which will also be used below. If X is a separated scheme of finite type over $\mathbb{C}$, if $\mathscr{F}$ is a $\Bbbk$-perverse sheaf on X, and if $M, N \in \mathsf{Mof}_\Bbbk$ are such that $\mathscr{F} \otimes^L_\Bbbk (M \otimes^L_\Bbbk N)$ is perverse, then for any $w, y \in W$ the complexes

$$\mathscr{F} \overset{L}{\boxtimes}_\Bbbk \big(\Delta^I_w(M) \star^I \nabla^I_y(N)\big) \quad \text{and} \quad \mathscr{F} \overset{L}{\boxtimes}_\Bbbk \big(\nabla^I_w(M) \star^I \Delta^I_y(N)\big)$$

are perverse sheaves on $X \times \mathrm{Fl}_G$.

4.1.4. Support of convolutions of standard and costandard objects. — Our next goal is to study the support of the perverse sheaves from Lemma 4.1.7. First we consider the case when $\Bbbk$ is a field.

Lemma 4.1.9. — *Assume that $\Bbbk$ is a field. Then, in the Grothendieck group $[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]$ of the triangulated category $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, for any $w, y \in W$ we have*

$$[\Delta^I_w(\Bbbk) \star^I \nabla^I_y(\Bbbk)] = [\nabla^I_w(\Bbbk) \star^I \Delta^I_y(\Bbbk)] = [\Delta^I_{wy}(\Bbbk)] = [\nabla^I_{wy}(\Bbbk)].$$

In particular, the perverse sheaf $\nabla^I_w(\Bbbk) \star^I \Delta^I_y(\Bbbk)$ is supported on $\overline{\mathrm{Fl}_{G,wy}}$, and its restriction to $\mathrm{Fl}_{G,wy}$ is $\underline{\Bbbk}_{\mathrm{Fl}_{G,wy}}[\ell(wy)]$.

Proof. — We have a natural isomorphism of abelian groups

$$\varphi : [D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)] \xrightarrow{\sim} \mathbb{Z}[W]$$

sending $[\mathscr{F}]$ to

$$\sum_{z \in W} \left(\sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim \mathsf{H}^{n+\ell(z)}(\mathrm{Fl}_{G,z}, j_z^* \mathscr{F}) \right) \cdot z,$$

i.e. sending $[\Delta^I_z(\Bbbk)]$ to z for any $z \in W$. For any $s \in S$, using the exact sequences

$$\mathscr{I}\mathscr{C}^I_e \hookrightarrow \Delta^I_s(\Bbbk) \twoheadrightarrow \mathscr{I}\mathscr{C}^I_s \quad \text{and} \quad \mathscr{I}\mathscr{C}^I_s \hookrightarrow \nabla^I_s(\Bbbk) \twoheadrightarrow \mathscr{I}\mathscr{C}^I_e$$

we see that $[\Delta^I_s(\Bbbk)] = [\nabla^I_s(\Bbbk)]$. Next, Lemma 4.1.4 implies that φ is in fact a ring homomorphism, and that it satisfies

$$\varphi([\nabla^I_z(\Bbbk)]) = \varphi([\Delta^I_z(\Bbbk)]) = z$$

for any $z \in W$. We deduce that

$$\varphi([\Delta^I_w(\Bbbk) \star^I \nabla^I_y(\Bbbk)]) = \varphi([\nabla^I_w(\Bbbk) \star^I \Delta^I_y(\Bbbk)]) = wy,$$

which proves the desired equalities.

Now if $\mathscr{F}$ is a perverse sheaf in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ and if $\mathrm{Fl}_{G,z}$ is open in its support then the complex $j_z^*(\mathscr{F})$ is concentrated in degree $-\ell(z)$; in particular the sum

$$\sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim \mathsf{H}^{n+\ell(z)}(\mathrm{Fl}_{G,z}, j_z^* \mathscr{F})$$

is equal to the rank of $j_z^*(\mathscr{F})$, which is necessarily nonzero. In the case at hand, we deduce that $\mathrm{Fl}_{G,wy}$ is the only I-orbit open in the support of $\Delta^I_w(\Bbbk) \star^I \nabla^I_y(\Bbbk)$, and that the restriction of this perverse sheaf to this stratum has rank 1. □

We next come back to the case when $\Bbbk$ is a general ring (satisfying our running assumptions), and extend Lemma 4.1.9 to this case, as follows.

Lemma 4.1.10. — *For any M in $\mathsf{Mof}_{\Bbbk}$, the perverse sheaf $\nabla^I_w(M) \star^I \Delta^I_y(\Bbbk)$ is supported on $\overline{\mathrm{Fl}_{G,wy}}$, and its restriction to $\mathrm{Fl}_{G,wy}$ is isomorphic to $\underline{M}_{\mathrm{Fl}_{G,wy}}[\ell(wy)]$.*

Proof. — To simplify notation, we denote by $F_{\Bbbk} : D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk} \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ the functor given by

$$F_{\Bbbk}(M) = \nabla^I_w(M) \star^I \Delta^I_y(\Bbbk).$$

Then for any M we have

$$F_{\Bbbk}(M) \cong M \overset{L}{\otimes}_{\Bbbk} F_{\Bbbk}(\Bbbk),$$

and if $\varphi : \Bbbk \to \Bbbk'$ is a ring homomorphism (where $\Bbbk$ and $\Bbbk'$ satisfy our running assumptions) there is a canonical isomorphism of functors

$$\Bbbk' \overset{L}{\otimes}_{\Bbbk} F_{\Bbbk}(-) \cong F_{\Bbbk'}(\Bbbk' \overset{L}{\otimes}_{\Bbbk} (-)) : D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk} \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk').$$

Here the first isomorphism lets us reduce the proof to the case $M = \Bbbk$, and the second one (applied to the unique ring homomorphism $\mathbb{Z} \to \Bbbk$) reduces it to the case $\Bbbk = \mathbb{Z}$.

Now let $\mathscr{F} := F_{\mathbb{Z}}(\mathbb{Z}) = \nabla^I_w(\mathbb{Z}) \star^I \Delta^I_y(\mathbb{Z})$. By Lemma 4.1.7, $\mathscr{F}$ is a perverse sheaf. Let $\mathrm{Fl}_{G,x}$ be an open I-orbit in its support. Then there is a nonzero and finitely generated $\mathbb{Z}$-module J such that

$$\mathscr{F}_{|\mathrm{Fl}_{G,x}} \cong \underline{J}_{\mathrm{Fl}_{G,x}}[\ell(x)].$$

For any prime number p we have

$$(\mathbb{F}_p \overset{L}{\otimes}_{\mathbb{Z}} \mathscr{F})_{|\mathrm{Fl}_{G,x}} \cong \underline{\mathbb{F}_p \overset{L}{\otimes}_{\mathbb{Z}} J}_{\mathrm{Fl}_{G,x}}[\ell(x)],$$

and if these objects are nonzero then $\mathrm{Fl}_{G,x}$ is open in the support of $\mathbb{F}_p \otimes^L_{\mathbb{Z}} \mathscr{F}$. In particular, choosing p such that $\mathbb{F}_p \otimes^L_{\mathbb{Z}} J \neq 0$ and using Lemma 4.1.9 we deduce that $x = wy$, and then using again this lemma we obtain that for any p the complex of $\mathbb{F}_p$-vector spaces $\mathbb{F}_p \otimes^L_{\mathbb{Z}} J$ is concentrated in degree 0, and of dimension 1. It follows that $J \cong \mathbb{Z}$, as desired. $\square$

4.2. Wakimoto sheaves

4.2.1. Wakimoto functors. — Recall that in Section 4.1 we have fixed a closed Weyl chamber Λ. We now consider another order on $\mathbf{X}^\vee$ determined by Λ, defined by

$$\lambda \trianglelefteq_\Lambda \mu \quad \text{iff} \quad \mu - \lambda \in \Lambda.$$

(This order should not be confused with the order $\preceq_\Lambda$ considered in §4.1.1.)

For $\lambda \in \mathbf{X}^\vee$ we set

$$\Lambda_\lambda = \{\mu \in \Lambda \mid \mu - \lambda \in \Lambda\} = \{\mu \in \Lambda \mid \lambda \trianglelefteq_\Lambda \mu\}.$$

We will consider Λ_λ as a (filtered) poset, for the order obtained from $\trianglelefteq_\Lambda$ by restriction.

Given $\lambda \in \mathbf{X}^\vee$ and $M \in D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$, we consider the inductive system in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ parametrized by Λ_λ which is defined as follows. Given $\mu \in \Lambda_\lambda$, the corresponding object is

$$\nabla^I_{\mathsf{t}(\mu)}(M) \star^I \Delta^I_{\mathsf{t}(\lambda-\mu)}(\Bbbk),$$

and if $\mu \trianglelefteq_\Lambda \nu$ the corresponding morphism is the composition

$$\nabla^I_{\mathsf{t}(\mu)}(M) \star^I \Delta^I_{\mathsf{t}(\lambda-\mu)}(\Bbbk) \xrightarrow{\sim} \nabla^I_{\mathsf{t}(\mu)}(M) \star^I \nabla^I_{\mathsf{t}(\nu-\mu)}(\Bbbk) \star^I \Delta^I_{\mathsf{t}(\mu-\nu)}(\Bbbk) \star^I \Delta^I_{\mathsf{t}(\lambda-\mu)}(\Bbbk) \\ \xrightarrow{\sim} \nabla^I_{\mathsf{t}(\nu)}(M) \star^I \Delta^I_{\mathsf{t}(\lambda-\nu)}(\Bbbk),$$

where the first morphism is induced by the isomorphism of Lemma 4.1.4(3), and the second one is induced by the isomorphisms of Lemma 4.1.4(1)–(2) (which are applicable by Lemma 4.1.2(1)). All the transition morphisms in this system are isomorphisms, so its inductive limit exists; we set

$$\mathbf{J}^\Lambda_\lambda(M) = \varinjlim_{\mu \in \Lambda_\lambda} \nabla^I_{\mathsf{t}(\mu)}(M) \star^I \Delta^I_{\mathsf{t}(\lambda-\mu)}(\Bbbk).$$

It is clear that this construction is functorial in M, so that we obtain the *Wakimoto functor*

$$\mathbf{J}^\Lambda_\lambda : D^{\mathrm{b}}\mathsf{Mof}_\Bbbk \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk).$$

Lemma 4.2.1. — *For any $\lambda \in \mathbf{X}^\vee$, the functor $\mathbf{J}^\Lambda_\lambda$ is t-exact.*

Proof. — This follows from Lemma 4.1.7. □

Remark 4.2.2. — In view of Remark 4.1.8(2), we have the following slight extension of Lemma 4.2.1: given a separated scheme X of finite type over $\mathbb{C}$, a perverse sheaf $\mathscr{F}$ on X, and a finitely generated $\Bbbk$-module M such that $\mathscr{F} \otimes^L_\Bbbk M$ is perverse, the complex $\mathscr{F} \boxtimes^L_\Bbbk \mathbf{J}^\Lambda_\lambda(M)$ is a perverse sheaf on $X \times \mathrm{Fl}_G$, for any $\lambda \in \mathbf{X}^\vee$.

Objects of the form $\mathbf{J}^\Lambda_\lambda(M)$ will be called *Wakimoto objects*. In the case when M is concentrated in degree 0 (so that $\mathbf{J}^\Lambda_\lambda(M)$ is a perverse sheaf) we will also speak of *Wakimoto sheaves*. These objects will play a crucial role throughout the rest of the book.

Proposition 4.2.3. — *For any $\lambda \in \mathbf{X}^\vee$ and $M \in \mathsf{Mof}_\Bbbk$, the perverse sheaf $\mathbf{J}^\Lambda_\lambda(M)$ is supported on $\overline{\mathrm{Fl}_{G,\mathsf{t}(\lambda)}}$, and its restriction to $\mathrm{Fl}_{G,\mathsf{t}(\lambda)}$ is isomorphic to $\underline{M}_{\mathrm{Fl}_{G,\mathsf{t}(\lambda)}}[\ell(\mathsf{t}(\lambda))]$. If $M = \Bbbk$, then this object is invertible in the monoidal category $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$.*

Proof. — The first claim follows from Lemma 4.1.10, and the second one from Lemma 4.1.4(3). □

Remark 4.2.4. — Assume that $\Bbbk$ is a field. From Lemma 4.1.3 and Lemma 4.1.9 we see that if $\omega \in \Omega$ is the unique element such that $\mathsf{t}(\lambda) \in \omega W_{\mathrm{Cox}}$, then the simple object $\mathscr{IC}^I_\omega$ is a composition factor of $\mathbf{J}^\Lambda_\lambda(\Bbbk)$ with multiplicity 1.

4.2.2. Full faithfulness. —

Proposition 4.2.5. — *Let $\lambda \in \mathbf{X}^\vee$.*

1. *The composition*

$$D^{\mathrm{b}}\mathsf{Mof}_\Bbbk \xrightarrow{\mathbf{J}^\Lambda_\lambda} D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk) \xrightarrow{\mathsf{For}} D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$$

is fully faithful. In particular, the functor

$$\mathscr{F} \mapsto \mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)}(\mathbf{J}^\Lambda_\lambda(\Bbbk), \mathscr{F})$$

is a left inverse to the fully faithful functor $\mathbf{J}^{\Lambda}_{\lambda} : \mathsf{Mof}_{\Bbbk} \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$.

2. *For any* $\lambda \in \mathbf{X}^{\vee}$ *and any* M, M' *in* $\mathsf{Mof}_{\Bbbk}$*, the functor* $\mathbf{J}^{\Lambda}_{\lambda}$ *induces an isomorphism*

$$\mathrm{Ext}^1_{\mathsf{Mof}_{\Bbbk}}(M, M') \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^{\Lambda}_{\lambda}(M), \mathbf{J}^{\Lambda}_{\lambda}(M')[1]).$$

In particular, the essential image of $\mathbf{J}^{\Lambda}_{\lambda} : \mathsf{Mof}_{\Bbbk} \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$ *is stable under extensions.*

Proof. — (1) Choose some $\mu \in \Lambda_{\lambda}$. Then since the functor of right convolution with $\Delta^I_{\mathsf{t}(\lambda-\mu)}(\Bbbk)$ is an autoequivalence of $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$ (see §4.1.2), the first claim follows from the fact that the composition of the functor $\nabla^I_{\mathsf{t}(\mu)}(-)$ with the forgetful functor $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$ is fully faithful, which is clear from adjunction.

The second claim follows, using the fact that the forgetful functor $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$ is fully faithful, see e.g. [**Ac3**, Proposition 6.2.15] or [**BR**, §1.16].

(2) As in (1), choosing $\mu \in \Lambda_{\lambda}$ we reduce the proof to the case $\lambda \in \Lambda$. In this case, for any M in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$ we have $\mathbf{J}^{\Lambda}_{\lambda}(M) = (j_{\mathsf{t}(\lambda)})_* \underline{M}_{\mathrm{Fl}_{G,\mathsf{t}(\lambda)}}[\ell(\mathsf{t}(\lambda))]$, so that by adjunction we have

$$\mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^{\Lambda}_{\lambda}(M), \mathbf{J}^{\Lambda}_{\lambda}(M')[1]) \cong \mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_{G,\mathsf{t}(\lambda)},\Bbbk)}(\underline{M}[\ell(\mathsf{t}(\lambda))], \underline{M'}[\ell(\mathsf{t}(\lambda)) + 1]).$$

Now, by general properties of t-structures (see e.g. [**BR**, Equation (1.4.1)]) we have

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_{G,\mathsf{t}(\lambda)},\Bbbk)}&(\underline{M}[\ell(\mathsf{t}(\lambda))], \underline{M'}[\ell(\mathsf{t}(\lambda)) + 1]) \\ &\cong \mathrm{Ext}^1_{\mathsf{Perv}_I(\mathrm{Fl}_{G,\mathsf{t}(\lambda)},\Bbbk)}(\underline{M}[\ell(\mathsf{t}(\lambda))], \underline{M'}[\ell(\mathsf{t}(\lambda))]).\end{aligned}$$

Hence the first claim follows from the observation that the functor $M \mapsto \underline{M}[\ell(\mathsf{t}(\lambda))]$ induces an equivalence of categories $\mathsf{Mof}_{\Bbbk} \xrightarrow{\sim} \mathsf{Perv}_I(\mathrm{Fl}_{G,\mathsf{t}(\lambda)}, \Bbbk)$.

For the second claim we simply have to remark that extensions of $\mathbf{J}^{\Lambda}_{\lambda}(M')$ by $\mathbf{J}^{\Lambda}_{\lambda}(M)$ are governed by the space

$$\mathrm{Ext}^1_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^{\Lambda}_{\lambda}(M), \mathbf{J}^{\Lambda}_{\lambda}(M')),$$

which as above identifies with

$$\mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^{\Lambda}_{\lambda}(M), \mathbf{J}^{\Lambda}_{\lambda}(M')[1]),$$

hence with $\mathrm{Ext}^1_{\mathsf{Mof}_{\Bbbk}}(M, M')$. ☐

In particular, part (1) in this lemma implies that $\mathbf{J}^{\Lambda}_{\lambda}$ induces an equivalence of categories between $\mathsf{Mof}_{\Bbbk}$ and its essential image in $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$, with quasi-inverse the functor $\mathscr{F} \mapsto \mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^{\Lambda}_{\lambda}(\Bbbk), \mathscr{F})$.

4.2.3. Wakimoto functors and convolution. — The construction of Wakimoto objects is compatible with convolution in the sense of the following lemma.

Lemma 4.2.6. — *For any* $\lambda, \lambda' \in \mathbf{X}^{\vee}$ *and any* M, M' *in* $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$ *there exists a canonical (in particular, bifunctorial) isomorphism*

$$\mathbf{J}^{\Lambda}_{\lambda}(M) \star^I \mathbf{J}^{\Lambda}_{\lambda'}(M') \cong \mathbf{J}^{\Lambda}_{\lambda+\lambda'}\big(M \overset{L}{\otimes}_{\Bbbk} M'\big).$$

Proof. — Let $\mu \in \Lambda_\lambda \cap \Lambda_{\lambda+\lambda'}$. From Lemma 4.1.4 we obtain canonical isomorphisms

$$\mathbf{J}^\Lambda_\lambda(M) \star^I \mathbf{J}^\Lambda_{\lambda'}(M') \star^I \nabla^I_{\mathsf{t}(\mu-\lambda-\lambda')}(\Bbbk) \cong \mathbf{J}^\Lambda_\lambda(M) \star^I \nabla^I_{\mathsf{t}(\mu-\lambda)}(M')$$
$$\cong \mathbf{J}^\Lambda_\lambda(M) \star^I \nabla^I_{\mathsf{t}(\mu-\lambda)}(\Bbbk) \star^I \nabla^I_e(M') \cong \nabla^I_{\mathsf{t}(\mu)}(M) \star^I \nabla^I_e(M') \cong \nabla^I_{\mathsf{t}(\mu)}\big(M \overset{L}{\otimes}_\Bbbk M'\big)$$

and

$$\mathbf{J}^\Lambda_{\lambda+\lambda'}\big(M \overset{L}{\otimes}_\Bbbk M'\big) \star^I \nabla^I_{\mathsf{t}(\mu-\lambda-\lambda')}(\Bbbk) \cong \nabla^I_{\mathsf{t}(\mu)}\big(M \overset{L}{\otimes}_\Bbbk M'\big).$$

Since the functor of right convolution with $\nabla^I_{\mathsf{t}(\mu-\lambda-\lambda')}(\Bbbk)$ is an equivalence, we deduce an isomorphism as in the lemma. One can check that this isomorphism does not depend on the choice of μ, and hence that it is canonical. □

As a special case of Lemma 4.2.6, we have

$$\mathbf{J}^\Lambda_\lambda(M) \cong \mathbf{J}^\Lambda_\lambda(\Bbbk) \star^I \nabla^I_e(M) \cong \mathbf{J}^\Lambda_\lambda(\Bbbk) \star^I \Delta^I_e(M).$$

Using Lemma 4.2.6 one can consider the collection $(\mathbf{J}^\Lambda_\lambda)_{\lambda \in \mathbf{X}^\vee}$ as a "monoidal collection of functors." More precisely, consider the category $(D^{\mathrm{b}}\mathsf{Mof}_\Bbbk)^{\mathbf{X}^\vee}$ whose objects are pairs (λ, M) with $\lambda \in \mathbf{X}^\vee$ and M in $D^{\mathrm{b}}\mathsf{Mof}_\Bbbk$, and morphisms from (λ, M) to (λ', M') are $\{0\}$ if $\lambda \neq \lambda'$, and $\mathrm{Hom}(M, M')$ otherwise. We turn this category into a monoidal category by setting

$$(\lambda, M) \star (\lambda', M') = (\lambda + \lambda', M \overset{L}{\otimes}_\Bbbk M').$$

Then the assignment $(\lambda, M) \mapsto \mathbf{J}^\Lambda_\lambda(M)$ defines a functor from $(D^{\mathrm{b}}\mathsf{Mof}_\Bbbk)^{\mathbf{X}^\vee}$ to $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, and one can check that the isomorphisms of Lemma 4.2.6 endow this functor with the structure of a monoidal functor.

4.3. Wakimoto filtrations

We continue with the closed Weyl chamber Λ from Sections 4.1–4.2.

4.3.1. Wakimoto functors and morphisms. — We now study morphisms between Wakimoto objects.

Lemma 4.3.1. — *For $\lambda, \lambda' \in \mathbf{X}^\vee$, M, M' in $D^{\mathrm{b}}\mathsf{Mof}_\Bbbk$ and $n \in \mathbb{Z}$, we have*

$$\mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^\Lambda_\lambda(M), \mathbf{J}^\Lambda_{\lambda'}(M')[n]) = 0$$

unless $\lambda' \preceq_\Lambda \lambda$.

Proof. — Choose $\nu \in \mathbf{X}^\vee$ such that $\lambda + \nu \in \Lambda$ and $\lambda' + \nu \in \Lambda$. Then it follows from Proposition 4.2.3 that right convolution with $\mathbf{J}^\Lambda_\nu(\Bbbk)$ is an equivalence of categories, so that we have

$$\mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^\Lambda_\lambda(M), \mathbf{J}^\Lambda_{\lambda'}(M')[n]) \cong \mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^\Lambda_\lambda(M) \star^I \mathbf{J}^\Lambda_\nu(\Bbbk), \mathbf{J}^\Lambda_{\lambda'}(M') \star^I \mathbf{J}^\Lambda_\nu(\Bbbk)[n]).$$

Using Lemma 4.2.6 and our choice of ν we deduce an isomorphism

$$\mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^\Lambda_\lambda(M), \mathbf{J}^\Lambda_{\lambda'}(M')[n]) \cong \mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\nabla^I_{\mathsf{t}(\lambda+\nu)}(M), \nabla^I_{\mathsf{t}(\lambda'+\nu)}(M')[n]).$$

Finally, using adjunction we obtain that

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_G,\Bbbk)}(\mathbf{J}^\Lambda_\lambda(M),\mathbf{J}^\Lambda_{\lambda'}(M')[n]) \cong \\ \mathrm{Hom}_{D^{\mathrm{b}}_I(\mathrm{Fl}_{G,\mathsf{t}(\lambda'+\nu)},\Bbbk)}(j^*_{\mathsf{t}(\lambda'+\nu)}\nabla^I_{\mathsf{t}(\lambda+\nu)}(M), \underline{M'}_{\mathrm{Fl}_{G,\mathsf{t}(\lambda'+\nu)}}[n+\ell(\mathsf{t}(\lambda'+\nu))]).\end{aligned}$$

Now, using Lemma 4.1.2(2) and (4.1.2) we see that $j^*_{\mathsf{t}(\lambda'+\nu)}\nabla^I_{\mathsf{t}(\lambda+\nu)}(M) = 0$ unless $\lambda'+\nu \preceq_\Lambda \lambda+\nu$, which is equivalent to $\lambda' \preceq_\Lambda \lambda$. □

4.3.2. Wakimoto filtrations. — If $\mathscr{F}$ belongs to $\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)$, a *($\Lambda$-)Wakimoto filtration* of $\mathscr{F}$ is a (finite) filtration whose subquotients are of the form $\mathbf{J}^\Lambda_\lambda(M)$ with $\lambda \in \mathbf{X}^\vee$ and M in $\mathsf{Mof}_\Bbbk$. We will say that $\mathscr{F}$ *admits a (Λ-)Wakimoto filtration* if such a filtration exists, and will denote by $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G,\Bbbk)$ the full additive subcategory of $\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)$ whose objects are those which admit a Wakimoto filtration. Note that $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G,\Bbbk)$ is *not* in general an abelian category.

We will say that a subset $\Omega \subset \mathbf{X}^\vee$ is an *ideal* (with respect to the order $\preceq_\Lambda$) if for any $\lambda \in \Omega$ and any $\mu \in \mathbf{X}^\vee$ such that $\mu \preceq_\Lambda \lambda$ we have $\mu \in \Omega$.

Lemma 4.3.2. — *If $\mathscr{F} \in \mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G,\Bbbk)$, then for any ideal $\Omega \subset \mathbf{X}^\vee$ there exists a unique subobject $\mathscr{F}_\Omega \subset \mathscr{F}$ which admits a Wakimoto filtration with subquotients of the form $\mathbf{J}^\Lambda_\lambda(M)$ with $\lambda \in \Omega$, and such that the quotient $\mathscr{F}/\mathscr{F}_\Omega$ admits a Wakimoto filtration with subquotients of the form $\mathbf{J}^\Lambda_\lambda(M)$ with $\lambda \in \mathbf{X}^\vee \smallsetminus \Omega$. Moreover, the assignment $\mathscr{F} \mapsto \mathscr{F}_\Omega$ is functorial.*

Proof. — We prove the existence by induction on the length of a Wakimoto filtration. If this filtration has length 0 then $\mathscr{F}=0$ and there is nothing to prove. Otherwise, consider an object $\mathscr{F}$ and an embedding $\mathbf{J}^\Lambda_\lambda(M) \hookrightarrow \mathscr{F}$ (with $M \in \mathsf{Mof}_\Bbbk$) whose cokernel $\mathscr{G}$ admits a Wakimoto filtration, and assume the result is known for $\mathscr{G}$. We will denote by $\mathscr{F}'$ the preimage of $\mathscr{G}_\Omega$ in $\mathscr{F}$. If $\lambda \in \Omega$, then $\mathscr{F}_\Omega := \mathscr{F}'$ satisfies the required properties. If $\lambda \notin \Omega$, then we consider the short exact sequence

$$\mathbf{J}^\Lambda_\lambda(M) \hookrightarrow \mathscr{F}' \twoheadrightarrow \mathscr{G}_\Omega.$$

Lemma 4.3.1 implies that $\mathrm{Ext}^1_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{G}_\Omega,\mathbf{J}^\Lambda_\lambda(M))=0$, so that this exact sequence must split. Choose a splitting $\mathscr{G}_\Omega \hookrightarrow \mathscr{F}'$; then the cokernel of the embedding $\mathscr{G}_\Omega \hookrightarrow \mathscr{F}$ fits in an exact sequence

$$\mathbf{J}^\Lambda_\lambda(M) \hookrightarrow \mathscr{F}/\mathscr{G}_\Omega \twoheadrightarrow \mathscr{G}/\mathscr{G}_\Omega,$$

so that $\mathscr{F}_\Omega := \mathscr{G}_\Omega$ satisfies the required properties.

Now, assume that we have two subobjects $\mathscr{F}_\Omega$ and $\mathscr{F}'_\Omega$ satisfying the desired conditions. Then the composition

$$\mathscr{F}_\Omega \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{F}/\mathscr{F}'_\Omega$$

vanishes by Lemma 4.3.1, so that $\mathscr{F}_\Omega \subset \mathscr{F}'_\Omega$. By symmetry we also have $\mathscr{F}'_\Omega \subset \mathscr{F}_\Omega$, so the two subobjects coincide.

The functoriality of the assignment $\mathscr{F} \mapsto \mathscr{F}_\Omega$ follows from similar arguments. □

Consider now two ideals $\Omega, \Gamma \subset \mathbf{X}^\vee$ such that $\Omega \subset \Gamma$.

Lemma 4.3.3. — *For any $\mathscr{F}$ in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$, we have $\mathscr{F}_\Omega \subset \mathscr{F}_\Gamma$, and moreover the quotient $\mathscr{F}_\Gamma/\mathscr{F}_\Omega$ admits a Wakimoto filtration with subquotients of the form $\mathbf{J}_\lambda^\Lambda(M)$ with $\lambda \in \Gamma \smallsetminus \Omega$.*

Proof. — The subobject $(\mathscr{F}_\Gamma)_\Omega$ of $\mathscr{F}$ is such that the cokernel $\mathscr{F}/(\mathscr{F}_\Gamma)_\Omega$ fits in a short exact sequence

$$(\mathscr{F}_\Gamma)/(\mathscr{F}_\Gamma)_\Omega \hookrightarrow \mathscr{F}/(\mathscr{F}_\Gamma)_\Omega \twoheadrightarrow \mathscr{F}/\mathscr{F}_\Gamma.$$

It follows that $\mathscr{F}_\Omega = (\mathscr{F}_\Gamma)_\Omega$, and the claims follow. □

It follows in particular from Lemma 4.3.3 and Proposition 4.2.5(2) that if we choose any bijection $\mathbb{Z} \xrightarrow{\sim} \mathbf{X}^\vee$, denoted by $i \mapsto \lambda_i$, such that

$$\lambda_i \preceq_\Lambda \lambda_j \quad \Longrightarrow \quad i \leq j \tag{4.3.1}$$

then any $\mathscr{F}$ in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$ has a functorial filtration

$$\cdots \subset \mathscr{F}_{-1} \subset \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}$$

indexed by $\mathbb{Z}$ such that each $\mathscr{F}_i/\mathscr{F}_{i-1}$ is isomorphic to $\mathbf{J}_{\lambda_i}^\Lambda(M_i)$ for some (unique) $M_i \in \mathsf{Mof}_\Bbbk$, and finitely many of the M_i's are nonzero.

4.3.3. The associated graded functor. — We can now define a kind of "associated graded of the Wakimoto filtration" as follows.

Lemma 4.3.4. — *Let $\lambda \in \mathbf{X}^\vee$ and $\mathscr{F} \in \mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$. Choose an ideal $\Omega \subset \mathbf{X}^\vee$ such that $\lambda \in \Omega$ and λ is maximal in Ω (in other words, $\Omega \smallsetminus \{\lambda\}$ is again an ideal). Then the quotient $\mathscr{F}_\Omega/\mathscr{F}_{\Omega \smallsetminus \{\lambda\}}$ is independent of the choice of Ω up to canonical isomorphism, and belongs to the essential image of $\mathbf{J}_\lambda^\Lambda : \mathsf{Mof}_\Bbbk \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$.*

Proof. — By Lemma 4.3.3 the quotient $\mathscr{F}_\Omega/\mathscr{F}_{\Omega \smallsetminus \{\lambda\}}$ admits a filtration with all subquotients in the essential image of $\mathbf{J}_\lambda^\Lambda : \mathsf{Mof}_\Bbbk \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$. Hence by Proposition 4.2.5(2) it itself belongs to this essential image.

Since Ω is an ideal containing λ, it must contain $\Gamma := \{\mu \in \mathbf{X}^\vee \mid \mu \preceq_\Lambda \lambda\}$. Hence by Lemma 4.3.3 we have $\mathscr{F}_\Gamma \subset \mathscr{F}_\Omega$. Similarly, setting $\Omega' := \Omega \smallsetminus \{\lambda\}$ and $\Gamma' := \Gamma \smallsetminus \{\lambda\}$ we have $\mathscr{F}_{\Gamma'} \subset \mathscr{F}_{\Omega'}$. We deduce that there is a canonical morphism

$$\mathscr{F}_\Gamma/\mathscr{F}_{\Gamma'} \to \mathscr{F}_\Omega/\mathscr{F}_{\Omega'}. \tag{4.3.2}$$

To finish the proof, it suffices to prove that this morphism is invertible. Now, as observed above both of these objects belong to the essential image of $\mathbf{J}_\lambda^\Lambda : \mathsf{Mof}_\Bbbk \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$. Since this functor is fully faithful (see Proposition 4.2.5(1)), by the Yoneda lemma, to prove that (4.3.2) is invertible it suffices to prove that for any $M \in \mathsf{Mof}_\Bbbk$ the induced map

$$\mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Omega/\mathscr{F}_{\Omega'}, \mathbf{J}_\lambda^\Lambda(M)) \to \mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Gamma/\mathscr{F}_{\Gamma'}, \mathbf{J}_\lambda^\Lambda(M))$$

is an isomorphism. Now Lemma 4.3.1 implies that the quotient morphisms $\mathscr{F}_\Omega \to \mathscr{F}_\Omega/\mathscr{F}_{\Omega'}$ and $\mathscr{F}_\Gamma \to \mathscr{F}_\Gamma/\mathscr{F}_{\Gamma'}$ induce isomorphisms

$$\mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Omega/\mathscr{F}_{\Omega'}, \mathbf{J}_\lambda^\Lambda(M)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Omega, \mathbf{J}_\lambda^\Lambda(M))$$

and

$$\mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Gamma/\mathscr{F}_{\Gamma'}, \mathbf{J}^\Lambda_\lambda(M)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Gamma, \mathbf{J}^\Lambda_\lambda(M)).$$

Another application of Lemma 4.3.1 implies that

$$\mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Omega/\mathscr{F}_\Gamma, \mathbf{J}^\Lambda_\lambda(M)) = \mathrm{Ext}^1_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Omega/\mathscr{F}_\Gamma, \mathbf{J}^\Lambda_\lambda(M)) = 0,$$

so from the long exact sequence associated to $\mathscr{F}_\Gamma \hookrightarrow \mathscr{F}_\Omega \twoheadrightarrow \mathscr{F}_\Omega/\mathscr{F}_\Gamma$, we obtain an isomorphism

$$\mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Omega, \mathbf{J}^\Lambda_\lambda(M)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}(\mathscr{F}_\Gamma, \mathbf{J}^\Lambda_\lambda(M)),$$

and the proof is complete. □

We can now define the functor

$$\mathrm{gr}^\Lambda_\lambda : \mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G,\Bbbk) \to \mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)$$

by

$$\mathrm{gr}^\Lambda_\lambda(\mathscr{F}) = \mathscr{F}_\Omega/\mathscr{F}_{\Omega\smallsetminus\{\lambda\}},$$

where $\Omega \subset \mathbf{X}^\vee$ is any ideal in which λ is maximal. From Lemma 4.3.4 we see that $\mathrm{gr}^\Lambda_\lambda$ is well defined and takes values in the essential image of the functor $\mathbf{J}^\Lambda_\lambda$, and from the considerations in §4.3.2 we see that for any given $\mathscr{F}$ in $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G,\Bbbk)$, there exist only finitely many λ's such that $\mathrm{gr}^\Lambda_\lambda(\mathscr{F}) \neq 0$. We will also set

$$\mathrm{Grad}^\Lambda_\lambda(\mathscr{F}) := \mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)}\big(\mathbf{J}^\Lambda_\lambda(\Bbbk), \mathrm{gr}^\Lambda_\lambda(\mathscr{F})\big),$$

so that we have a canonical isomorphism

$$\mathrm{gr}^\Lambda_\lambda(\mathscr{F}) \cong \mathbf{J}^\Lambda_\lambda(\mathrm{Grad}^\Lambda_\lambda(\mathscr{F})), \tag{4.3.3}$$

see Proposition 4.2.5(1). Observe that $\mathscr{F}$ belongs to the subcategory of $\mathsf{Perv}_I(\mathrm{Fl}_G,\Bbbk)$ generated under extensions by the objects $\mathbf{J}^\Lambda_\lambda(\mathrm{Grad}^\Lambda_\lambda(\mathscr{F}))$, where λ runs over those values such that $\mathrm{gr}^\Lambda_\lambda(\mathscr{F}) \neq 0$.

It is clear that for any $\lambda, \mu \in \mathbf{X}^\vee$ and M in $\mathsf{Mof}_\Bbbk$ we have

$$\mathrm{Grad}^\Lambda_\mu(\mathbf{J}^\Lambda_\lambda(M)) \cong \begin{cases} M & \text{if } \mu = \lambda; \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.4}$$

It will sometimes be convenient to combine all the $\mathrm{Grad}^\Lambda_\lambda$ (for $\lambda \in \mathbf{X}$) into a single object. Let $\mathsf{Mof}^{\mathbf{X}^\vee}_\Bbbk$ be the (abelian) category of $\mathbf{X}^\vee$-graded finitely generated $\Bbbk$-modules, and define a functor

$$\mathrm{Grad}^\Lambda_{\mathbf{X}^\vee} : \mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G,\Bbbk) \to \mathsf{Mof}^{\mathbf{X}^\vee}_\Bbbk \tag{4.3.5}$$

by

$$\mathrm{Grad}^\Lambda_{\mathbf{X}^\vee}(\mathscr{F}) := \bigoplus_{\lambda\in\mathbf{X}^\vee} \mathrm{Grad}^\Lambda_\lambda(\mathscr{F}),$$

where the right-hand side has the obvious $\mathbf{X}^\vee$-grading.

Remark 4.3.5. — It is not clear from the definition whether the functor $\mathrm{gr}^\Lambda_\lambda$ is exact. We will address this question in Remark 4.6.4 below.

4.3.4. Stability under direct summands. — In this subsection, we will prove that $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$ is closed under taking direct summands. To do this, we will need the following auxiliary notion: a perverse sheaf $\mathscr{F} \in \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$ is said to admit a *(Λ-)dominant Wakimoto filtration* if it admits a finite filtration whose subquotients are of the form $\mathbf{J}_\lambda^\Lambda(M) \cong \nabla^I_{\mathsf{t}(\lambda)}(M)$ with $\lambda \in \Lambda$ and $M \in \mathsf{Mof}_\Bbbk$.

Lemma 4.3.6. — *A perverse sheaf $\mathscr{F} \in \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$ admits a Λ-Wakimoto filtration if and only if there is some $\lambda \in \Lambda$ such that $\nabla^I_{\mathsf{t}(\lambda)}(\Bbbk) \star^I \mathscr{F}$ is perverse and admits a Λ-dominant Wakimoto filtration.*

Proof. — If $\mathscr{F}$ admits a Wakimoto filtration, say with subquotients denoted by $\mathbf{J}^\Lambda_{\lambda_1}(M_1), \dots, \mathbf{J}^\Lambda_{\lambda_r}(M_r)$, then choose $\lambda \in \Lambda$ such that $\lambda + \lambda_1, \dots, \lambda + \lambda_r$ all lie in Λ. In this setting Lemma 4.2.6 implies that $\nabla^I_{\mathsf{t}(\lambda)}(\Bbbk) \star^I \mathscr{F}$ is perverse and admits a dominant Wakimoto filtration.

Conversely, let $\mathscr{G} = \nabla^I_{\mathsf{t}(\lambda)}(\Bbbk) \star^I \mathscr{F}$, and assume that $\mathscr{G}$ is perverse and admits a dominant Wakimoto filtration. Since $\lambda \in \Lambda$ we have $\nabla^I_{\mathsf{t}(\lambda)}(\Bbbk) \cong \mathbf{J}^\Lambda_\lambda(\Bbbk)$, and using Lemma 4.2.6 we see that

$$\mathscr{F} \cong \mathbf{J}^\Lambda_{-\lambda}(\Bbbk) \star^I \mathscr{G}.$$

Using again Lemma 4.2.6 we obtain that $\mathscr{F}$ admits a Wakimoto filtration. $\square$

Lemma 4.3.7. — *Let $\mathscr{F} \in D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$. We have that $\mathscr{F}$ is a perverse sheaf admitting a Λ-dominant Wakimoto filtration if and only if*

$$(4.3.6) \qquad {}^{\mathrm{p}}\mathscr{H}^n(j_w^!(\mathscr{F})) \neq 0 \quad \Longrightarrow \quad n = 0 \text{ and } w \in \{\mathsf{t}(\mu) : \mu \in \Lambda\}.$$

Proof. — The condition (4.3.6) is clearly stable under extensions, and it is satisfied by $\mathbf{J}^\Lambda_\lambda(M) \cong \nabla^I_{\mathsf{t}(\lambda)}(M)$ for any $\lambda \in \Lambda$ and any $M \in \mathsf{Mof}_\Bbbk$. Thus, if a perverse sheaf $\mathscr{F}$ admits a dominant Wakimoto filtration, it satisfies (4.3.6).

For the opposite implication, suppose $\mathscr{F}$ is a perverse sheaf satisfying (4.3.6). Of course, there can be only finitely many $w \in W$ such that $j_w^! \mathscr{F} \neq 0$. Moreover, if $\mathrm{Fl}_{G,w}$ is open in the support of $\mathscr{F}$, then $j_w^! \mathscr{F} \neq 0$.

We proceed by induction on the number of w's such that $j_w^! \mathscr{F} \neq 0$. If there are no such w, then we must have $\mathscr{F} = 0$, and there is nothing to prove. Otherwise, let v be a maximal element of W such that $j_v^! \mathscr{F} \neq 0$, and set $M := \mathrm{Hom}(\Delta^I_v(\Bbbk), \mathscr{F}) \cong \mathrm{Hom}(\underline{\Bbbk}_{\mathrm{Fl}_{G,v}}[\ell(v)], j_v^! \mathscr{F})$. We then have an isomorphism $\mathscr{F}_{|\mathrm{Fl}_{G,v}} \cong \underline{M}_{\mathrm{Fl}_{G,v}}[\ell(v)]$, and a canonical map $\theta : \mathscr{F} \to \nabla^I_v(M)$. (The maximality of v implies that $\mathrm{Fl}_{G,v}$ is open in the support of $\mathscr{F}$, so $j_v^! \mathscr{F} \cong j_v^* \mathscr{F} = \mathscr{F}_{|\mathrm{Fl}_{G,v}}$.) Complete the map θ to a distinguished triangle

$$\mathscr{F}' \to \mathscr{F} \xrightarrow{\theta} \nabla^I_v(M) \xrightarrow{[1]} .$$

Since θ is an isomorphism over $\mathrm{Fl}_{G,v}$, we have

$$j_w^! \mathscr{F}' \cong \begin{cases} j_w^! \mathscr{F} & \text{if } w \neq v, \\ 0 & \text{if } w = v. \end{cases}$$

Therefore, $\mathscr{F}'$ satisfies (4.3.6) as well, but there are fewer w's for which $j_w^! \mathscr{F}' \neq 0$. By induction, $\mathscr{F}'$ is a perverse sheaf with a dominant Wakimoto filtration. By assumption, $v = \mathsf{t}(\mu)$ for some $\mu \in \Lambda$, so $\nabla^I_v(M) \cong \mathbf{J}^\Lambda_\mu(M)$. The distinguished triangle above shows that $\mathscr{F}$ is also a perverse sheaf with a dominant Wakimoto filtration, which finishes the proof. □

Corollary 4.3.8. — *The subcategory* $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G, \Bbbk) \subset \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$ *is stable under direct summands.*

Proof. — Condition (4.3.6) is clearly stable direct summands, so this follows from Lemmas 4.3.6 and 4.3.7. □

4.4. Central sheaves admit Wakimoto filtrations

4.4.1. An existence criterion. — Our goal in this subsection is to explain the proof of the following criterion which guarantees the existence of a Wakimoto filtration.

Proposition 4.4.1. — *Let* $\mathscr{F}$ *in* $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$*. Assume that*

1. *for any* $\nu \in -\Lambda$ *we have an isomorphism*

$$\Delta^I_{\mathsf{t}(\nu)}(\Bbbk) \star^I \mathscr{F} \cong \mathscr{F} \star^I \Delta^I_{\mathsf{t}(\nu)}(\Bbbk), \tag{4.4.1}$$

   and moreover these objects are perverse;

2. *for any* $\nu \in \Lambda$ *the object* $\nabla^I_{\mathsf{t}(\nu)}(\Bbbk) \star^I \mathscr{F}$ *is perverse.*

Then $\mathscr{F}$ *admits a* (Λ-)*Wakimoto filtration.*

The proof will require some preliminaries.

Lemma 4.4.2. — *Let* $X \subset \mathrm{Fl}_G$ *be a finite union of* I*-orbits. Then there exists a finite subset* $A_X \subset W$ *such that for any* $x \in W$ *we have*

$$m'(q^{-1}(\mathrm{Fl}_{G,x}) \times^I X) \subset \bigcup_{y \in A_X} \mathrm{Fl}_{G,x\cdot y}, \quad m'(q^{-1}(X) \times^I \mathrm{Fl}_{G,x}) \subset \bigcup_{y \in A_X} \mathrm{Fl}_{G,y\cdot x}.$$

Proof. — Of course we can assume that $X = \mathrm{Fl}_{G,w}$ for some $w \in W$. In this case, we prove the statement by induction on $\ell(w)$. In case $\ell(w) = 0$, the subset $A = \{w\}$ satisfies the required properties, since for any $x \in W$ the morphism m' induces isomorphisms

$$q^{-1}(\mathrm{Fl}_{G,x}) \times^I \mathrm{Fl}_{G,w} \xrightarrow{\sim} \mathrm{Fl}_{G,xw}, \quad q^{-1}(\mathrm{Fl}_{G,w}) \times^I \mathrm{Fl}_{G,x} \xrightarrow{\sim} \mathrm{Fl}_{G,wx}.$$

Now we assume that $\ell(w) > 0$, and write $w = sw' = w''t$ with $\ell(w') = \ell(w'') = \ell(w) - 1$ and $s, t \in S$. By induction we have finite subsets $A', A'' \subset W$ such that

$$m'(q^{-1}(\mathrm{Fl}_{G,x}) \times^I \mathrm{Fl}_{G,w'}) \subset \bigcup_{y \in A'} \mathrm{Fl}_{G,x\cdot y}, \quad m'(q^{-1}(\mathrm{Fl}_{G,w''}) \times^I \mathrm{Fl}_{G,x}) \subset \bigcup_{y \in A''} \mathrm{Fl}_{G,y\cdot x}$$

for any $x \in W$. Then the morphism m' induces isomorphisms

$$q^{-1}(\mathrm{Fl}_{G,s}) \times^I \mathrm{Fl}_{G,w'} \xrightarrow{\sim} \mathrm{Fl}_{G,w} \quad \text{and} \quad q^{-1}(\mathrm{Fl}_{G,w''}) \times^I \mathrm{Fl}_{G,t} \xrightarrow{\sim} \mathrm{Fl}_{G,w}.$$

Since moreover we have, for any $x \in W$,

$$m'(q^{-1}(\mathrm{Fl}_{G,x}) \times^I \mathrm{Fl}_{G,s}) \subset \mathrm{Fl}_{G,x} \sqcup \mathrm{Fl}_{G,xs},$$
$$m'(q^{-1}(\mathrm{Fl}_{G,t}) \times^I \mathrm{Fl}_{G,x}) \subset \mathrm{Fl}_{G,x} \sqcup \mathrm{Fl}_{G,tx},$$

we deduce that the subset $A := A' \cup sA' \cup A'' \cup A''t$ satisfies the required property for w. □

We now define the $*$-support and the !-support of an object $\mathscr{F}$ of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ as follows:

$$*\text{-}\mathrm{Supp}(\mathscr{F}) = \{w \in W \mid j_w^*(\mathscr{F}) \neq 0\};$$
$$!\text{-}\mathrm{Supp}(\mathscr{F}) = \{w \in W \mid j_w^!(\mathscr{F}) \neq 0\}.$$

The main point of introducing these notions is given by the following lemma.

Lemma 4.4.3. — *For any $\mathscr{F}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, the object $\mathscr{F}$ belongs to the subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ generated under extensions by the objects $(j_w)_! j_w^* \mathscr{F}$ where w runs over $*$-$\mathrm{Supp}(\mathscr{F})$, as well as to the subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ generated under extensions by the objects $(j_w)_* j_w^! \mathscr{F}$ where w runs over !-$\mathrm{Supp}(\mathscr{F})$.*

Proof. — We proceed by induction on the smallest closed union X of I-orbits over which $\mathscr{F}$ is supported. If $X = \varnothing$ then $\mathscr{F} = 0$, and there is nothing to prove.

Now, assume that $X \neq \varnothing$, and choose $w \in W$ such that $\mathrm{Fl}_{G,w}$ is open in X. Then we have $j_w^* \mathscr{F} \cong j_w^! \mathscr{F} \neq 0$, so that w belongs both to $*$-$\mathrm{Supp}(\mathscr{F})$ and to !-$\mathrm{Supp}(\mathscr{F})$. If $i : X \smallsetminus \mathrm{Fl}_{G,w} \hookrightarrow X$ is the closed embedding, then we have distinguished triangles

$$i_! i^! \mathscr{F} \to \mathscr{F} \to (j_w)_* j_w^! \mathscr{F} \xrightarrow{[1]} \quad \text{and} \quad (j_w)_! j_w^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \xrightarrow{[1]} .$$

Moreover, for any $x \in W$ such that $\mathrm{Fl}_{G,x} \subset X \smallsetminus \mathrm{Fl}_{G,w}$ we have

$$j_x^* i_* i^* \mathscr{F} \cong j_x^* \mathscr{F}, \qquad j_x^! i_! i^! \mathscr{F} \cong j_x^! \mathscr{F}.$$

Using these isomorphisms and induction (with the first claim applied to $i_* i^* \mathscr{F}$, and the second one applied to $i_! i^! \mathscr{F}$), we deduce the desired claims for $\mathscr{F}$. □

Proposition 4.4.4. — *For any $\mathscr{F}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, there exists a finite subset $A_{\mathscr{F}} \subset W$ such that for any $x \in W$ and any M in $D^{\mathrm{b}}\mathsf{Mof}_{\Bbbk}$ we have*

$$*\text{-}\mathrm{Supp}(\Delta^I_x(M) \star^I \mathscr{F}) \subset x \cdot A_{\mathscr{F}}, \quad !\text{-}\mathrm{Supp}(\nabla^I_x(M) \star^I \mathscr{F}) \subset x \cdot A_{\mathscr{F}},$$
$$*\text{-}\mathrm{Supp}(\mathscr{F} \star^I \Delta^I_x(M)) \subset A_{\mathscr{F}} \cdot x, \quad !\text{-}\mathrm{Supp}(\mathscr{F} \star^I \nabla^I_x(M)) \subset A_{\mathscr{F}} \cdot x.$$

Proof. — Let $X \subset \mathrm{Fl}_G$ be a closed finite union of I-orbits such that $\mathscr{F}$ is supported on X. Then the base change theorem shows that $j_w^*(\Delta^I_x(M) \star^I \mathscr{F}) = 0$ unless $\mathrm{Fl}_{G,w} \subset m'(q^{-1}(\mathrm{Fl}_{G,x}) \times^I X)$; in other words we have

$$*\text{-}\mathrm{Supp}(\Delta^I_x(M) \star^I \mathscr{F}) \subset \{w \in W \mid \mathrm{Fl}_{G,w} \subset m'(q^{-1}(\mathrm{Fl}_{G,x}) \times^I X)\}.$$

Similar considerations for the objects $\nabla^I_x(M) \star^I \mathscr{F}$, $\mathscr{F} \star^I \Delta^I_x(M)$ and $\mathscr{F} \star^I \nabla^I_x(M)$ show that if A_X is a finite subset of W as in Lemma 4.4.2 (for our given X), then the subset $A_{\mathscr{F}} = A_X$ satisfies the required properties. □

Proof of Proposition 4.4.1. — Let $A_{\mathscr{F}} \subset W$ be a finite subset as in Proposition 4.4.4, and choose (as we may) $\nu \in -\Lambda$ such that

$$\mathsf{t}(\nu) \cdot A_{\mathscr{F}} \subset \{\mathsf{t}(\mu) \cdot x : \mu \in -\Lambda^\circ,\ x \in W_{\mathrm{f}}\},$$
$$A_{\mathscr{F}} \cdot \mathsf{t}(\nu) \subset \{x \cdot \mathsf{t}(\mu) : \mu \in -\Lambda^\circ,\ x \in W_{\mathrm{f}}\}.$$

Then, in particular, we have

$$(\mathsf{t}(\nu) \cdot A_{\mathscr{F}}) \cap (A_{\mathscr{F}} \cdot \mathsf{t}(\nu)) \subset \{\mathsf{t}(\mu) : \mu \in -\Lambda^\circ\}, \tag{4.4.2}$$

see e.g. [**Hu1**, §13.2, Lemma A].

Since $\nu \in -\Lambda$, we have $\mathbf{J}^\Lambda_\nu(\Bbbk) = \Delta^I_{\mathsf{t}(\nu)}(\Bbbk)$. Hence (4.4.1), (4.4.2), and the choice of $A_{\mathscr{F}}$ imply that

$$*\text{-Supp}(\mathbf{J}^\Lambda_\nu(\Bbbk) \star^I \mathscr{F}) \subset \{\mathsf{t}(\mu) : \mu \in -\Lambda^\circ\}.$$

On the other hand, we have assumed that $\mathbf{J}^\Lambda_\nu(\Bbbk) \star^I \mathscr{F}$ is perverse, so that for any $w \in W$ the complex $j_w^*(\mathbf{J}^\Lambda_\nu(\Bbbk) \star^I \mathscr{F})$ is concentrated in degrees $\leq -\ell(w)$. These two observations, together with Lemma 4.4.3, ensure that the object $\mathbf{J}^\Lambda_\nu(\Bbbk) \star^I \mathscr{F}$ belongs to the full subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ generated under extensions by objects of the form $\Delta^I_{\mathsf{t}(\mu)}(M)[n]$ with $\mu \in -\Lambda^\circ$, $M \in \mathsf{Mof}_\Bbbk$ and $n \in \mathbb{Z}_{\geq 0}$, or in other words by objects of the form $\mathbf{J}^\Lambda_\mu(M)[n]$ with $\mu \in -\Lambda^\circ$, $M \in \mathsf{Mof}_\Bbbk$ and $n \in \mathbb{Z}_{\geq 0}$.

Convolving on the left with $\mathbf{J}^\Lambda_{-\nu}(\Bbbk)$ and using Lemma 4.2.6, we deduce that $\mathscr{F}$ can be obtained from some collection of objects $(\mathbf{J}^\Lambda_{\mu_i}(M_i)[n_i])_{i \in I}$ where I is a finite set, $\mu_i \in \mathbf{X}^\vee$, $n_i \in \mathbb{Z}_{\geq 0}$, and $M_i \in \mathsf{Mof}_\Bbbk$, by taking successive extensions. If we choose some $\eta \in \Lambda$ such that $\mu_i + \eta \in \Lambda$ for any i, it follows (using again Lemma 4.2.6) that the object $\mathbf{J}^\Lambda_\eta(\Bbbk) \star^I \mathscr{F}$ belongs to the full subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ generated under extensions by objects of the form $\mathbf{J}^\Lambda_\mu(M)[n]$ with $\mu \in \Lambda$, $M \in \mathsf{Mof}_\Bbbk$ and $n \in \mathbb{Z}_{\geq 0}$, or in other words by objects of the form $\nabla^I_{\mathsf{t}(\mu)}(M)[n]$ with $\mu \in \Lambda$, $M \in \mathsf{Mof}_\Bbbk$ and $n \in \mathbb{Z}_{\geq 0}$. We then have $!\text{-Supp}(\mathbf{J}^\Lambda_\eta(\Bbbk) \star^I \mathscr{F}) \subset \{\mathsf{t}(\mu) : \mu \in \Lambda\}$, and moreover for each $\mu \in \Lambda$ the complex $j^!_{\mathsf{t}(\mu)}(\mathbf{J}^\Lambda_\eta(\Bbbk) \star^I \mathscr{F})$ is concentrated in degrees $\leq -\ell(\mathsf{t}(\mu))$. On the other hand, by assumption $\mathbf{J}^\Lambda_\eta(\Bbbk) \star^I \mathscr{F}$ is perverse, so that this complex is also concentrated in degrees $\geq -\ell(\mathsf{t}(\mu))$. Hence it is concentrated in degree $-\ell(\mathsf{t}(\mu))$. By Lemma 4.3.7, $\mathbf{J}^\Lambda_\eta(\Bbbk) \star^I \mathscr{F}$ admits a dominant Wakimoto filtration, and then by Lemma 4.3.6, $\mathscr{F}$ admits a Wakimoto filtration. □

4.4.2. Central sheaves admit filtrations by Wakimoto sheaves. — We can now prove the main result of this section.

Theorem 4.4.5. — *For any $\mathscr{F}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, the perverse sheaf $\mathsf{Z}(\mathscr{F})$ admits a Wakimoto filtration.*

Proof. — We will apply the criterion of Proposition 4.4.1 to the perverse sheaf $\mathsf{Z}(\mathscr{F})$. Theorem 3.2.3 guarantees the existence of isomorphisms (4.4.1), so to conclude it suffices to prove that for any $w \in W$ the objects

$$\Delta^I_w(\Bbbk) \star^I \mathsf{Z}(\mathscr{F}) \quad \text{and} \quad \nabla^I_w(\Bbbk) \star^I \mathsf{Z}(\mathscr{F})$$

are perverse. For this, in view of Corollary 3.2.5 it suffices to prove that the complexes

$$\mathscr{F} \overset{L}{\boxtimes}_{\Bbbk} \Delta^I_w(\Bbbk) \quad \text{and} \quad \mathscr{F} \overset{L}{\boxtimes}_{\Bbbk} \nabla^I_w(\Bbbk)$$

are perverse sheaves. However, the first complex can be obtained from the perverse sheaf $\mathscr{F}$ by shifted pullback along the natural projection $\mathrm{Gr}_G \times \mathrm{Fl}_{G,w} \to \mathrm{Gr}_G$ (a smooth morphism) followed by !-pushforward along the embedding $\mathrm{Gr}_G \times \mathrm{Fl}_{G,w} \hookrightarrow \mathrm{Gr}_G \times \mathrm{Fl}_G$ (an affine embedding). Since both of these operations are t-exact (see [**Ac3**, Corollary 3.5.9 and Proposition 3.6.1] respectively), it follows that $\mathscr{F} \boxtimes^L_{\Bbbk} \Delta^I_w(\Bbbk)$ is indeed perverse. Similar considerations show that $\mathscr{F} \boxtimes^L_{\Bbbk} \nabla^I_w(\Bbbk)$ is perverse, which finishes the proof. □

4.5. Cohomology of Wakimoto-filtered perverse sheaves

4.5.1. Cohomology. — In this subsection we study the consequences of the existence of a Wakimoto filtration on cohomology. For this we must first describe the cohomology of Wakimoto sheaves.

Lemma 4.5.1. — *For any $\lambda \in \mathbf{X}^\vee$ and $M \in \mathsf{Mof}_{\Bbbk}$, we have canonical isomorphisms*

$$\mathsf{H}^n(\mathrm{Fl}_G, \mathbf{J}^\Lambda_\lambda(M)) \cong \begin{cases} M & \textit{if } n = -\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle; \\ 0 & \textit{otherwise.} \end{cases}$$

In order to prove this lemma we will need some preparation. It is well known that the algebra $\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)$ identifies canonically with $\mathrm{Sym}_{\Bbbk}(\Bbbk \otimes_{\mathbb{Z}} \mathbf{X})$. For $\mathscr{F}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$, we can consider its ordinary cohomology $\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F})$, as well as its equivariant cohomology $\mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F})$. By construction the latter $\Bbbk$-module is a graded module over $\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)$; this action will be called the *left* action. But since Fl_G is obtained by taking the quotient of $\mathrm{L}G$ by I, $\mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F})$ has a second $\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)$-module structure, called the *right* action. The right action is inherited by $\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F})$.

Consider now two objects $\mathscr{F}$ and $\mathscr{G}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$. Then there exists a canonical morphism of graded $\Bbbk$-modules

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \otimes_{\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)} \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{G}) \to \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F} \star^I \mathscr{G}) \tag{4.5.1}$$

where the tensor product is taken with respect to the right $\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)$-action on $\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F})$ and the left $\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)$-action on $\mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{G})$. This morphism sends $f \otimes g$ (where $f \in \mathsf{H}^n(\mathrm{Fl}_G, \mathscr{F})$ is regarded as a morphism $\underline{\Bbbk}_{\mathrm{Fl}_G} \to \mathscr{F}[n]$ and $g \in \mathsf{H}^m_I(\mathrm{Fl}_G, \mathscr{G})$ is regarded as a morphism $\underline{\Bbbk}_{\mathrm{Fl}_G} \to \mathscr{G}[m]$) to the morphism

$$f \mathbin{\widetilde{\boxtimes}} g : \underline{\Bbbk}_{\mathrm{L}G \times^I \mathrm{Fl}_G} = \underline{\Bbbk}_{\mathrm{Fl}_G} \mathbin{\widetilde{\boxtimes}} \underline{\Bbbk}_{\mathrm{Fl}_G} \to \mathscr{F} \mathbin{\widetilde{\boxtimes}} \mathscr{G}[n+m],$$

regarded as an element in

$$\mathsf{H}^{n+m}(\mathrm{L}G \times^I \mathrm{Fl}_G, \mathscr{F} \mathbin{\widetilde{\boxtimes}} \mathscr{G}) \cong \mathsf{H}^{n+m}(\mathrm{Fl}_G, \mathscr{F} \star^I \mathscr{G}).$$

(This construction is similar to that in (3.2.11).)

Lemma 4.5.2. — *In case $\mathscr{G} = \nabla^I_w(\Bbbk)$ for some $w \in W$, the morphism* (4.5.1) *is an isomorphism for any $\mathscr{F}$ in $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$.*

Proof. — If $\mathscr{G} = \nabla^I_w(\Bbbk)$ then we have

$$\mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{G}) = \mathsf{H}^{\bullet+\ell(w)}_I(\mathrm{Fl}_{G,w}; \Bbbk),$$

which is canonically isomorphic to $\mathsf{H}^{\bullet+\ell(w)}_I(\mathrm{pt}; \Bbbk)$. On the other hand,

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F} \star^I \mathscr{G}) = \mathsf{H}^\bullet(\mathrm{L}G \times^I \mathrm{Fl}_G, \mathscr{F} \mathbin{\widetilde{\boxtimes}} \mathscr{G})$$

identifies canonically with the total cohomology of the shifted pullback of $\mathscr{F}$ under the morphism $\mathrm{L}G \times^I \mathrm{Fl}_{G,w} \to \mathrm{Fl}_G$ defined by $[a, b] \mapsto aI$. The desired claim is then clear. □

Proof of Lemma 4.5.1. — Choose some $\mu \in \Lambda_\lambda$, and recall that we have a canonical isomorphism

$$\mathbf{J}^\Lambda_\lambda(M) \star^I \nabla^I_{\mathsf{t}(\mu-\lambda)}(\Bbbk) \cong \nabla^I_{\mathsf{t}(\mu)}(M).$$

Applying Lemma 4.5.2 we obtain a canonical isomorphism

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathbf{J}^\Lambda_\lambda(M)) \otimes_{\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)} \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \nabla^I_{\mathsf{t}(\mu-\lambda)}(\Bbbk)) \xrightarrow{\sim} \mathsf{H}^\bullet(\mathrm{Fl}_G, \nabla^I_{\mathsf{t}(\mu)}(M)).$$

Since (as in the proof of Lemma 4.5.2) $\mathsf{H}^\bullet_I(\mathrm{Fl}_G, \nabla^I_{\mathsf{t}(\mu-\lambda)}(\Bbbk))$ is canonically isomorphic to $\mathsf{H}^{\bullet+\ell(\mathsf{t}(\mu-\lambda))}_I(\mathrm{pt}; \Bbbk)$ and since

$$\mathsf{H}^m(\mathrm{Fl}_G, \nabla^I_{\mathsf{t}(\mu)}(M)) = \begin{cases} M & \text{if } m = -\ell(\mathsf{t}(\mu)); \\ 0 & \text{otherwise,} \end{cases}$$

we deduce isomorphisms

$$\mathsf{H}^m(\mathrm{Fl}_G, \mathbf{J}^\Lambda_\lambda(M)) \cong \begin{cases} M & \text{if } m = -\ell(\mathsf{t}(\mu)) + \ell(\mathsf{t}(\mu-\lambda)); \\ 0 & \text{otherwise.} \end{cases} \tag{4.5.2}$$

Now since μ and $\mu - \lambda$ belong to Λ, using (4.1.5) and (4.1.3) we see that

$$\ell(\mathsf{t}(\mu)) = \ell(\mathsf{t}(x_\Lambda^{-1}(\mu))) = \langle x_\Lambda^{-1}(\mu), 2\rho\rangle$$

and similarly that

$$\ell(\mathsf{t}(\mu - \lambda)) = \langle x_\Lambda^{-1}(\mu - \lambda), 2\rho\rangle,$$

so that

$$\ell(\mathsf{t}(\mu)) - \ell(\mathsf{t}(\mu - \lambda)) = \langle x_\Lambda^{-1}(\lambda), 2\rho\rangle. \tag{4.5.3}$$

One can check that the isomorphism (4.5.2) does not depend on the choice of μ, and hence is canonical. □

Remark 4.5.3. — Lemma 4.5.1 shows that the functor $\mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, -)$ is left inverse to the functor

$$\mathbf{J}^\Lambda_\lambda : \mathsf{Mof}_\Bbbk \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk).$$

We deduce that there exists a canonical isomorphism

$$\mathrm{Grad}^\Lambda_\lambda(\mathscr{F}) \cong \mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, \mathrm{gr}^\Lambda_\lambda(\mathscr{F}))$$

for any $\mathscr{F}$ in $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G, \Bbbk)$. In fact, if $\Omega = \{\mu \in \mathbf{X}^\vee \mid \mu \preceq_\Lambda \lambda\}$, we similarly have a canonical isomorphism

$$\mathrm{Grad}^\Lambda_\lambda(\mathscr{F}) \cong \mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{F}_\Omega). \tag{4.5.4}$$

As a consequence we obtain the following property.

Proposition 4.5.4. — *For any $\mathscr{F}$ in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$ and any $n \in \mathbb{Z}$ there exists a canonical (in particular, functorial) isomorphism of $\Bbbk$-modules*

$$\mathsf{H}^n(\mathrm{Fl}_G, \mathscr{F}) \cong \bigoplus_{\substack{\lambda \in \mathbf{X}^\vee \\ \langle x_\Lambda^{-1}(\lambda), 2\rho\rangle = -n}} \mathrm{Grad}_\lambda^\Lambda(\mathscr{F}). \tag{4.5.5}$$

Taking the sum over all n, this proposition yields a canonical isomorphism

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \xrightarrow{\sim} \mathrm{Grad}_{\mathbf{X}^\vee}^\Lambda(\mathscr{F}). \tag{4.5.6}$$

Proof. — We set $X = \{\lambda \in \mathbf{X}^\vee \mid \mathrm{Grad}_\lambda^\Lambda(\mathscr{F}) \neq 0\}$. We can of course assume that $\mathscr{F}$ is supported on a connected component Y of Fl_G; then, in view of Proposition 4.2.3, for any $\lambda \in X$ we have $\mathrm{Fl}_{G,\mathrm{t}(\lambda)} \subset Y$. It follows that X is contained in some $\mathbb{Z}\mathfrak{R}^\vee$-coset in $\mathbf{X}^\vee$, so that $\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle$ has the same parity for all $\lambda \in X$. In this setting, we proceed by induction on the cardinality of

$$\{\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle : \lambda \in X\} \subset \mathbb{Z}.$$

If this set is empty then $\mathscr{F} = 0$ and there is nothing to prove. Otherwise let n be the smallest integer in this set, and let $X_{\min} = \{\lambda \in X \mid \langle x_\Lambda^{-1}(\lambda), 2\rho\rangle = n\}$. Since the function $\mu \mapsto \langle x_\Lambda^{-1}(\mu), 2\rho\rangle$ is order-preserving with respect to $\preceq_\Lambda$ and the standard order on $\mathbb{Z}$, all the elements in $X_{\min}$ are minimal in X. In particular, if $\Omega = \bigcup_{\lambda \in X_{\min}} \{\mu \in \mathbf{X}^\vee \mid \mu \preceq_\Lambda \lambda\}$, then $\mathscr{F}_\Omega$ has a Wakimoto filtration with subquotients $\mathrm{gr}_\lambda^\Lambda(\mathscr{F})$ where λ runs over $X_{\min}$, and $\mathscr{F}/\mathscr{F}_\Omega$ has a Wakimoto filtration with subquotients $\mathrm{gr}_\mu^\Lambda(\mathscr{F})$ where μ runs over $X \smallsetminus X_{\min}$. Now since the elements in $X_{\min}$ are pairwise incomparable, Lemma 4.3.1 and (4.3.3) ensure that there exists a canonical isomorphism

$$\mathscr{F}_\Omega \xrightarrow{\sim} \bigoplus_{\lambda \in X_{\min}} \mathbf{J}_\lambda^\Lambda\big(\mathrm{Grad}_\lambda^\Lambda(\mathscr{F})\big).$$

In particular, by Lemma 4.5.1 we have

$$\mathsf{H}^m(\mathrm{Fl}_G, \mathscr{F}_\Omega) \cong \begin{cases} \bigoplus_{\lambda \in X_{\min}} \mathrm{Grad}_\lambda^\Lambda(\mathscr{F}) & \text{if } m = -n; \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by induction we have

$$\mathsf{H}^m(\mathrm{Fl}_G, \mathscr{F}/\mathscr{F}_\Omega) \cong \begin{cases} \bigoplus_{\substack{\mu \in X \\ \langle x_\Lambda^{-1}(\mu), 2\rho\rangle = -m}} \mathrm{Grad}_\mu^\Lambda(\mathscr{F}) & \text{if } m \leq -n-2; \\ 0 & \text{otherwise.} \end{cases}$$

From these computations we obtain that the exact sequence $\mathscr{F}_\Omega \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{F}/\mathscr{F}_\Omega$ induces isomorphisms

$$\mathsf{H}^m(\mathrm{Fl}_G, \mathscr{F}) \cong \begin{cases} \mathsf{H}^m(\mathrm{Fl}_G, \mathscr{F}_\Omega) & \text{if } m = -n; \\ \mathsf{H}^m(\mathrm{Fl}_G, \mathscr{F}/\mathscr{F}_\Omega) & \text{otherwise,} \end{cases}$$

which completes the proof. □

4.5.2. Equivariant cohomology. — Proposition 4.5.4 also has consequences for the *equivariant* cohomology of perverse sheaves admitting Wakimoto filtrations. We first start with the following lemma, which should be compared with Lemma 3.3.9(2). (Note however that we do not have to impose any assumption on $\Bbbk$ in the present case.)

Lemma 4.5.5. — *For any $\mathscr{F}$ in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$, there exists a noncanonical isomorphism of $\mathsf{H}_I^\bullet(\mathrm{pt}; \Bbbk)$-modules*

$$\mathsf{H}_I^\bullet(\mathrm{Fl}_G, \mathscr{F}) \cong \mathsf{H}_I^\bullet(\mathrm{pt}; \Bbbk) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}). \tag{4.5.7}$$

Moreover, the forgetful functor induces an isomorphism

$$\Bbbk \otimes_{\mathsf{H}_I^\bullet(\mathrm{pt};\Bbbk)} \mathsf{H}_I^\bullet(\mathrm{Fl}_G, \mathscr{F}) \xrightarrow{\sim} \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}).$$

Proof. — Of course we can assume that $\mathscr{F}$ is supported on a single connected component of Fl_G. There exists a convergent spectral sequence

$$E_2^{p,q} = \mathsf{H}_I^p(\mathrm{pt}; \Bbbk) \otimes_\Bbbk \mathsf{H}^q(\mathrm{Fl}_G, \mathscr{F}) \ \Rightarrow\ \mathsf{H}_I^{p+q}(\mathrm{Fl}_G, \mathscr{F}),$$

and to prove both claims it suffices to prove that this spectral sequence degenerates at the E_2-page. However, we have $\mathsf{H}_I^p(\mathrm{pt}; \Bbbk) = 0$ unless p is even (see §4.5.1). On the other hand, it follows from our assumption that all the coweights λ such that $\mathrm{Grad}_\lambda(\mathscr{F}) \neq 0$ belong to the same $\mathbb{Z}\mathfrak{R}^\vee$-coset, so the parity of $\langle x_\Lambda^{-1}(\lambda), 2\rho \rangle$ is constant for these coweights. Thus, Proposition 4.5.4 implies that the integers q such that $\mathsf{H}^q(\mathrm{Fl}_G, \mathscr{F}) \neq 0$ all have the same parity, so that the spectral sequence indeed degenerates for parity reasons. □

Let us explain the noncanonicity of (4.5.7) in more detail. The degeneration of the spectral sequence in the preceding proof means that $\mathsf{H}_I^\bullet(\mathrm{Fl}_G, \mathscr{F})$ admits a filtration (as an $\mathsf{H}_I^\bullet(\mathrm{pt}; \Bbbk)$-module) whose subquotients are of the form $\mathsf{H}_I^\bullet(\mathrm{pt}; \Bbbk) \otimes_\Bbbk \mathsf{H}^q(\mathrm{Fl}_G, \mathscr{F})$. The isomorphism (4.5.7) is obtained by choosing a splitting of this filtration.

To get a better description of equivariant cohomology one can proceed as follows. For brevity, given $\mathscr{F}$ in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$ and $\lambda \in \mathbf{X}^\vee$, we set

$$\mathscr{F}_{\preceq_\Lambda \lambda} := \mathscr{F}_{\{\mu \in \mathbf{X}^\vee \mid \mu \preceq_\Lambda \lambda\}}. \tag{4.5.8}$$

Proposition 4.5.6. — *For any $\mathscr{F}$ in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$, the natural morphism of $\mathsf{H}_I^\bullet(\mathrm{pt}; \Bbbk)$-modules*

$$\bigoplus_{\lambda \in \mathbf{X}^\vee} \mathsf{H}_I^\bullet(\mathrm{pt}; \Bbbk) \otimes_\Bbbk \mathsf{H}_I^{-\langle x_\Lambda^{-1}(\lambda), 2\rho \rangle}(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda}) \to \mathsf{H}_I^\bullet(\mathrm{Fl}_G, \mathscr{F})$$

is an isomorphism.

Proof. — We begin by analyzing the left-hand side. By Proposition 4.5.4, we have

$$\mathsf{H}^n(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda}) \cong \begin{cases} \mathrm{Grad}^\Lambda_\lambda(\mathscr{F}_{\preceq_\Lambda \lambda}) \cong \mathrm{Grad}^\Lambda_\lambda(\mathscr{F}) & \text{if } n = -\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle, \\ \displaystyle\bigoplus_{\substack{\mu \in \mathbf{X}^\vee,\ \mu \prec_\Lambda \lambda \\ \langle x_\Lambda^{-1}(\mu), 2\rho\rangle = -n}} \mathrm{Grad}^\Lambda_\mu(\mathscr{F}) & \text{if } n > -\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle, \\ 0 & \text{if } n < -\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle. \end{cases}$$

Since $\mathsf{H}^n(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda})$ vanishes if $n < -\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle$, Lemma 4.5.5 implies that the morphism

$$\mathsf{H}_I^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda}) \to \mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda}) \tag{4.5.9}$$

induced by the forgetful map is an isomorphism. Moreover, both sides vanish for all but finitely many λ (namely, those λ such that $\mathrm{Grad}^\Lambda_\lambda(\mathscr{F}) \neq 0$). In particular, the direct sum appearing in the statement of the proposition has only finitely many nonzero terms.

Now we can prove the statement. As usual one can assume that $\mathscr{F}$ is supported on a single connected component of Fl_G, and we proceed by induction on the number of λ's such that $\mathrm{Grad}^\Lambda_\lambda(\mathscr{F}) \neq 0$. (Note that our assumption guarantees that the parity of $-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle$ is constant on these coweights.) If this number is 0 then $\mathscr{F} = 0$ and there is nothing to prove. If it is 1, then $\mathscr{F} \cong \mathbf{J}^\Lambda_\mu(M)$ for some $\mu \in \mathbf{X}^\vee$ and $M \in \mathsf{Mof}_\Bbbk$. The summand in the left-hand side parametrized by λ vanishes unless $\lambda = \mu$, and the claim follows easily from Lemma 4.5.1 and Lemma 4.5.5. If this number is at least 2, we choose μ minimal such that $\mathrm{Grad}^\Lambda_\mu(\mathscr{F}) \neq 0$, and consider the exact sequence

$$\mathscr{F}_{\preceq_\Lambda \mu} \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{F}/\mathscr{F}_{\preceq_\Lambda \mu}.$$

Lemma 4.5.1, Lemma 4.5.5 and parity considerations imply that applying $\mathsf{H}^\bullet_I(\mathrm{Fl}_G, -)$ we obtain an exact sequence

$$\mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \mu}) \hookrightarrow \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}) \twoheadrightarrow \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}/\mathscr{F}_{\preceq_\Lambda \mu}).$$

On the other hand, the identifications above show that this exact sequence also induces an exact sequence

$$\begin{aligned} &\bigoplus_{\lambda \in \mathbf{X}^\vee} \mathsf{H}^\bullet_I(\mathrm{pt}; \Bbbk) \otimes_\Bbbk \mathsf{H}_I^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, (\mathscr{F}_{\preceq_\Lambda \mu})_{\preceq_\Lambda \lambda}) \hookrightarrow \\ &\qquad \bigoplus_{\lambda \in \mathbf{X}^\vee} \mathsf{H}^\bullet_I(\mathrm{pt}; \Bbbk) \otimes_\Bbbk \mathsf{H}_I^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda}) \twoheadrightarrow \\ &\qquad\qquad \bigoplus_{\lambda \in \mathbf{X}^\vee} \mathsf{H}^\bullet_I(\mathrm{pt}; \Bbbk) \otimes_\Bbbk \mathsf{H}_I^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, (\mathscr{F}/\mathscr{F}_{\preceq_\Lambda \mu})_{\preceq_\Lambda \lambda}). \end{aligned}$$

We conclude using the 5-lemma. □

Using the calculations carried out in the preceding proof, we can rephrase Proposition 4.5.6 as follows: there is a canonical isomorphism

$$\bigoplus_{\lambda \in \mathbf{X}^\vee} \mathsf{H}^\bullet_I(\mathrm{pt}; \Bbbk) \otimes_\Bbbk \mathrm{Grad}^\Lambda_\lambda(\mathscr{F}) \to \mathsf{H}^\bullet_I(\mathrm{Fl}_G, \mathscr{F}).$$

4.5.3. Cohomology with support. — Let $\dot{x}_\Lambda$ be a lift of x_Λ in $N_G(T)$. We then set

$$B_\Lambda^+ := \dot{x}_\Lambda \cdot B^+ \cdot \dot{x}_\Lambda^{-1},$$

so that the roots appearing in the Lie algebra of B_Λ^+ are those in $\mathfrak{R}^\Lambda$. We also denote by U_Λ^+ the unipotent radical of B_Λ^+. Let $\chi \in \mathbf{X}^\vee$ be a cocharacter such that $x_\Lambda^{-1}(\chi)$ is regular dominant. As in §1.2.2.2, the action of $\mathbb{G}_\mathrm{m}$ on G via conjugation with χ defines an attractor scheme, which is the subgroup B_Λ^+, and a fixed points scheme, equal to T. We can also consider the action of $\mathbb{G}_\mathrm{m}$ on Fl_G via χ, which turns to be Zariski locally linearizable (see [**HR2**, Lemma 4.3]), and the associated ind-schemes

$$(\mathrm{Fl}_G)^+, \quad (\mathrm{Fl}_G)^0,$$

see Theorem 1.1.6. Here $(\mathrm{Fl}_G)^0$ is discrete, with underlying topological space W (seen as a subspace in Fl_G via the assignment $\mathsf{t}(\lambda)w \mapsto x^\lambda \dot{w}$ for $\lambda \in \mathbf{X}^\vee$ and $w \in W_\mathrm{f}$, where $\dot{w}$ is any lift of w in $N_G(T)$). As in §1.1.2.2 of Proposition 1.2.8 the natural morphism $(\mathrm{Fl}_G)^+ \to (\mathrm{Fl}_G)^0$ induces a bijection between sets of connected components, so that the connected component of $(\mathrm{Fl}_G)^+$ are in a natural bijection with W. We will denote by

$$\mathfrak{S}_w^\Lambda$$

the connected component associated with w. If $w = \mathsf{t}(\lambda) \cdot x$ with $\lambda \in \mathbf{X}^\vee$ and $x \in W_\mathrm{f}$, and if $\dot{x}$ is a lift of x in $N_G(T)$, then we have

$$(\mathfrak{S}_w^\Lambda)(\mathbb{C}) := (\mathrm{L}U_\Lambda^+)(\mathbb{C}) \cdot z^\lambda \dot{x} I.$$

The corresponding inclusion will be denoted

$$\sigma_w^\Lambda : \mathfrak{S}_w^\Lambda \to \mathrm{Fl}_G.$$

(As in §1.2.3.1, this morphism is representable by a locally closed immersion.)

Lemma 4.5.7. — *For any $\lambda \in \Lambda$ we have $\mathrm{Fl}_{G,\mathsf{t}(\lambda)} = (I \cap \mathrm{L}U_\Lambda^+) \cdot x^\lambda I$.*

Proof. — For $\alpha \in \mathfrak{R}$ and $m \in \mathbb{Z}$ we denote by $U_{\alpha,m} \subset \mathrm{L}G$ the subgroup which identifies, for any choice of isomorphism $u_\alpha : \mathbb{G}_\mathrm{a} \xrightarrow{\sim} U_\alpha$ (where U_α is the root subgroup of G associated with α), with the image of the morphism $t \mapsto u_\alpha(tx^m)$. If we fix an arbitrary order on $\mathfrak{R}_+$ and $-\mathfrak{R}_+$ and set, for any $\lambda \in \mathbf{X}^\vee$,

$$I^\lambda := \left(\prod_{\substack{\alpha \in \mathfrak{R}_+ \\ \langle \lambda, \alpha^\vee \rangle > 0}} \prod_{m=1}^{\langle \lambda, \alpha^\vee \rangle} U_{\alpha,m} \right) \times \left(\prod_{\substack{\alpha \in -\mathfrak{R}_+ \\ \langle \lambda, \alpha^\vee \rangle > 0}} \prod_{m=0}^{\langle \lambda, \alpha^\vee \rangle - 1} U_{\alpha,m} \right), \tag{4.5.10}$$

then it is well known that the I-action on the point $x^\lambda I \in \mathrm{Fl}_G$ induces an isomorphism

$$I^\lambda \xrightarrow{\sim} \mathrm{Fl}_{G,\mathsf{t}(\lambda)}.$$

In (4.5.10), only the roots α such that $\langle \lambda, \alpha^\vee \rangle > 0$ contribute. If we assume that $\lambda \in \Lambda$, then these roots all belong to $\mathfrak{R}^\Lambda$, which implies the desired claim. $\square$

Lemma 4.5.8. — *For any $\lambda \in \Lambda$, $w \in W$ and $\mathscr{F}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ we have a canonical isomorphism*

$$\mathsf{H}^{\bullet}_{\mathfrak{S}^{\Lambda}_{\mathsf{t}(\lambda)w}}\left(\mathrm{Fl}_G, \nabla^I_{\mathsf{t}(\lambda)}(\Bbbk) \star^I \mathscr{F}\right) \cong \mathsf{H}^{\bullet+\ell(\mathsf{t}(\lambda))}_{\mathfrak{S}^{\Lambda}_w}(\mathrm{Fl}_G, \mathscr{F}).$$

Proof. — By definition we have

$$\mathsf{H}^{\bullet}_{\mathfrak{S}^{\Lambda}_{\mathsf{t}(\lambda)w}}\left(\mathrm{Fl}_G, \nabla^I_{\mathsf{t}(\lambda)}(\Bbbk) \star^I \mathscr{F}\right) = \mathsf{H}^{\bullet}\left(\mathfrak{S}^{\Lambda}_{\mathsf{t}(\lambda)w}, (\sigma^{\Lambda}_{\mathsf{t}(\lambda)w})^!(\nabla^I_{\mathsf{t}(\lambda)}(\Bbbk) \star^I \mathscr{F})\right).$$

Let $X \subset \mathrm{Fl}_G$ be a closed finite union of I-orbits over which $\mathscr{F}$ is supported. Then from the definition we see that $\nabla^I_{\mathsf{t}(\lambda)}(\Bbbk) \star^I \mathscr{F}$ is the $*$-pushforward of the complex $\underline{\Bbbk} \mathbin{\widetilde{\boxtimes}} \mathscr{F}[\ell(\mathsf{t}(\lambda))]$ under the map $q^{-1}(\mathrm{Fl}_{G,\mathsf{t}(\lambda)}) \times^I X \to \mathrm{Fl}_G$ induced by m'. Now since $\lambda \in \Lambda$, by Lemma 4.5.7 we have

$$\mathrm{Fl}_{G,\mathsf{t}(\lambda)} = (I \cap \mathrm{L}U^+_{\Lambda}) \cdot x^{\lambda} I.$$

We deduce isomorphisms

$$q^{-1}(\mathrm{Fl}_{G,\mathsf{t}(\lambda)}) \times^I X \cong I \times^{I \cap x^{\lambda} I x^{-\lambda}} x^{\lambda} X \cong (I \cap \mathrm{L}U^+_{\Lambda}) \times^{I \cap x^{\lambda} I x^{-\lambda} \cap \mathrm{L}U^+_{\Lambda}} x^{\lambda} X,$$

under which $\underline{\Bbbk} \mathbin{\widetilde{\boxtimes}} \mathscr{F}[\ell(\mathsf{t}(\lambda))]$ corresponds to $\underline{\Bbbk} \mathbin{\widetilde{\boxtimes}} (x^{\lambda})_* \mathscr{F}[\ell(\mathsf{t}(\lambda))]$ (where by abuse we denote by $x^{\lambda} : \mathrm{Fl}_G \to \mathrm{Fl}_G$ the left multiplication by x^{λ}). The preimage of $\mathfrak{S}^{\Lambda}_{\mathsf{t}(\lambda)w}$ in the right-hand side is then

$$\begin{aligned}(I \cap \mathrm{L}U^+_{\Lambda}) \times^{I \cap x^{\lambda} I x^{-\lambda} \cap \mathrm{L}U^+_{\Lambda}} (x^{\lambda} X \cap \mathfrak{S}^{\Lambda}_{\mathsf{t}(\lambda)w}) \\ = (I \cap \mathrm{L}U^+_{\Lambda}) \times^{I \cap x^{\lambda} I x^{-\lambda} \cap \mathrm{L}U^+_{\Lambda}} x^{\lambda} \cdot (X \cap \mathfrak{S}^{\Lambda}_w)\end{aligned}$$

since $\mathfrak{S}^{\Lambda}_{\mathsf{t}(\lambda)w} = x^{\lambda} \cdot \mathfrak{S}^{\Lambda}_w$. It follows that

$$\begin{aligned}(4.5.11) \quad & \mathsf{H}^{\bullet}_{\mathfrak{S}^{\Lambda}_{\mathsf{t}(\lambda)w}}\left(\mathrm{Fl}_G, \nabla^I_{\mathsf{t}(\lambda)} \star^I \mathscr{F}\right) \cong \\ & \mathsf{H}^{\bullet}((I \cap \mathrm{L}U^+_{\Lambda}) \times^{I \cap x^{\lambda} I x^{-\lambda} \cap \mathrm{L}U^+_{\Lambda}} x^{\lambda} \cdot (X \cap \mathfrak{S}^{\Lambda}_w), \underline{\Bbbk} \mathbin{\widetilde{\boxtimes}} (x^{\lambda})_* (\sigma^{\Lambda}_w)^! \mathscr{F}[\ell(\mathsf{t}(\lambda))]).\end{aligned}$$

The scheme on the right-hand side of this formula is a bundle over

$$(I \cap \mathrm{L}U^+_{\Lambda})/(I \cap x^{\lambda} I x^{-\lambda} \cap \mathrm{L}U^+_{\Lambda}) \cong \mathrm{Fl}_{G,\mathsf{t}(\lambda)},$$

which is contractible, so the Leray–Serre spectral sequence identifies (4.5.11) with

$$\mathsf{H}^{\bullet}\left(x^{\lambda} \cdot (X \cap \mathfrak{S}^{\Lambda}_w), (x^{\lambda})_* (\sigma^{\Lambda}_w)^! \mathscr{F}[\ell(\mathsf{t}(\lambda))]\right).$$

This, in turn, is identified with $\mathsf{H}^{\bullet+\ell(\mathsf{t}(\lambda))}_{\mathfrak{S}^{\Lambda}_w}(\mathrm{Fl}_G, \mathscr{F})$, so we are done. □

Lemma 4.5.9. — *For any $\lambda \in \mathbf{X}^{\vee}$ and $M \in \mathsf{Mof}_{\Bbbk}$ we have canonical isomorphisms*

$$\mathsf{H}^n_{\mathfrak{S}^{\Lambda}_w}(\mathrm{Fl}_G, \mathbf{J}^{\Lambda}_{\lambda}(M)) \cong \begin{cases} M & \text{if } n = -\langle x^{-1}_{\Lambda}(\lambda), 2\rho\rangle \text{ and } w = \mathsf{t}(\lambda); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — Choose $\mu \in \Lambda_\lambda$. Then by Lemma 4.2.6 we have a canonical isomorphism

$$\nabla^I_{\mathsf{t}(\mu-\lambda)}(\Bbbk) \star^I \mathbf{J}^\Lambda_\lambda(M) \cong \nabla^I_{\mathsf{t}(\mu)}(M).$$

Using Lemma 4.5.8 twice, we obtain canonical isomorphisms

$$\begin{aligned}\mathsf{H}^n_{\mathfrak{S}^\Lambda_w}(\mathrm{Fl}_G, \mathbf{J}^\Lambda_\lambda(M)) &\cong \mathsf{H}^{n-\ell(\mathsf{t}(\mu-\lambda))}_{\mathfrak{S}^\Lambda_{\mathsf{t}(\mu-\lambda)w}}(\mathrm{Fl}_G, \nabla^I_{\mathsf{t}(\mu)}(M)) \\ &\cong \mathsf{H}^{n-\ell(\mathsf{t}(\mu-\lambda))+\ell(\mathsf{t}(\mu))}_{\mathfrak{S}^\Lambda_{\mathsf{t}(-\lambda)w}}(\mathrm{Fl}_G, \nabla^I_e(M)).\end{aligned}$$

Now $\mathrm{Fl}_{G,e} \subset \mathfrak{S}^\Lambda_e$, so the right-hand side vanishes unless $w = \mathsf{t}(\lambda)$ and $n = \ell(\mathsf{t}(\mu - \lambda)) - \ell(\mathsf{t}(\mu))$, in which case it is isomorphic to M. Using (4.5.3) we deduce the desired description. One can check that our isomorphism does not depend on the choice of μ, so it is canonical. □

Proposition 4.5.10. — *Let $\mathscr{F} \in \mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G, \Bbbk)$.*

1. *For any $\lambda \in \mathbf{X}^\vee$, we have canonical isomorphisms*

$$\mathsf{H}^n_{\mathfrak{S}^\Lambda_{\mathsf{t}(\lambda)}}(\mathrm{Fl}_G, \mathscr{F}) \cong \begin{cases}\mathrm{Grad}^\Lambda_\lambda(\mathscr{F}) & \textit{if } n = -\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle; \\ 0 & \textit{otherwise.}\end{cases}$$

2. *For any $w \in W \smallsetminus \{\mathsf{t}(\lambda) : \lambda \in \mathbf{X}^\vee\}$ we have*

$$\mathsf{H}^n_{\mathfrak{S}^\Lambda_w}(\mathrm{Fl}_G, \mathscr{F}) = 0$$

for any $n \in \mathbb{Z}$.

Proof. — The claims are immediate from Lemma 4.5.9. □

Remark 4.5.11. — For any $\lambda \in \mathbf{X}^\vee$ we have $\pi(\mathfrak{S}^\Lambda_{\mathsf{t}(\lambda)}) = \mathrm{L}U_{x_\Lambda w_\circ} \cdot L_\lambda$, where we use the notation of Remark 1.3.7. In particular, if $\mathfrak{S}^\Lambda_{\mathsf{t}(\lambda)} \subset \overline{\mathfrak{S}^\Lambda_{\mathsf{t}(\mu)}}$ then

$$\mathrm{L}U_{x_\Lambda w_\circ} \cdot L_\lambda \subset \overline{\mathrm{L}U_{x_\Lambda w_\circ} \cdot L_\mu},$$

so that (by [**BR**, Proposition 1.3.4]) we have $\lambda \preceq_\Lambda \mu$. If $\lambda \neq \mu$, this guarantees that $\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle \neq \langle x_\Lambda^{-1}(\mu), 2\rho\rangle$. Using this and standard arguments (see e.g. [**BR**, Proof of Theorem 1.5.9]), from Proposition 4.5.10 we deduce that for any $\mathscr{F}$ in $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G, \Bbbk)$, for any locally closed union X of orbits $\mathfrak{S}_w$, and for any $n \in \mathbb{Z}$ we have a canonical isomorphism

$$\mathsf{H}^n_X(\mathrm{Fl}_G, \mathscr{F}) \cong \bigoplus_{\substack{\lambda \in \mathbf{X}^\vee,\ \mathfrak{S}^\Lambda_{\mathsf{t}(\lambda)} \subset X, \\ \langle x_\Lambda^{-1}(\lambda), 2\rho\rangle = -n}} \mathrm{Grad}^\Lambda_\lambda(\mathscr{F}).$$

4.6. Consequences

The results proved in Section 4.5 have a number of significant consequences for the functors $\mathrm{Grad}^\Lambda_\lambda$ and $\mathrm{gr}^\Lambda_\lambda$ (and also for the functor Z) that we explain in the present section.

4.6.1. First consequences. —

4.6.1.1. *Acyclicity.* — Recall the functor $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}$ from (4.3.5). The first observation is that this functor "detects" acyclic chain complexes of Wakimoto-filtered perverse sheaves, in the following sense.

Proposition 4.6.1. — *Let*

$$\mathscr{F} = (\cdots \to \mathscr{F}^{-1} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots) \tag{4.6.1}$$

be a bounded chain complex over $\mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk)$*. If this chain complex is acyclic (when regarded as a chain complex over* $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$*) then the chain complex*

$$\cdots \to \mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}^{-1}) \to \mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}^0) \to \mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}^1) \to \cdots \tag{4.6.2}$$

is an acyclic complex of $\mathbf{X}^\vee$*-graded* $\Bbbk$*-modules. Conversely, if* $n \in \mathbb{Z}$ *and if the chain complex* (4.6.2) *has no cohomology in degree* n*, then the complex* (4.6.1) *has no cohomology in degree* n*.*

Proof. — Suppose first that (4.6.1) is acyclic. Define a functor

$$f : D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}\mathsf{Mof}^{\mathbf{X}^\vee}_{\Bbbk}$$

by

$$f(\mathscr{G}) := \bigoplus_{\lambda \in \mathbf{X}} R\Gamma_{\mathfrak{S}^{\Lambda}_{\mathrm{t}(\lambda)}}(\mathscr{G})[-\langle x_{\Lambda}^{-1}(\lambda), 2\rho\rangle].$$

By Proposition 4.5.10, if $\mathscr{G} \in \mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk)$, then the chain complex $f(\mathscr{G})$ is concentrated in degree 0, and there is a natural isomorphism

$$f(\mathscr{G}) \cong \mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{G}).$$

Thus, the chain complex (4.6.2) can be identified with

$$\cdots \to f(\mathscr{F}^{-1}) \to f(\mathscr{F}^0) \to f(\mathscr{F}^1) \to \cdots .$$

To see that this complex is acyclic, apply Lemma 4.6.2 below (with $\mathscr{A}_1 = \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$ and $\mathscr{A}_2 = \mathsf{Mof}^{\mathbf{X}^\vee}_{\Bbbk}$).

Conversely, fix $n \in \mathbb{Z}$, and assume that (4.6.2) has no cohomology in degree n. To show that (4.6.1) has no cohomology in degree n, we proceed by induction on the size of the set

$$X = \{\lambda \in \mathbf{X} \mid \text{for some } i \in \mathbb{Z},\ \mathrm{Grad}^{\Lambda}_{\lambda}(\mathscr{F}^i) \neq 0\}.$$

(Note that this set is finite because only finitely many $\mathscr{F}^i$'s are nonzero.) If X is empty, then $\mathscr{F}^i = 0$ for all i, and there is nothing to prove.

If X is a singleton, say $X = \{\lambda\}$, then by Lemma 4.3.4 all $\mathscr{F}^i$'s are in the essential image of the functor $\mathbf{J}^{\Lambda}_{\lambda} : \mathsf{Mof}_{\Bbbk} \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$. Since the latter functor is fully faithful (see Proposition 4.2.5), the entire chain complex (4.6.1) is obtained by applying $\mathbf{J}^{\Lambda}_{\lambda}$ to some chain complex

$$\cdots \to M^{-1} \to M^0 \to M^1 \to \cdots \tag{4.6.3}$$

in $\mathsf{Mof}_{\Bbbk}$. By (4.3.4), the complex (4.6.2) identifies with (4.6.3); in particular, (4.6.3) has no cohomology in degree n. Since $\mathbf{J}^{\Lambda}_{\lambda}$ is exact (see Lemma 4.2.1), we conclude that (4.6.1) has no cohomology in degree n as well.

Now suppose that X contains more than one element, and choose an element $\lambda \in X$ that is minimal with respect to $\preceq_\Lambda$. By Lemma 4.3.2 we can form two new chain complexes as follows:

$$\mathscr{F}_{\preceq_\Lambda \lambda} := (\cdots \to \mathscr{F}^{-1}_{\preceq_\Lambda \lambda} \to \mathscr{F}^{0}_{\preceq_\Lambda \lambda} \to \mathscr{F}^{1}_{\preceq_\Lambda \lambda} \to \cdots),$$
$$\mathscr{F}/\mathscr{F}_{\preceq_\Lambda \lambda} := (\cdots \to \mathscr{F}^{-1}/\mathscr{F}^{-1}_{\preceq_\Lambda \lambda} \to \mathscr{F}^{0}/\mathscr{F}^{0}_{\preceq_\Lambda \lambda} \to \mathscr{F}^{1}/\mathscr{F}^{1}_{\preceq_\Lambda \lambda} \to \cdots),$$

and we have a short exact sequence of chain complexes

$$0 \to \mathscr{F}_{\preceq_\Lambda \lambda} \to \mathscr{F} \to \mathscr{F}/\mathscr{F}_{\preceq_\Lambda \lambda} \to 0.$$

Using the long exact sequence of cohomology associated with this short exact sequence, we see that to conclude it is enough to show that both $\mathscr{F}_{\preceq_\Lambda \lambda}$ and $\mathscr{F}/\mathscr{F}_{\preceq_\Lambda \lambda}$ have no cohomology in degree n. It is immediate from the definitions that

$$\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}) \cong \mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}_{\preceq_\Lambda \lambda}) \oplus \mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}/\mathscr{F}_{\preceq_\Lambda \lambda});$$

since (4.6.2) is assumed to have no cohomology in degree n, this implies that the chain complexes $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}_{\preceq_\Lambda \lambda})$ and $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}/\mathscr{F}_{\preceq_\Lambda \lambda})$ also have no cohomology in degree n. By induction we deduce that $\mathscr{F}_{\preceq_\Lambda \lambda}$ and $\mathscr{F}/\mathscr{F}_{\preceq_\Lambda \lambda}$ have no cohomology in degree n, which concludes the proof. □

Lemma 4.6.2. — *Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be triangulated categories equipped with t-structures, and let $\mathscr{A}_1 \subset \mathscr{D}_1$ and $\mathscr{A}_2 \subset \mathscr{D}_2$ be their respective hearts. Let*

$$0 \to X_1 \to X_2 \to \cdots \to X_n \to 0$$

be an exact sequence in $\mathscr{A}_1$, with $n \geq 2$. Let $f : \mathscr{D}_1 \to \mathscr{D}_2$ be a triangulated functor.

1. *If $f(X_1), f(X_2), \dots, f(X_{n-1})$ all lie in $\mathscr{A}_2$, then $f(X_n)$ lies in $\mathscr{D}_2^{\leq 0}$.*
2. *If $f(X_2), f(X_3), \dots, f(X_n)$ all lie in $\mathscr{A}_2$, then $f(X_1)$ lies in $\mathscr{D}_2^{\geq 0}$.*
3. *If all terms $f(X_1), \dots, f(X_n)$ lie in $\mathscr{A}_2$, then the sequence*

$$0 \to f(X_1) \to f(X_2) \to \cdots \to f(X_n) \to 0$$

is an exact sequence in $\mathscr{A}_2$.

Proof. — (1) We proceed by induction on n. If $n = 2$, the claim is obvious. If $n \geq 3$, let Y be the kernel of $X_{n-1} \to X_n$, so that we have two exact sequences

$$0 \to X_1 \to X_2 \to \cdots X_{n-2} \to Y \to 0, \qquad 0 \to Y \to X_{n-1} \to X_n \to 0 \tag{4.6.4}$$

in $\mathscr{A}_1$. By induction, the first exact sequence shows that $f(Y) \in \mathscr{D}_2^{\leq 0}$. The second short exact sequence comes from some distinguished triangle $Y \to X_{n-1} \to X_n \xrightarrow{[1]}$ in $\mathscr{D}_1$. Apply f to this distinguished triangle; the claim follows.

(2) This is similar to part (1).

(3) We proceed by induction on n. If $n = 2$, the claim is obvious. If $n \geq 3$, define Y as above, and consider the two exact sequences (4.6.4). Parts (1) and (2) together imply that $f(Y) \in \mathscr{A}_2$. By induction, the sequences

$$0 \to f(X_1) \to f(X_2) \to \cdots f(X_{n-2}) \to f(Y) \to 0,$$
$$0 \to f(Y) \to f(X_{n-1}) \to f(X_n) \to 0$$

are both exact. The claim follows by concatenating these sequences. □

Remark 4.6.3. — The "converse" implication in Proposition 4.6.1 also holds for unbounded complexes, in the sense that if $n \in \mathbb{Z}$ and if $\mathscr{F}$ is any chain complex over $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$, if the complex obtained by applying the functor $\mathrm{Grad}^\Lambda_{\mathbf{X}^\vee}$ to all terms has no cohomology in degree n, then the same holds for the complex $\mathscr{F}$. In fact, it suffices to apply the property for bounded complexes to the complex $\mathscr{F}'$ whose i-th term is $\mathscr{F}^i$ if $i \in \{n-1, n, n+1\}$, and 0 otherwise.

We can now answer the question raised in Remark 4.3.5.

Remark 4.6.4. — Although $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$ is not an abelian category, it inherits from $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$ the structure of a (Quillen) exact category. Since a short exact sequence in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$ can be regarded as a three-term acyclic chain complex, Proposition 4.6.1 implies that

$$\mathrm{Grad}^\Lambda_{\mathbf{X}^\vee} : \mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk) \to \mathsf{Mof}_\Bbbk^{\mathbf{X}^\vee}$$

is an exact functor of exact categories. Using (4.3.3) and the exactness of the functor $\mathbf{J}^\Lambda_\lambda$ (see Lemma 4.2.1), we see that

$$\mathrm{gr}^\Lambda_\lambda : \mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk) \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$$

is also an exact functor of exact categories.

Since an isomorphism in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$ can be regarded as a two-term acyclic chain complex, Proposition 4.6.1 also tells us that $\mathrm{Grad}^\Lambda_{\mathbf{X}^\vee}$ is conservative.

4.6.1.2. *Cohomology with support for central sheaves.* — We now turn our attention to central sheaves. The ind-subschemes $\mathfrak{S}^\Lambda_w$ are analogues in Fl_G of the "semi-infinite orbits" S_λ considered in §1.2.3. More specifically, consider a cocharacter χ as in §4.5.3, and the associated action of $\mathbb{G}_\mathrm{m}$ on Gr_G. As in §1.2.3 the fixed points for this action identify (at the level of reduced ind-schemes) with $\mathbf{X}^\vee$, hence the connected components of the associated attractor ind-scheme are parametrized by $\mathbf{X}^\vee$, and we denote by S^Λ_λ the connected component associated with λ. Comparing with the setting of §1.2.3, we see that for any $\lambda \in \mathbf{X}^\vee$ we have

$$S^\Lambda_\lambda = \dot{x}_\Lambda \cdot S_{x_\Lambda^{-1}(\lambda)}.$$

With this notation, it is easily seen that for any $\lambda \in \mathbf{X}^\vee$ we have

$$\pi^{-1}(S^\Lambda_\lambda) = \bigsqcup_{w \in W_\mathrm{f}} \mathfrak{S}^\Lambda_{\mathsf{t}(\lambda)w}. \tag{4.6.5}$$

Proposition 4.6.5. — *For any $\mathscr{F}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ we have*

$$\mathsf{H}^n_{S^\Lambda_\lambda}(\mathrm{Gr}_G, \mathscr{F}) \cong \begin{cases} \mathrm{Grad}^\Lambda_\lambda(\mathsf{Z}(\mathscr{F})) & \text{if } n = -\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — Recall from Lemma 2.5.1 that we have $\mathscr{F} \cong \pi_* \mathsf{Z}(\mathscr{F})$. Using the base change theorem we deduce a canonical isomorphism

$$\mathsf{H}^n_{S^\Lambda_\lambda}(\mathrm{Gr}_G, \mathscr{F}) \cong \mathsf{H}^n_{\pi^{-1}(S^\Lambda_\lambda)}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{F})).$$

In view of Theorem 4.4.5 and (4.6.5), the result follows from Remark 4.5.11 applied with $X = \pi^{-1}(S^{\Lambda}_{\lambda})$. □

Remark 4.6.6. — 1. The vanishing statement in Proposition 4.6.5 is not new: it is one of the fundamental ingredients in the Mirković–Vilonen proof of the geometric Satake equivalence, see Proposition 1.3.5. (More specifically, Proposition 1.3.5 corresponds to the case when $\Lambda = -\mathbf{X}^{\vee}_{+}$. For the more general case, see Remark 1.3.7.)

2. Combining Proposition 4.6.5 with Proposition 4.5.10(1) we obtain isomorphisms

$$\mathsf{H}^n_{\mathfrak{S}^{\Lambda}_{w\mathsf{t}(\lambda)}}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{F})) \cong \begin{cases} \mathsf{H}^n_{S^{\Lambda}_{\lambda}}(\mathrm{Gr}_G, \mathscr{F}) & \text{if } w = 1; \\ 0 & \text{otherwise.} \end{cases}$$

In this form, this result can be seen as an application of a general compatibility result between hyperbolic localization and nearby cycles proved in [**Rc3**]; see [**HR2**, Theorem A] for this point of view.

4.6.1.3. *Support.* — Proposition 4.6.5 tells us in particular that a Wakimoto sheaf with label λ appears in a Wakimoto filtration of $\mathsf{Z}(\mathscr{F})$ iff $\mathsf{H}^{\bullet}_{S^{\Lambda}_{\lambda}}(\mathrm{Gr}_G, \mathscr{F}) \neq 0$. We will apply this observation in the following special case. Assume that $\lambda \in \mathbf{X}^{\vee}_{+}$ is a dominant coweight. Then we have perverse sheaves $\mathcal{J}_*(\lambda, \Bbbk)$ and $\mathcal{J}_!(\lambda, \Bbbk)$, see §1.3.1. For these objects, it is known that $\mathsf{H}^{\bullet}_{S^{\Lambda}_{\mu}}(\mathrm{Gr}_G, \mathcal{J}_*(\lambda, \Bbbk)) \neq 0$ iff $\mathsf{H}^{\bullet}_{S^{\Lambda}_{\mu}}(\mathrm{Gr}_G, \mathcal{J}_!(\lambda, \Bbbk)) \neq 0$, and that this happens iff μ belongs to the intersection of the convex hull of $W_{\mathrm{f}} \cdot \lambda$ with $\lambda + \mathbb{Z}\mathfrak{R}^{\vee}$, cf. [**BR**, Theorem 1.5.2 and Proposition 1.11.1], or equivalently iff $\lambda - \mu^+ \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+$, where μ^+ is the dominant W_{f}-translate of μ.

Corollary 4.6.7. — *Let $\mathscr{F} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, and let $X \subset \mathrm{Gr}_G$ be a closed finite union of L^+G-orbits on which $\mathscr{F}$ is supported. Then $\mathsf{Z}(\mathscr{F})$ is supported on*

$$\bigcup_{\substack{\mu \in \mathbf{X}^{\vee}_{+} \\ \mathrm{Gr}^{\mu}_G \subset X \text{ open}}} \bigcup_{\eta \in W_{\mathrm{f}}\mu} \overline{\mathrm{Fl}_{G,\mathsf{t}(\eta)}}.$$

Proof. — First we consider the case $\mathscr{F} = \mathcal{J}_!(\lambda, \Bbbk)$ and $X = \overline{\mathrm{Gr}^{\lambda}_G}$. Then, as explained above, it is known that $\mathsf{Z}(\mathscr{F})$ admits a filtration with subquotients of the form $\mathbf{J}^{\Lambda}_{\mu}(M)$ where μ runs over the weights such that $\lambda - \mu^+ \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+$. Since $\mathbf{J}^{\Lambda}_{\mu}(M)$ is supported on $\overline{\mathrm{Fl}_{G,\mathsf{t}(\mu)}}$ (see Proposition 4.2.3), it follows that $\mathsf{Z}(\mathscr{F})$ is supported on the union of the closures $\overline{\mathrm{Fl}_{G,\mathsf{t}(\mu)}}$ where μ runs over these weights. To conclude the proof in this case, it thus suffices to show that any of these closures is contained in

$$\bigcup_{\nu \in W_{\mathrm{f}}\lambda} \overline{\mathrm{Fl}_{G,\mathsf{t}(\nu)}}.$$

However, if μ is as above, and if ν is a W_{f}-translate of λ which belongs to the same closed Weyl chamber as μ, then using Lemma 4.1.2(2) one can check that $\mathsf{t}(\mu) \leq_{\mathrm{Bru}} \mathsf{t}(\nu)$, so that $\overline{\mathrm{Fl}_{G,\mathsf{t}(\mu)}} \subset \overline{\mathrm{Fl}_{G,\mathsf{t}(\nu)}}$, which finishes the proof.

We now consider the general case. To fix notation, we denote by $A \subset \mathbf{X}^{\vee}_{+}$ the set of dominant coweights ν such that Gr^{ν}_G is open in X, and by $A' \subset \mathbf{X}^{\vee}_{+}$ the set of dominant coweights η such that $\mathrm{Gr}_G^\eta \subset X$. Then η belongs to A' iff there exists $\nu \in A$ such that $\nu - \eta \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+$.

As explained in [**BR**, §§1.12.1–1.12.2], there exists an object $\mathscr{P}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(X, \Bbbk)$ which admits a (finite) filtration with subquotients of the form $\mathcal{J}_!(\eta, \Bbbk)$ with $\eta \in A'$, integers $n, m \in \mathbb{Z}_{\geq 0}$, and an exact sequence

$$\mathscr{P}^{\oplus n} \to \mathscr{P}^{\oplus m} \to \mathscr{F} \to 0$$

in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. By exactness of Z we deduce an exact sequence

$$\mathsf{Z}(\mathscr{P})^{\oplus n} \to \mathsf{Z}(\mathscr{P})^{\oplus m} \to \mathsf{Z}(\mathscr{F}) \to 0$$

in $\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$. This exact sequence reduces the proof to the case $\mathscr{F} = \mathscr{P}$, and then to the case $\mathscr{F} = \mathcal{J}_!(\eta, \Bbbk)$ with $\eta \in A'$. In this case we have seen that $\mathsf{Z}(\mathcal{J}_!(\eta, \Bbbk))$ is supported on

$$\bigcup_{\nu \in W_{\mathrm{f}}\eta} \overline{\mathrm{Fl}_{G,\mathsf{t}(\nu)}}.$$

If $\mu \in A$ is such that $\mu - \eta \in \mathbb{Z}_{\geq 0}\mathfrak{R}_+$, then as above we have

$$\bigcup_{\nu \in W_{\mathrm{f}}\eta} \overline{\mathrm{Fl}_{G,\mathsf{t}(\nu)}} \subset \bigcup_{\nu \in W_{\mathrm{f}}\mu} \overline{\mathrm{Fl}_{G,\mathsf{t}(\nu)}},$$

which completes the proof. □

Remark 4.6.8. — In the particular cases considered before the statement, Corollary 4.6.7 says that $\mathsf{Z}(\mathcal{J}_!(\lambda, \Bbbk))$ and $\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk))$ are supported on

$$\bigcup_{\nu \in W_{\mathrm{f}}\lambda} \overline{\mathrm{Fl}_{G,\mathsf{t}(\nu)}}.$$

The orbits which are open in this subset are exactly those of the form $\mathrm{Fl}_{G,\mathsf{t}(\nu)}$ with $\nu \in W_{\mathrm{f}}\lambda$, and all of them are of dimension $\ell(\mathsf{t}(\lambda)) = \langle \lambda, 2\rho \rangle$ (see (4.1.5)). A closer look at the proof of the corollary shows (using also Proposition 4.2.3) that the restriction of $\mathsf{Z}(\mathcal{J}_!(\lambda, \Bbbk))$ and $\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk))$ to each of these orbits is the constant sheaf placed in degree $-\langle \lambda, 2\rho \rangle$.

4.6.2. Monodromy and Wakimoto filtrations. — Recall (see §2.4.5) that as part of the nearby cycles formalism, there is a natural *monodromy map*

$$\mathsf{m}_{\mathscr{A}} : \mathsf{Z}(\mathscr{A}) \to \mathsf{Z}(\mathscr{A})$$

for every $\mathscr{A} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$. In Proposition 2.4.6(1) we have proved that this monodromy automorphism is unipotent. Our goal in this subsection is to provide an alternative proof of this fact based on Theorem 4.4.5 and the idea in Remark 9.3.7.

Lemma 4.6.9. — *For any $\lambda \in \mathbf{X}^\vee$, the induced map*

$$\mathsf{m}_{\mathscr{A},\lambda} := \mathrm{Grad}_\lambda^\Lambda(\mathsf{m}_{\mathscr{A}}) : \mathrm{Grad}_\lambda^\Lambda(\mathsf{Z}(\mathscr{A})) \to \mathrm{Grad}_\lambda^\Lambda(\mathsf{Z}(\mathscr{A}))$$

is the identity map.

Proof. — By Proposition 4.5.4, it is enough to show that for any $n \in \mathbb{Z}$, the map

$$\mathsf{H}^n(\mathrm{Fl}_G, \mathsf{m}_{\mathscr{A}}) : \mathsf{H}^n(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \to \mathsf{H}^n(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}))$$

is the identity map. By Lemma 2.5.1 we have natural isomorphisms

$$\mathsf{H}^n(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \cong \mathsf{H}^n(\mathrm{Gr}_G, \pi_*\mathsf{Z}(\mathscr{A})) \cong \mathsf{H}^n(\mathrm{Gr}_G, \mathsf{Z}^{\mathrm{sph}}(\mathscr{A})),$$

which identify $\mathsf{H}^n(\mathrm{Fl}_G, \mathsf{m}_{\mathscr{A}})$ with $\mathsf{H}^n(\mathrm{Gr}_G, \mathsf{m}^{\mathrm{sph}}_{\mathscr{A}})$. As explained in §2.5.4, $\mathsf{m}^{\mathrm{sph}}_{\mathscr{A}}$ is the identity map, which implies that $\mathsf{H}^n(\mathrm{Fl}_G, \mathsf{m}_{\mathscr{A}})$ is also the identity map, as desired. □

Proposition 4.6.10. — *For $\mathscr{A} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, the map $\mathsf{m}_{\mathscr{A}} : \mathsf{Z}(\mathscr{A}) \to \mathsf{Z}(\mathscr{A})$ is unipotent.*

Proof. — Choose a bijection $\mathbb{Z} \xrightarrow{\sim} \mathbf{X}^\vee$ as in (4.3.1), and let $(\mathsf{Z}(\mathscr{A})_i)_{i\in\mathbb{Z}}$ be the corresponding Λ-Wakimoto filtration of $\mathsf{Z}(\mathscr{A})$ (see §4.3.2). By Lemma 4.6.9, we have $(\mathrm{id} - \mathsf{m}_{\mathscr{A}})(\mathsf{Z}(\mathscr{A})_i) \subset \mathsf{Z}(\mathscr{A})_{i-1}$ for any $i \in \mathbb{Z}$. Since the filtration has only finitely many nonzero steps, it follows that $(\mathrm{id} - \mathsf{m}_{\mathscr{A}})^N = 0$ for some $N > 0$. □

4.6.3. Highest weight arrows. — We conclude this section with the construction and study of some "highest weight arrows" (somewhat similar to the projection from an induced module over a connected reductive group to its highest weight line, see §1.5.1) that will be used in Chapter 6.

This endeavor will require some background on I-orbits on Gr_G. For $\mu \in \mathbf{X}^\vee$, set

$$\mathrm{Gr}_{G,\mu} := I \cdot L_\mu,$$

and denote by $j_\mu : \mathrm{Gr}_{G,\mu} \to \mathrm{Gr}_G$ the embedding. (This notation should not be confused with $j_{\mathsf{t}(\mu)}$, which denotes the embedding of $\mathrm{Fl}_{G,\mathsf{t}(\mu)}$ in Fl_G.) Then for any $\lambda \in \mathbf{X}^\vee_+$ we have

$$\mathrm{Gr}^\lambda_G = \bigsqcup_{\mu \in W_{\mathrm{f}}\lambda} \mathrm{Gr}_{G,\mu};$$

in particular we have

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \bigsqcup_{\mu \in \mathbf{X}^\vee} \mathrm{Gr}_{G,\mu}.$$

Note also that each $\mathrm{Gr}_{G,\mu}$ is isomorphic to an affine space.

Let us now fix $\lambda \in \mathbf{X}^\vee_+$, and let μ be the unique W_{f}-translate of λ which belongs to Λ. By Lemma 2.5.1, we have a canonical isomorphism

$$\pi_*\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)) \cong \mathcal{J}_*(\lambda, \Bbbk).$$

On the other hand, since μ belongs to Λ we have

$$\mathbf{J}^\Lambda_\mu(\Bbbk) \cong \nabla^I_{\mathsf{t}(\mu)}(\Bbbk).$$

Since π restricts to a trivial fibration $\mathrm{Fl}_{G,\mathsf{t}(\mu)} \to \mathrm{Gr}_{G,\mu}$ with fibers isomorphic to $\mathbb{A}^{\ell(\mathsf{t}(\mu)) - \dim(\mathrm{Gr}_{G,\mu})}$, we deduce a canonical isomorphism

$$\pi_*\mathbf{J}^\Lambda_\mu(\Bbbk) \cong (j_\mu)_*\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))].$$

(Note that here the right-hand side is *not* perverse in general.) We use these isomorphisms in the following statement.

Lemma 4.6.11. — *For λ and μ as above, the functor π_* induces an isomorphism*

$$\mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Fl}_G,\Bbbk)}\big(\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk)),\mathbf{J}^{\Lambda}_{\mu}(\Bbbk)\big) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Gr}_G,\Bbbk)}\big(\mathcal{J}_*(\lambda,\Bbbk),(j_\mu)_*\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]\big),$$

and moreover both sides are free $\Bbbk$-modules of rank 1.

Proof. — Since (as explained above) we have $\mathbf{J}^{\Lambda}_{\mu}(\Bbbk)\cong\nabla^{I}_{\mathsf{t}(\mu)}(\Bbbk)$, by adjunction we have

$$\mathrm{Hom}\big(\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk)),\mathbf{J}^{\Lambda}_{\mu}(\Bbbk)\big)\cong\mathrm{Hom}\big((j_{\mathsf{t}(\mu)})^*\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk)),\underline{\Bbbk}_{\mathrm{Fl}_{G,\mathsf{t}(\mu)}}[\ell(\mathsf{t}(\mu))]\big).$$

By Remark 4.6.8 we know that

$$(j_{\mathsf{t}(\mu)})^*\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk))\cong\underline{\Bbbk}_{\mathrm{Fl}_{G,\mathsf{t}(\mu)}}[\ell(\mathsf{t}(\mu))].$$

After fixing such an isomorphism, we deduce that the $\Bbbk$-module

$$\mathrm{Hom}\big(\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk)),(j_{\mathsf{t}(\mu)})_*\underline{\Bbbk}_{\mathrm{Fl}_{G,\mathsf{t}(\mu)}}[\ell(\mathsf{t}(\mu))]\big)$$

is free of rank 1, and spanned by the morphism

$$\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk))\to(j_{\mathsf{t}(\mu)})_*(j_{\mathsf{t}(\mu)})^*\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk))\cong(j_{\mathsf{t}(\mu)})_*\underline{\Bbbk}_{\mathrm{Fl}_{G,\mathsf{t}(\mu)}}[\ell(\mathsf{t}(\mu))] \tag{4.6.6}$$

induced by adjunction.

On the other hand, again by adjunction we have

$$\mathrm{Hom}\big(\mathcal{J}_*(\lambda,\Bbbk),(j_\mu)_*\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]\big)\cong\mathrm{Hom}((j_\mu)^*\mathcal{J}_*(\lambda,\Bbbk),\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]).$$

Now we have $\mathrm{Gr}_{G,\mu}\subset\mathrm{Gr}^{\lambda}_G$, and the restriction of $\mathcal{J}_*(\lambda,\Bbbk)$ to the subset Gr^{λ}_G (which is open in its support) is free of rank 1 (with a canonical generator). Since $\ell(\mathsf{t}(\lambda))=\ell(\mathsf{t}(\mu))$ (see (4.1.5)), we deduce a (canonical) isomorphism $(j_\mu)^*\mathcal{J}_*(\lambda,\Bbbk)\cong\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]$, so that the $\Bbbk$-module $\mathrm{Hom}(\mathcal{J}_*(\lambda,\Bbbk),(j_\mu)_*\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))])$ is also free of rank 1, and spanned by the adjunction morphism

$$\mathcal{J}_*(\lambda,\Bbbk)\to(j_\mu)_*(j_\mu)^*\mathcal{J}_*(\lambda,\Bbbk)\cong(j_\mu)_*\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]. \tag{4.6.7}$$

These considerations show that proving the lemma amounts to proving that the functor π_* sends the morphism (4.6.6) to a multiple of (4.6.7) by an invertible scalar. Now, recall that we have a canonical isomorphism

$$\Bbbk\overset{L}{\otimes}_{\mathbb{Z}}\mathcal{J}_*(\lambda,\mathbb{Z})\cong\mathcal{J}_*(\lambda,\Bbbk),$$

see [**BR**, Proposition 1.11.13]. Since all the functors considered above commute with the functor $\Bbbk\otimes^{L}_{\mathbb{Z}}(-)$ (see Proposition 9.1.6(2) for the functor Z), it suffices to prove the claim above in case $\Bbbk=\mathbb{Z}$. Using then the functor $\mathbb{F}_p\otimes^{L}_{\mathbb{Z}}(-)$ we can further reduce the proof to showing that, in case $\Bbbk$ is a finite field, the functor π_* does not kill the morphism (4.6.6). In fact we will prove this claim for any choice of coefficients.

For this we observe that we have

$$\mathsf{H}^\bullet(\mathrm{Fl}_G,-)\cong\mathsf{H}^\bullet(\mathrm{Gr}_G,-)\circ\pi_*.$$

Hence the claim will follow if we prove that the functor $\mathsf{H}^\bullet(\mathrm{Fl}_G,-)$ does not kill the first map in (4.6.6). Let $\Gamma:=\{\nu\in\mathbf{X}^\vee\mid\nu\prec_\Lambda\mu\}$. Then the considerations preceding Corollary 4.6.7 show that all the coweights η such that $\mathrm{Grad}^{\Lambda}_{\eta}(\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk)))\neq0$ belong to $\Gamma \cup \{\mu\}$. In particular, setting $\mathscr{F} := (\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)))_\Gamma$ we have an exact sequence of perverse sheaves

$$0 \to \mathscr{F} \to \mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)) \to \mathrm{gr}^\Lambda_\mu(\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk))) \to 0. \tag{4.6.8}$$

Here the perverse sheaf $\mathrm{gr}^\Lambda_\mu(\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)))$ is of the form $\mathbf{J}^\Lambda_\mu(M) = \nabla^I_{\mathsf{t}(\mu)}(M)$ for some M in $\mathsf{Mof}_\Bbbk$, and $\mathscr{F}$ is supported on the image of the closed embedding

$$k : \left(\bigcup_{\nu \in W_\mathrm{f}\lambda} \overline{\mathrm{Fl}_{G,\mathsf{t}(\nu)}} \right) \smallsetminus \mathrm{Fl}_{G,\mathsf{t}(\mu)} \hookrightarrow \bigcup_{\nu \in W_\mathrm{f}\lambda} \overline{\mathrm{Fl}_{G,\mathsf{t}(\nu)}}.$$

Hence (4.6.8) is induced by the distinguished triangle

$$k_! k^! (\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk))) \to \mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)) \to (j_{\mathsf{t}(\mu)})_* (j_{\mathsf{t}(\mu)})^* \mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)) \xrightarrow{[1]}$$

provided by the gluing formalism. In particular, we have proved that the first map in (4.6.6) identifies with the surjection $\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)) \twoheadrightarrow \mathrm{gr}^\Lambda_\mu(\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)))$. By definition, the latter map is not killed by the functor $\mathrm{Grad}^\Lambda_\mu$. Using Proposition 4.5.4 we deduce that this map is not killed by the functor $\mathsf{H}^\bullet(\mathrm{Fl}_G, -)$ either, which finishes the proof. □

As seen in the course of the proof of Lemma 4.6.11, there exists a *canonical* morphism $f^\Lambda_\lambda : \mathcal{J}_*(\lambda, \Bbbk) \to (j_\mu)_* \underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]$. We will denote by

$$\mathfrak{f}^\Lambda_\lambda : \mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)) \to \mathbf{J}^\Lambda_\mu(\Bbbk)$$

the inverse image of this map under the isomorphism of Lemma 4.6.11. From the proof of this lemma we know that $\mathfrak{f}^\Lambda_\lambda$ is a surjective morphism of perverse sheaves, and moreover that its kernel admits a Wakimoto filtration whose subquotients have labels in the intersection of the convex hull of $W_\mathrm{f} \cdot \lambda$ with $\lambda + \mathbb{Z}\mathfrak{R}$, and distinct from μ. In particular, this map induces a canonical isomorphism

$$\mathrm{Grad}^\Lambda_\mu(\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk))) \xrightarrow{\sim} \Bbbk. \tag{4.6.9}$$

This construction can also be phrased in the following terms. We have a morphism of functors

$$(j_\mu)^* \pi_* \to (\pi_\mu)_* (j_{\mathsf{t}(\mu)})^*$$

(where π_μ is the restriction of π to $\mathrm{Fl}_{G,\mathsf{t}(\mu)}$) obtained by adjunction from the natural morphism $\pi_* \to \pi_* (j_{\mathsf{t}(\mu)})_* (j_{\mathsf{t}(\mu)})^* \cong (\pi \circ j_{\mathsf{t}(\mu)})_* (j_{\mathsf{t}(\mu)})^* \cong (j_\mu)_* (\pi_\mu)_* (j_{\mathsf{t}(\mu)})^*$. From the proof of Lemma 4.6.11, one can see that applying this morphism to $\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk))$ yields an isomorphism

$$(j_\mu)^* \mathcal{J}_*(\lambda, \Bbbk) \xrightarrow{\sim} (\pi_\mu)_* (j_{\mathsf{t}(\mu)})^* \mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)). \tag{4.6.10}$$

Now the left-hand side has a canonical trivialization (i.e., an isomorphism with $\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]$). Since π_μ is a trivial fibration with fibers isomorphic to an affine space, and since $(j_{\mathsf{t}(\mu)})^* \mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk))$ is known to be isomorphic to a constant sheaf (with stalks $\Bbbk$) concentrated in degree $-\ell(\mathsf{t}(\mu))$, we deduce a *canonical* isomorphism $(j_{\mathsf{t}(\mu)})^* \mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)) \cong \underline{\Bbbk}_{\mathrm{Fl}_{G,\mathsf{t}(\mu)}}[\ell(\mathsf{t}(\mu))]$. By adjunction, this isomorphism provides the morphism $\mathfrak{f}^\Lambda_\lambda$.

4.6.4. Highest weight arrows, convolution, and commutativity. — Consider now two dominant coweights $\lambda_1, \lambda_2 \in \mathbf{X}^\vee_+$, and denote by μ_1, μ_2 their W_{f}-translates in Λ. Recall the morphism

$$a_{\lambda_1,\lambda_2} : \mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G} \mathcal{J}_*(\lambda_2, \Bbbk) \to \mathcal{J}_*(\lambda_1 + \lambda_2, \Bbbk)$$

constructed in §1.5.2. Applying Z and using the isomorphism $\phi_{\mathcal{J}_*(\lambda_1,\Bbbk),\mathcal{J}_*(\lambda_2,\Bbbk)}$ from Theorem 3.4.1, we deduce a canonical morphism

$$\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)) \to \mathsf{Z}(\mathcal{J}_*(\lambda_1 + \lambda_2, \Bbbk)). \tag{4.6.11}$$

This map will be used in the proof of the following statement.

Proposition 4.6.12. — *The following diagram commutes:*

$$\begin{tikzcd}
\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)) \arrow[rr, "\sigma_{\mathcal{J}_*(\lambda_1,\Bbbk),\mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))}"] \arrow[d, "\mathfrak{f}^\Lambda_{\lambda_1} \star \mathfrak{f}^\Lambda_{\lambda_2}"'] & & \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \arrow[d, "\mathfrak{f}^\Lambda_{\lambda_2} \star \mathfrak{f}^\Lambda_{\lambda_1}"] \\
\mathbf{J}^\Lambda_{\mu_1}(\Bbbk) \star^I \mathbf{J}^\Lambda_{\mu_2}(\Bbbk) \arrow[r, "\textit{Lemma 4.2.6}", "\sim"'] & \mathbf{J}^\Lambda_{\mu_1+\mu_2}(\Bbbk) & \mathbf{J}^\Lambda_{\mu_2}(\Bbbk) \star^I \mathbf{J}^\Lambda_{\mu_1}(\Bbbk). \arrow[l, "\textit{Lemma 4.2.6}"', "\sim"]
\end{tikzcd}$$

Proof. — For the duration of the proof we set

$$\lambda := \lambda_1 + \lambda_2, \qquad \mu := \mu_1 + \mu_2.$$

Then μ is the unique W_{f}-translate of λ which belongs to Λ.

The statement claims that two elements in

$$\mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Fl}_G,\Bbbk)}(\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)), \mathbf{J}^\Lambda_\mu(\Bbbk))$$

are equal. Now, by definition, the kernel and cokernel of a_{λ_1,λ_2} are supported on $\overline{\mathrm{Gr}^\lambda_G} \smallsetminus \mathrm{Gr}^\lambda_G$. Hence, by Corollary 4.6.7 (and by exactness of the functor Z), the kernel and cokernel of (4.6.11) are supported on a closed subset which does not intersect $\mathrm{Fl}_{G,\mathsf{t}(\mu)}$. It follows that a_{λ_1,λ_2} and (4.6.11) induce isomorphisms

$$\mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Fl}_G,\Bbbk)}(\mathsf{Z}(\mathcal{J}_*(\lambda, \Bbbk)), \mathbf{J}^\Lambda_\mu(\Bbbk)) \xrightarrow{\sim} \\ \mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Fl}_G,\Bbbk)}(\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)), \mathbf{J}^\Lambda_\mu(\Bbbk))$$

and

$$\mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Gr}_G,\Bbbk)}\big(\mathcal{J}_*(\lambda, \Bbbk), (j_\mu)_* \underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]\big) \xrightarrow{\sim} \\ \mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Gr}_G,\Bbbk)}\big(\mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G} \mathcal{J}_*(\lambda_2, \Bbbk), (j_\mu)_* \underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]\big).$$

In view of Lemma 4.6.11, it follows that the functor π_* induces an isomorphism

$$\mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Fl}_G,\Bbbk)}\big(\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)), \mathbf{J}^\Lambda_\mu(\Bbbk)\big) \\ \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}_c(\mathrm{Gr}_G,\Bbbk)}\big(\mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G} \mathcal{J}_*(\lambda_2, \Bbbk), (j_\mu)_* \underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))]\big),$$

so that to prove the proposition it suffices to prove that the image of our diagram under the functor π_* commutes.

For this, we will break the diagram up into three pieces (two of which will be identical, modulo switching λ_1 and λ_2). First we will prove the commutativity of the diagram

(4.6.12)
$$\begin{tikzcd} \pi_*\big(\mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk))\star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))\big) \arrow[rr, "\pi_*\sigma_{\mathcal{J}_*(\lambda_1,\Bbbk),\mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))}"] \arrow[dr] & & \pi_*\big(\mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))\star^I \mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk))\big) \arrow[dl] \\ & \pi_*\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk)), & \end{tikzcd}$$

where both diagonal maps are images of (4.6.11) (for the pairs (λ_1,λ_2) and (λ_2,λ_1) respectively) under the functor π_*. By definition the map on the left can be written as a composition

$$\pi_*\big(\mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk))\star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))\big) \xrightarrow[\sim]{\pi_*\phi} \pi_*\big(\mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk)\star^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_2,\Bbbk))\big) \xrightarrow{\pi_*\mathsf{Z}(a_{\lambda_1,\lambda_2})} \pi_*\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk))$$

where we write ϕ for $\phi_{\mathcal{J}_*(\lambda_1,\Bbbk),\mathcal{J}_*(\lambda_2,\Bbbk)}$. The map on the right in (4.6.12) has a similar description (switching the roles of λ_1 and λ_2). Now the functor $\pi_*\mathsf{Z}(-)$ is isomorphic to the identity functor by Lemma 2.5.1. Since, as discussed in §3.5.9, the following diagram commutes:

$$\begin{tikzcd} \pi_*\big(\mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk))\star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))\big) \arrow[r, "\pi_*\sigma_{\mathcal{J}_*(\lambda_1,\Bbbk),\mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))}"] \arrow[d, "\pi_*\phi"'] & \pi_*\big(\mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))\star^I \mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk))\big) \arrow[d, "\pi_*\phi"] \\ \pi_*\mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk)\star^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_2,\Bbbk)) \arrow[d, "\wr"'] & \pi_*\mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk)\star^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_1,\Bbbk)) \arrow[d, "\wr"] \\ \mathcal{J}_*(\lambda_1,\Bbbk)\star^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_2,\Bbbk) \arrow[r, "\sigma^{\mathrm{Com}}_{\mathcal{J}_*(\lambda_1,\Bbbk),\mathcal{J}_*(\lambda_2,\Bbbk)}"] & \mathcal{J}_*(\lambda_2,\Bbbk)\star^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_1,\Bbbk), \end{tikzcd}$$

the commutativity of (4.6.12) follows from Corollary 1.5.7.

Now we consider the following diagram:

(4.6.13)
$$\begin{tikzcd} \pi_*\big(\mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk))\star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))\big) \arrow[r] \arrow[d, "\pi_*(\mathfrak{f}^\Lambda_{\lambda_1}\star\mathfrak{f}^\Lambda_{\lambda_2})"'] & \pi_*\mathsf{Z}(\mathcal{J}_*(\lambda,\Bbbk)) \arrow[d, "\pi_*\mathfrak{f}^\Lambda_\lambda"] \\ \pi_*(\mathbf{J}^\Lambda_{\mu_1}(\Bbbk)\star^I \mathbf{J}^\Lambda_{\mu_2}(\Bbbk)) \arrow[r, "\sim"] & \pi_*(\mathbf{J}^\Lambda_\mu(\Bbbk)) \cong (j_\mu)_*\underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))], \end{tikzcd}$$

where the upper line is induced by (4.6.11). The path in this diagram through the top right corner is the morphism deduced by adjunction from the composition of natural morphisms

$$(j_\mu)^*\pi_*(\mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk))\star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2,\Bbbk))) \xrightarrow{(j_\mu)^*\pi_*\phi} (j_\mu)^*\pi_*(\mathsf{Z}(\mathcal{J}_*(\lambda_1,\Bbbk)\star^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_2,\Bbbk))) \xrightarrow[\sim]{\text{Lemma 2.5.1}} (j_\mu)^*(\mathcal{J}_*(\lambda_1,\Bbbk)\star^{\mathrm{L}^+G}\mathcal{J}_*(\lambda_2,\Bbbk)) \cong \underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))].$$

On the other hand, we have

$$\pi_*(\mathfrak{f}^{\Lambda}_{\lambda_1} \star \mathfrak{f}^{\Lambda}_{\lambda_2}) = \pi_*(m')_*(\mathfrak{f}^{\Lambda}_{\lambda_1} \mathbin{\widetilde{\boxtimes}} \mathfrak{f}^{\Lambda}_{\lambda_2}) = m_*\varpi_*(\mathfrak{f}^{\Lambda}_{\lambda_1} \mathbin{\widetilde{\boxtimes}} \mathfrak{f}^{\Lambda}_{\lambda_2})$$

where $\varpi := \pi \mathbin{\widetilde{\times}} \pi : \mathrm{L}G \times^I \mathrm{Fl}_G \to \mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G$. Here $\mathfrak{f}^{\Lambda}_{\lambda_1}$ and $\mathfrak{f}^{\Lambda}_{\lambda_2}$ are obtained from appropriate trivializations of $(j_{\mathsf{t}(\mu_1)})^*\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk))$ and $(j_{\mathsf{t}(\mu_2)})^*\mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk))$ respectively. Therefore, $\mathfrak{f}^{\Lambda}_{\lambda_1} \mathbin{\widetilde{\boxtimes}} \mathfrak{f}^{\Lambda}_{\lambda_2}$ is deduced from the induced trivialization of the restriction of $\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk))$ to $\mathrm{Fl}_{G,\mathsf{t}(\mu_1)} \mathbin{\widetilde{\times}} \mathrm{Fl}_{G,\mathsf{t}(\mu_2)}$. (Here we write $\mathrm{Fl}_{G,\mathsf{t}(\mu_1)} \mathbin{\widetilde{\times}} \mathrm{Fl}_{G,\mathsf{t}(\mu_2)}$ for $q^{-1}(\mathrm{Fl}_{G,\mathsf{t}(\mu_1)}) \times^I \mathrm{Fl}_{G,\mathsf{t}(\mu_2)}$; similar notation will be also used for twisted products over L^+G.) It is well known that the morphism $\mathrm{Gr}^{\lambda_1}_G \mathbin{\widetilde{\times}} \mathrm{Gr}^{\lambda_2}_G \to \overline{\mathrm{Gr}^{\lambda}_G}$ restricts to an isomorphism on the preimage of Gr^{λ}_G. It follows that if we denote by X the preimage of $\mathrm{Gr}_{G,\mu}$, then this morphism restricts to an isomorphism $X \xrightarrow{\sim} \mathrm{Gr}_{G,\mu}$, and moreover we have $\varpi(\mathrm{Fl}_{G,\mathsf{t}(\mu_1)} \mathbin{\widetilde{\times}} \mathrm{Fl}_{G,\mathsf{t}(\mu_2)}) = X$. From this we obtain that the trivialization of $\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk))$ involved in the description of $\mathfrak{f}^{\Lambda}_{\lambda_1} \mathbin{\widetilde{\boxtimes}} \mathfrak{f}^{\Lambda}_{\lambda_2}$ is obtained from a canonical isomorphism

$$\begin{aligned}(j_X)^*\varpi_*(\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)))& \\ \xrightarrow{\sim} (\varpi_{\mu_1,\mu_2})_*(j_{\mathsf{t}(\mu_1)} \mathbin{\widetilde{\times}} j_{\mathsf{t}(\mu_2)})^*&(\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)))\end{aligned}$$

(where $j_X : X \to \mathrm{L}G \times^{\mathrm{L}^+G} \mathrm{Gr}_G$ is the embedding, and ϖ_{μ_1,μ_2} is the restriction of ϖ to $\mathrm{Fl}_{G,\mathsf{t}(\mu_1)} \mathbin{\widetilde{\times}} \mathrm{Fl}_{G,\mathsf{t}(\mu_2)})$, together with the canonical trivialization of the restriction of

$$\varpi_*(\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk))) \cong \mathcal{J}_*(\lambda_1, \Bbbk) \mathbin{\widetilde{\boxtimes}} \mathcal{J}_*(\lambda_2, \Bbbk)$$

(see (3.4.4)) to X, as in (4.6.10). We then obtain that the path in (4.6.13) through the bottom left corner is obtained by adjunction from the composition of natural morphisms

$$\begin{aligned}(j_\mu)^*\pi_*(\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk))) &\xrightarrow{\sim} (j_\mu)^* m_* \varpi_*(\mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \mathbin{\widetilde{\boxtimes}} \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk))) \\ &\xrightarrow[\sim]{(3.4.4)} (j_\mu)^* m_*(\mathcal{J}_*(\lambda_1, \Bbbk) \mathbin{\widetilde{\boxtimes}} \mathcal{J}_*(\lambda_2, \Bbbk)) \\ &\qquad = (j_\mu)^*(\mathcal{J}_*(\lambda_1, \Bbbk) \star^{\mathrm{L}^+G} \mathcal{J}_*(\lambda_2, \Bbbk)) \cong \underline{\Bbbk}_{\mathrm{Gr}_{G,\mu}}[\ell(\mathsf{t}(\mu))].\end{aligned}$$

These two constructions coincide by Lemma 3.4.3, from which we deduce the commutativity of (4.6.13).

Finally, pasting the commutative diagram (4.6.12) with two copies of (4.6.13) (one for the pair (λ_1, λ_2), and one for the pair (λ_2, λ_1)), we obtain the commutativity of the diagram of the proposition. □

Remark 4.6.13. — The arguments in the proof above (see in particular diagram (4.6.13)) show that the following diagram commutes:

$$\begin{tikzcd} \mathsf{Z}(\mathcal{J}_*(\lambda_1, \Bbbk)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda_2, \Bbbk)) \arrow[r, "(4.6.11)"] \arrow[d, "\mathfrak{f}^{\Lambda}_{\lambda_1} \star \mathfrak{f}^{\Lambda}_{\lambda_2}"'] & \mathsf{Z}(\mathcal{J}_*(\lambda_1 + \lambda_2, \Bbbk)) \arrow[d, "\mathfrak{f}^{\Lambda}_{\lambda_1+\lambda_2}"] \\ \mathbf{J}^{\Lambda}_{\mu_1}(\Bbbk) \star^I \mathbf{J}^{\Lambda}_{\mu_2}(\Bbbk) \arrow[r, "\sim"] & \mathbf{J}^{\Lambda}_{\mu_1+\mu_2}(\Bbbk). \end{tikzcd}$$

4.7. Associated graded of the Wakimoto filtration and convolution

This section is dedicated to the study of the compatibility of the functors $\mathrm{gr}^{\Lambda}_{\lambda}$ or $\mathrm{Grad}^{\Lambda}_{\lambda}$ with the convolution bifunctor (and the associated constraints).

4.7.1. Associated graded of a convolution. — A first difficulty that arises in our study is that the bifunctor $\star^I$ is *not* t-exact. The situation is slightly better for Wakimoto-filtered perverse sheaves, as shown in the following lemma.

Lemma 4.7.1. — *If $\mathscr{F}, \mathscr{G}$ belong to $\mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk)$, then the object $\mathscr{F} \star^I \mathscr{G}$ is concentrated in nonpositive perverse degrees.*

Proof. — It suffices to prove the claim when $\mathscr{F}$ and $\mathscr{G}$ are Wakimoto sheaves, in which case it follows from Lemma 4.2.6 and Lemma 4.2.1. □

Recall the bifunctor $\star^I_0$ defined in Remark 3.2.2. In terms of this bifunctor, Lemma 4.7.1 has the following consequence.

Lemma 4.7.2. — *For any $\mathscr{F}$ in $\mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk)$, the functors*

$$\mathscr{F} \star^I_0 (-) : \mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk) \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$$

and

$$(-) \star^I_0 \mathscr{F} : \mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk) \to \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)$$

are right exact.

Note also that Lemma 4.2.1 and Lemma 4.2.6 ensure that for $\lambda, \lambda' \in \mathbf{X}^{\vee}$ and M, M' in $\mathsf{Mof}_{\Bbbk}$ we have

$$\mathbf{J}^{\Lambda}_{\lambda}(M) \star^I_0 \mathbf{J}^{\Lambda}_{\lambda'}(M') \cong \mathbf{J}^{\Lambda}_{\lambda+\lambda'}\big(M \otimes_{\Bbbk} M'\big). \tag{4.7.1}$$

It is not clear to us whether the perverse sheaf $\mathscr{F} \star^I_0 \mathscr{G}$ admits a Wakimoto filtration for general $\mathscr{F}, \mathscr{G}$ in $\mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk)$, and we will not be able to go further without assuming this is the case. Note that this property holds at least in the following cases (which will be sufficient for our needs):

1. if $\mathrm{Grad}^{\Lambda}_{\lambda}(\mathscr{F})$ is flat over $\Bbbk$ for any $\lambda \in \mathbf{X}^{\vee}$, or if $\mathrm{Grad}^{\Lambda}_{\lambda}(\mathscr{G})$ is flat over $\Bbbk$ for any $\lambda \in \mathbf{X}^{\vee}$, in which case we have $\mathscr{F} \star^I_0 \mathscr{G} = \mathscr{F} \star^I \mathscr{G}$ (see Proposition 4.7.5 below for more details);
2. if $\mathscr{F} = \mathsf{Z}(\mathscr{A})$ and $\mathscr{G} = \mathsf{Z}(\mathscr{B})$ for some $\mathscr{A}$, $\mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, since in this case we have an isomorphism $\phi^0_{\mathscr{A},\mathscr{B}} : \mathsf{Z}(\mathscr{A}) \star^I_0 \mathsf{Z}(\mathscr{B}) \xrightarrow{\sim} \mathsf{Z}(\mathscr{A} \star^{\mathrm{L}^+G}_0 \mathscr{B})$, see Section 3.4.

Remark 4.7.3. — We mentioned in Remark 3.2.2 that the bifunctor $\star^I_0$ does not seem to admit an associativity constraint. If $\mathscr{F}, \mathscr{G}, \mathscr{H}$ belong to $\mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk)$, and if both $\mathscr{F} \star^I_0 \mathscr{G}$ and $\mathscr{G} \star^I_0 \mathscr{H}$ admit Wakimoto filtrations, then Lemma 4.7.1 implies (as for the bifunctor $\star^{\mathrm{L}^+G}$) that there exists a canonical isomorphism

$$(\mathscr{F} \star^I_0 \mathscr{G}) \star^I_0 \mathscr{H} \cong \mathscr{F} \star^I_0 (\mathscr{G} \star^I_0 \mathscr{H}).$$

This allows us to define a monoidal structure on $\mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk)$ in case $\Bbbk$ is a field (but not in the general case, since $\mathsf{Perv}^{\Lambda}_I(\mathrm{Fl}_G, \Bbbk)$ does not seem to be stable under $\star^I_0$).

We fix $\mathscr{F}, \mathscr{G}$ in $\mathsf{Perv}_I^\Lambda(\mathrm{Fl}_G, \Bbbk)$ and $\lambda, \mu \in \mathbf{X}^\vee$. Recall the notation $\mathscr{F}_{\preceq_\Lambda \lambda}$ introduced in (4.5.8). In the following lemma we consider the morphism

$$\mathscr{F}_{\preceq_\Lambda \lambda} \star_0^I \mathscr{G}_{\preceq_\Lambda \mu} \to \mathscr{F} \star_0^I \mathscr{G} \tag{4.7.2}$$

induced by the embeddings $\mathscr{F}_{\preceq_\Lambda \lambda} \hookrightarrow \mathscr{F}$ and $\mathscr{G}_{\preceq_\Lambda \mu} \hookrightarrow \mathscr{G}$.

Lemma 4.7.4. — *Assume that $\mathscr{F} \star_0^I \mathscr{G}$ admits a Wakimoto filtration. Then the morphism* (4.7.2) *factors through $(\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu}$, and there exists a unique morphism*

$$\mathrm{gr}_\lambda^\Lambda(\mathscr{F}) \star_0^I \mathrm{gr}_\mu^\Lambda(\mathscr{G}) \to \mathrm{gr}_{\lambda+\mu}^\Lambda(\mathscr{F} \star_0^I \mathscr{G})$$

such that the following diagram (where the vertical maps are the natural ones, and the horizontal maps are induced by the morphisms considered above) commutes:

$$\begin{array}{ccc}
\mathscr{F}_{\preceq_\Lambda \lambda} \star_0^I \mathscr{G}_{\preceq_\Lambda \mu} & \longrightarrow & (\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu} \\
\downarrow & & \downarrow \\
\mathrm{gr}_\lambda^\Lambda(\mathscr{F}) \star_0^I \mathrm{gr}_\mu^\Lambda(\mathscr{G}) & \longrightarrow & \mathrm{gr}_{\lambda+\mu}^\Lambda(\mathscr{F} \star_0^I \mathscr{G}).
\end{array}$$

Proof. — We consider the composition

$$\mathscr{F}_{\preceq_\Lambda \lambda} \star_0^I \mathscr{G}_{\preceq_\Lambda \mu} \to \mathscr{F} \star_0^I \mathscr{G} \twoheadrightarrow (\mathscr{F} \star_0^I \mathscr{G})/(\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu}. \tag{4.7.3}$$

Here the right-hand side admits a filtration whose subquotients are of the form $\mathbf{J}_\nu^\Lambda(M_\nu)$ with $\nu \not\preceq_\Lambda \lambda+\mu$ and M_ν in $\mathsf{Mof}_\Bbbk$, and by Lemma 4.7.1 we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)}(\mathscr{F}_{\preceq_\Lambda \lambda} \star_0^I \mathscr{G}_{\preceq_\Lambda \mu}, (\mathscr{F} \star_0^I \mathscr{G})/(\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu}) \\
&\qquad \cong \mathrm{Hom}_{D_I^{\mathrm{b}}(\mathrm{Fl}_G, \Bbbk)}(\mathscr{F}_{\preceq_\Lambda \lambda} \star^I \mathscr{G}_{\preceq_\Lambda \mu}, (\mathscr{F} \star_0^I \mathscr{G})/(\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu}).
\end{aligned}$$

Now $\mathscr{F}_{\preceq_\Lambda \lambda} \star^I \mathscr{G}_{\preceq_\Lambda \mu}$ belongs to the triangulated subcategory of $D_I^{\mathrm{b}}(\mathrm{Fl}_G, \Bbbk)$ generated by the objects of the form $\mathbf{J}_\eta^\Lambda(M)$ with $\eta \preceq_\Lambda \lambda+\mu$ (see Lemma 4.2.6), so that

$$\mathrm{Hom}_{D_I^{\mathrm{b}}(\mathrm{Fl}_G, \Bbbk)}(\mathscr{F}_{\preceq_\Lambda \lambda} \star^I \mathscr{G}_{\preceq_\Lambda \mu}, (\mathscr{F} \star_0^I \mathscr{G})/(\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu}) = 0$$

by Lemma 4.3.1, proving that (4.7.3) vanishes. This implies the first claim of the lemma.

The proof of the second claim is similar: we observe that we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk)}(\mathscr{F}_{\preceq_\Lambda \lambda} \star_0^I \mathscr{G}_{\preceq_\Lambda \mu}, \mathrm{gr}_{\lambda+\mu}^\Lambda(\mathscr{F} \star_0^I \mathscr{G})) \\
&\qquad \cong \mathrm{Hom}_{D_I^{\mathrm{b}}(\mathrm{Fl}_G, \Bbbk)}(\mathscr{F}_{\preceq_\Lambda \lambda} \star^I \mathscr{G}_{\preceq_\Lambda \mu}, \mathrm{gr}_{\lambda+\mu}^\Lambda(\mathscr{F} \star_0^I \mathscr{G}))
\end{aligned}$$

and similarly with $\mathscr{F}_{\preceq_\Lambda \lambda}$ and $\mathscr{G}_{\preceq_\Lambda \mu}$ replaced by $\mathrm{gr}_\lambda^\Lambda(\mathscr{F})$ and $\mathrm{gr}_\mu^\Lambda(\mathscr{G})$ respectively, and that the cones of the maps

$$\mathscr{F}_{\preceq_\Lambda \lambda} \star^I \mathscr{G}_{\preceq_\Lambda \mu} \to \mathrm{gr}_\lambda^\Lambda(\mathscr{F}) \star^I \mathscr{G}_{\preceq_\Lambda \mu}$$

and

$$\mathrm{gr}_\lambda^\Lambda(\mathscr{F}) \star^I \mathscr{G}_{\preceq_\Lambda \mu} \to \mathrm{gr}_\lambda^\Lambda(\mathscr{F}) \star^I \mathrm{gr}_\mu^\Lambda(\mathscr{G})$$

have no nonzero maps to $\mathrm{gr}_{\lambda+\mu}^\Lambda(\mathscr{F} \star_0^I \mathscr{G})$. Hence the composition

$$\mathscr{F}_{\preceq_\Lambda \lambda} \star_0^I \mathscr{G}_{\preceq_\Lambda \mu} \to (\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu} \to \mathrm{gr}_{\lambda+\mu}^\Lambda(\mathscr{F} \star_0^I \mathscr{G})$$

must factor through a (unique) morphism

$$\mathrm{gr}^\Lambda_\lambda(\mathscr{F}) \star^I_0 \mathrm{gr}^\Lambda_\mu(\mathscr{G}) \to \mathrm{gr}^\Lambda_{\lambda+\mu}(\mathscr{F} \star^I_0 \mathscr{G}),$$

which finishes the proof. □

Taking direct sums of the morphisms provided by Lemma 4.7.4, we obtain for $\nu \in \mathbf{X}^\vee$ a bifunctorial morphism

$$\bigoplus_{\lambda+\mu=\nu} \mathrm{gr}^\Lambda_\lambda(\mathscr{F}) \star^I_0 \mathrm{gr}^\Lambda_\mu(\mathscr{G}) \to \mathrm{gr}^\Lambda_\nu(\mathscr{F} \star^I_0 \mathscr{G}) \tag{4.7.4}$$

for $\mathscr{F}, \mathscr{G}$ in $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G, \Bbbk)$ such that $\mathscr{F} \star^I_0 \mathscr{G}$ admits a Wakimoto filtration.

Proposition 4.7.5. — *Let $\mathscr{F},\mathscr{G}$ in $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G, \Bbbk)$. Assume either that $\mathrm{Grad}^\Lambda_\eta(\mathscr{F})$ is flat over $\Bbbk$ for any $\eta \in \mathbf{X}^\vee$, or that $\mathrm{Grad}^\Lambda_\eta(\mathscr{G})$ is flat over $\Bbbk$ for any $\eta \in \mathbf{X}^\vee$. Then $\mathscr{F} \star^I_0 \mathscr{G} = \mathscr{F} \star^I \mathscr{G}$, this object admits a Wakimoto filtration, and the morphism* (4.7.4) *is an isomorphism for any $\nu \in \mathbf{X}^\vee$.*

Proof. — Of course the two cases are similar; to fix notation we assume that $\mathrm{Grad}^\Lambda_\eta(\mathscr{F})$ is flat over $\Bbbk$ for any $\eta \in \mathbf{X}^\vee$. We will proceed by induction on the length of a Wakimoto filtration of $\mathscr{F}$. So, we consider $\eta \in \mathbf{X}^\vee$, a flat $\Bbbk$-module M, and an exact sequence $\mathbf{J}^\Lambda_\eta(M) \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{F}'$ where $\mathscr{F}'$ belongs to $\mathsf{Perv}^\Lambda_I(\mathrm{Fl}_G, \Bbbk)$ and $\mathrm{Grad}^\Lambda_\xi(\mathscr{F}')$ is flat for any ξ, and assume the claim is known for the pair $(\mathscr{F}',\mathscr{G})$. Then applying the functor $(-) \star^I \mathscr{G}$ we obtain a distinguished triangle

$$\mathbf{J}^\Lambda_\eta(M) \star^I \mathscr{G} \to \mathscr{F} \star^I \mathscr{G} \to \mathscr{F}' \star^I \mathscr{G} \xrightarrow{[1]} . \tag{4.7.5}$$

Here, by assumption the right-hand side is perverse and admits a Wakimoto filtration. On the other hand, taking a Wakimoto filtration of $\mathscr{G}$ and applying the (triangulated) functor $\mathbf{J}^\Lambda_\eta(M) \star^I (-)$ we see that the left-hand side also satisfies these properties (because $\mathbf{J}^\Lambda_\eta(M) \star^I \mathbf{J}^\Lambda_\xi(M')$ is a Wakimoto sheaf for any $\xi \in \mathbf{X}^\vee$ and M' in $\mathsf{Mof}_\Bbbk$ since M is flat, see Lemma 4.2.6). We deduce that $\mathscr{F} \star^I \mathscr{G}$ is perverse and that (4.7.5) is an exact sequence of perverse sheaves, which implies in addition that $\mathscr{F} \star^I \mathscr{G}$ admits a Wakimoto filtration.

Then, if $\nu \in \mathbf{X}^\vee$, by exactness of the functor gr^Λ_ν (see Remark 4.6.4) we have an exact sequence

$$\mathrm{gr}^\Lambda_\nu(\mathbf{J}^\Lambda_\eta(M) \star^I \mathscr{G}) \hookrightarrow \mathrm{gr}^\Lambda_\nu(\mathscr{F} \star^I \mathscr{G}) \twoheadrightarrow \mathrm{gr}^\Lambda_\nu(\mathscr{F}' \star^I \mathscr{G}).$$

For any $\lambda \in \mathbf{X}^\vee$ we also have an exact sequence

$$\mathrm{gr}^\Lambda_\lambda(\mathbf{J}^\Lambda_\eta(M)) \hookrightarrow \mathrm{gr}^\Lambda_\lambda(\mathscr{F}) \twoheadrightarrow \mathrm{gr}^\Lambda_\lambda(\mathscr{F}')$$

where all terms are of the form $\mathbf{J}^\Lambda_\lambda(M')$ with M' flat; hence Lemma 4.2.6 ensures that this exact sequence induces for any $\mu \in \mathbf{X}^\vee$ an exact sequence of perverse sheaves

$$\mathrm{gr}^\Lambda_\lambda(\mathbf{J}^\Lambda_\eta(M)) \star^I \mathrm{gr}^\Lambda_\mu(\mathscr{G}) \hookrightarrow \mathrm{gr}^\Lambda_\lambda(\mathscr{F}) \star^I \mathrm{gr}^\Lambda_\mu(\mathscr{G}) \twoheadrightarrow \mathrm{gr}^\Lambda_\lambda(\mathscr{F}') \star^I \mathrm{gr}^\Lambda_\mu(\mathscr{G}).$$

The invertibility of (4.7.4) for the pair $(\mathscr{F},\mathscr{G})$ then follows from that for the pair $(\mathbf{J}^\Lambda_\eta(M),\mathscr{G})$ (which is obvious), that for the pair $(\mathscr{F}',\mathscr{G})$ (which is true by assumption), and the 5-lemma. □

Remark 4.7.6. — Of course, if $\Bbbk$ is a field the assumption of Proposition 4.7.5 is always satisfied. In this case, this statement implies that the functor $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}$ considered in §4.7.2 below is a monoidal functor.

Corollary 4.7.7. — *For any $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, the morphism* (4.7.4) *is an isomorphism when $\mathscr{F} = \mathsf{Z}(\mathscr{A})$ and $\mathscr{G} = \mathsf{Z}(\mathscr{B})$.*

Proof. — Recall that the existence of a Wakimoto filtration on $\mathscr{F} \star^I_0 \mathscr{G}$ is automatic in this setting, see the comments preceding Remark 4.7.3. As in [**BR**, proof of Lemma 1.10.10], there exist objects $\mathscr{A}_1, \mathscr{A}_2$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ and an exact sequence

$$\mathscr{A}_2 \to \mathscr{A}_1 \to \mathscr{A} \to 0$$

such that $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}_1)$ and $\mathsf{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}_2)$ are free over $\Bbbk$. Then we have an exact sequence

$$\mathsf{Z}(\mathscr{A}_2) \to \mathsf{Z}(\mathscr{A}_1) \to \mathsf{Z}(\mathscr{A}) \to 0,$$

which induces for any $\eta \in \mathbf{X}^\vee$ an exact sequence

$$\mathrm{gr}^{\Lambda}_{\eta}(\mathsf{Z}(\mathscr{A}_2)) \to \mathrm{gr}^{\Lambda}_{\eta}(\mathsf{Z}(\mathscr{A}_1)) \to \mathrm{gr}^{\Lambda}_{\eta}(\mathscr{F}) \to 0,$$

see Remark 4.6.4, and moreover the first two terms in this sequence are of the form $\mathbf{J}^{\Lambda}_{\eta}(M)$ with M projective (and hence flat), see Proposition 4.5.4.

We consider the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\lambda+\mu=\nu} \mathrm{gr}^{\Lambda}_{\lambda}\big(\mathsf{Z}(\mathscr{A}_2)\big) \star^I_0 \mathrm{gr}^{\Lambda}_{\mu}(\mathscr{G}) \arrow[r] \arrow[d] & \mathrm{gr}^{\Lambda}_{\nu}\big(\mathsf{Z}(\mathscr{A}_2) \star^I_0 \mathscr{G}\big) \arrow[d] \\
\bigoplus_{\lambda+\mu=\nu} \mathrm{gr}^{\Lambda}_{\lambda}\big(\mathsf{Z}(\mathscr{A}_1)\big) \star^I_0 \mathrm{gr}^{\Lambda}_{\mu}(\mathscr{G}) \arrow[r] \arrow[d] & \mathrm{gr}^{\Lambda}_{\nu}\big(\mathsf{Z}(\mathscr{A}_1) \star^I_0 \mathscr{G}\big) \arrow[d] \\
\bigoplus_{\lambda+\mu=\nu} \mathrm{gr}^{\Lambda}_{\lambda}(\mathscr{F}) \star^I_0 \mathrm{gr}^{\Lambda}_{\mu}(\mathscr{G}) \arrow[r] \arrow[d] & \mathrm{gr}^{\Lambda}_{\nu}(\mathscr{F} \star^I_0 \mathscr{G}) \arrow[d] \\
0 & 0
\end{tikzcd}$$

where each horizontal map is induced by (4.7.4). Here the upper two arrows are invertible by Proposition 4.7.5, the left column is exact by Lemma 4.7.2, and the right column is exact by Lemma 4.7.2 again and Remark 4.6.4. We deduce that the lower horizontal arrow is an isomorphism too, which finishes the proof. □

4.7.2. Monoidal structure on $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}$**.** — In view of (4.7.1) and the full faithfulness of the functor $\mathrm{Grad}^{\Lambda}_{\lambda+\mu}$ on the image of $\mathbf{J}^{\Lambda}_{\lambda+\mu}$ (see Proposition 4.2.5(1)), the map constructed in Lemma 4.7.4 can also be seen as defining a bifunctorial morphism

$$\mathrm{Grad}^{\Lambda}_{\lambda}(\mathscr{F}) \otimes_{\Bbbk} \mathrm{Grad}^{\Lambda}_{\mu}(\mathscr{G}) \to \mathrm{Grad}^{\Lambda}_{\lambda+\mu}(\mathscr{F} \star^I_0 \mathscr{G}). \tag{4.7.6}$$

Our goal in this subsection is to give an alternative description of this morphism in terms of equivariant cohomology.

First, for any $\mathscr{F}_1, \mathscr{F}_2$ in $\mathsf{Perv}_I^{\Lambda}(\mathrm{Fl}_G, \Bbbk)$, combining the morphism (4.5.1) with the morphism induced by the truncation map $\mathscr{F}_1 \star^I \mathscr{F}_2 \to \mathscr{F}_1 \star_0^I \mathscr{F}_2$ we obtain a canonical morphism of graded $\Bbbk$-modules

$$\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}_1) \otimes_{\mathsf{H}_I^\bullet(\mathrm{pt};\Bbbk)} \mathsf{H}_I^\bullet(\mathrm{Fl}_G, \mathscr{F}_2) \to \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}_1 \star_0^I \mathscr{F}_2). \tag{4.7.7}$$

Now, in the setting of Lemma 4.7.4, in view of Remark 4.5.3 we have

$$\begin{aligned} \mathrm{Grad}_\lambda^\Lambda(\mathscr{F}) &\cong \mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda}), \\ \mathrm{Grad}_\mu^\Lambda(\mathscr{G}) &\cong \mathsf{H}^{-\langle x_\Lambda^{-1}(\mu), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{G}_{\preceq_\Lambda \mu}), \\ \mathrm{Grad}_{\lambda+\mu}^\Lambda(\mathscr{F} \star_0^I \mathscr{G}) &\cong \mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda+\mu), 2\rho\rangle}(\mathrm{Fl}_G, (\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu}). \end{aligned}$$

As noted in §4.5.2 (see especially (4.5.9)), the forgetful map

$$\mathsf{H}_I^{-\langle x_\Lambda^{-1}(\mu), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{G}_{\preceq_\Lambda \mu}) \to \mathsf{H}^{-\langle x_\Lambda^{-1}(\mu), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{G}_{\preceq_\Lambda \mu})$$

is an isomorphism, so that we also have

$$\mathrm{Grad}_\mu^\Lambda(\mathscr{G}) \cong \mathsf{H}_I^{-\langle x_\Lambda^{-1}(\mu), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{G}_{\preceq_\Lambda \mu}).$$

Apply (4.7.7) to obtain a canonical morphism

$$\begin{aligned} &\mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda}) \otimes \mathsf{H}_I^{-\langle x_\Lambda^{-1}(\mu), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{G}_{\preceq_\Lambda \mu}) \\ &\qquad\qquad \to \mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda+\mu), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda} \star_0^I \mathscr{G}_{\preceq_\Lambda \mu}). \end{aligned}$$

Combining this with the morphism $\mathscr{F}_{\preceq_\Lambda \lambda} \star_0^I \mathscr{G}_{\preceq_\Lambda \mu} \to (\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu}$ induced by (4.7.2) we obtain a canonical morphism

$$\begin{aligned} &\mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{F}_{\preceq_\Lambda \lambda}) \otimes \mathsf{H}_I^{-\langle x_\Lambda^{-1}(\mu), 2\rho\rangle}(\mathrm{Fl}_G, \mathscr{G}_{\preceq_\Lambda \mu}) \\ &\qquad\qquad \to \mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda+\mu), 2\rho\rangle}(\mathrm{Fl}_G, (\mathscr{F} \star_0^I \mathscr{G})_{\preceq_\Lambda \lambda+\mu}), \end{aligned}$$

which coincides with (4.7.6) under the identifications considered above.

Gathering the morphisms (4.7.6) for all values of λ and μ we obtain, for any $\mathscr{F}, \mathscr{G}$ in $\mathsf{Perv}_I^{\Lambda}(\mathrm{Fl}_G, \Bbbk)$ such that $\mathscr{F} \star_0^I \mathscr{G}$ admits a Wakimoto filtration, a canonical morphism of $\mathbf{X}^\vee$-graded $\Bbbk$-modules

$$\mathrm{Grad}_{\mathbf{X}^\vee}^\Lambda(\mathscr{F}) \otimes_\Bbbk \mathrm{Grad}_{\mathbf{X}^\vee}^\Lambda(\mathscr{G}) \to \mathrm{Grad}_{\mathbf{X}^\vee}^\Lambda(\mathscr{F} \star_0^I \mathscr{G}). \tag{4.7.8}$$

From this point of view, Proposition 4.7.5 and Corollary 4.7.7 say that (4.7.8) is an isomorphism

- if either $\mathrm{Grad}_{\mathbf{X}^\vee}^\Lambda(\mathscr{F})$ or $\mathrm{Grad}_{\mathbf{X}^\vee}^\Lambda(\mathscr{G})$ is flat;
- or if $\mathscr{F} = \mathsf{Z}(\mathscr{A})$ and $\mathscr{G} = \mathsf{Z}(\mathscr{B})$ for some $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$.

(In both cases, the fact that $\mathscr{F} \star_0^I \mathscr{G}$ admits a Wakimoto filtration is automatic.)

Lemma 4.7.8. — *For any $\mathscr{F}, \mathscr{G}$ in $\mathsf{Perv}_I^{\Lambda}(\mathrm{Fl}_G, \Bbbk)$ such that $\mathscr{F} \star_0^I \mathscr{G}$ admits a Wakimoto filtration the following diagram commutes, where in the left column the upper* *arrow is induced by the forgetful map from equivariant cohomology to cohomology and the middle arrow is the natural map:*

$$\begin{tikzcd}
\mathsf{H}^\bullet(\mathrm{Fl}_G,\mathscr{F})\otimes_{\Bbbk}\mathsf{H}^\bullet(\mathrm{Fl}_G,\mathscr{G}) \arrow[r, "(4.5.6)", "\sim"'] & \mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F})\otimes_{\Bbbk}\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{G}) \arrow[ddd, "(4.7.8)"] \\
\mathsf{H}^\bullet(\mathrm{Fl}_G,\mathscr{F})\otimes_{\Bbbk}\mathsf{H}^\bullet_I(\mathrm{Fl}_G,\mathscr{G}) \arrow[u] \arrow[d] & \\
\mathsf{H}^\bullet(\mathrm{Fl}_G,\mathscr{F})\otimes_{\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)}\mathsf{H}^\bullet_I(\mathrm{Fl}_G,\mathscr{G}) \arrow[d, "(4.7.7)"'] & \\
\mathsf{H}^\bullet(\mathrm{Fl}_G,\mathscr{F}\star^I_0\mathscr{G}) \arrow[r, "(4.5.6)", "\sim"'] & \mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F}\star^I_0\mathscr{G}).
\end{tikzcd}$$

Proof. — By Proposition 4.5.6, in this diagram one can replace $\mathsf{H}^\bullet_I(\mathrm{Fl}_G,\mathscr{G})$ by

$$\bigoplus_{\lambda\in\mathbf{X}^\vee}\mathsf{H}^\bullet_I(\mathrm{pt};\Bbbk)\otimes_{\Bbbk}\mathsf{H}_I^{-\langle x_\Lambda^{-1}(\lambda),2\rho\rangle}(\mathrm{Fl}_G,\mathscr{G}_{\preceq_\Lambda\lambda}).$$

Similarly, by Proposition 4.5.4 and (4.5.4) one can replace $\mathsf{H}^\bullet(\mathrm{Fl}_G,\mathscr{F})$ by

$$\bigoplus_{\lambda\in\mathbf{X}^\vee}\mathsf{H}^{-\langle x_\Lambda^{-1}(\lambda),2\rho\rangle}(\mathrm{Fl}_G,\mathscr{F}_{\preceq_\Lambda\lambda}).$$

Then the statement exactly becomes the fact that the construction given at the beginning of this subsection corresponds to (4.7.6) under the identification given by (4.5.6). $\square$

4.8. Comparison of the functors Grad and F

Consider the functor

$$\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}\circ\mathsf{Z}:\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk)\to\mathsf{Mof}_{\Bbbk}$$

(where we omit the forgetful functor from $\mathsf{Mof}_{\Bbbk}^{\mathbf{X}^\vee}$ to $\mathsf{Mof}_{\Bbbk}$.) Combining (2.5.6) and Proposition 4.5.4 we obtain a canonical isomorphism

$$\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}\circ\mathsf{Z}\xrightarrow{\sim}\mathsf{F}, \tag{4.8.1}$$

where the right-hand side is the fiber functor for the Satake category (see §1.3.2). Our goal in this section is to show that several structures defined independently on the left- and right-hand sides of (4.8.1) are intertwined by this isomorphism.

4.8.1. Grading. — We first consider the gradings on the left- and right-hand sides of (4.8.1).

Lemma 4.8.1. — *For any $\mathscr{A}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G,\Bbbk)$ and $\lambda\in\mathbf{X}^\vee$, the isomorphism* (4.8.1) *induces an isomorphism*

$$\mathrm{Grad}^{\Lambda}_{x_\Lambda w_\circ(\lambda)}\circ\mathsf{Z}(\mathscr{A})\xrightarrow{\sim}\mathsf{F}_\lambda(\mathscr{A}).$$

Proof. — The isomorphism under consideration is the composition

$$\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee} \circ \mathsf{Z}(\mathscr{A}) \xrightarrow{\sim} \mathsf{H}^\bullet(\mathrm{Gr}_G, \mathsf{Z}(\mathscr{A})) \xrightarrow{\sim} \mathsf{F}(\mathscr{A}) \tag{4.8.2}$$

where the first isomorphism is provided by Proposition 4.5.4, and the second one by (2.5.6). Recall (see §1.3.2, and in particular Remark 1.3.7) that the submodule $\mathsf{F}_\lambda(\mathscr{A}) \subset \mathsf{F}(\mathscr{A})$ is the image of the canonical (injective) morphism

$$\mathsf{H}^{\langle\lambda,2\rho\rangle}_{\overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}}}(\mathrm{Gr}_G, \mathscr{A}) \to \mathsf{H}^{\langle\lambda,2\rho\rangle}(\mathrm{Gr}_G, \mathscr{A}).$$

By base change, under the second isomorphism in (4.8.2), this subspace corresponds to the image of the canonical (injective) morphism

$$\mathsf{H}^{\langle\lambda,2\rho\rangle}_{\pi^{-1}(\overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}})}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \to \mathsf{H}^{\langle\lambda,2\rho\rangle}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})).$$

On the other hand, setting $\Omega := \{\mu \in \mathbf{X}^\vee \mid \mu \preceq_\Lambda x_\Lambda w_\circ(\lambda)\}$, $\mathrm{Grad}^{\Lambda}_{x_\Lambda w_\circ(\lambda)} \circ \mathsf{Z}(\mathscr{A})$ corresponds (under the first isomorphism in (4.8.2)) to the image of the canonical (injective) morphism

$$\mathsf{H}^{\langle\lambda,2\rho\rangle}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})_\Omega) \to \mathsf{H}^{\langle\lambda,2\rho\rangle}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})).$$

To compare these two submodules, we consider the commutative diagram

$$\begin{tikzcd}
\mathsf{H}^{\langle\lambda,2\rho\rangle}_{\pi^{-1}(\overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}})}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})_\Omega) \arrow[r] \arrow[d] & \mathsf{H}^{\langle\lambda,2\rho\rangle}_{\pi^{-1}(\overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}})}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \arrow[d] \\
\mathsf{H}^{\langle\lambda,2\rho\rangle}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})_\Omega) \arrow[r] & \mathsf{H}^{\langle\lambda,2\rho\rangle}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})),
\end{tikzcd}$$

where all the maps are the natural ones. In order to prove that our two submodules coincide, it suffices to prove that the two maps with domain the top left corner are surjective. These morphisms are part of long exact sequences, whose next terms are

$$\mathsf{H}^{\langle\lambda,2\rho\rangle}_{\pi^{-1}(\overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}})}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})/\mathsf{Z}(\mathscr{A})_\Omega)$$

and

$$\mathsf{H}^{\langle\lambda,2\rho\rangle}_{\mathrm{Fl}_G \smallsetminus \pi^{-1}(\overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}})}(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})_\Omega)$$

respectively. It therefore suffices to prove that these $\Bbbk$-modules vanish.

If $\mu \in \mathbf{X}^\vee$ and $\mathfrak{S}^\Lambda_{\mathsf{t}(\mu)} \subset \pi^{-1}(\overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}})$, then we also have $\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_\mu \subset \overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}}$, so that (as in Remark 4.5.11) we have $\mu \in \Omega$. Using the formula in Remark 4.5.11 we deduce that the first module vanishes. On the other hand, if $\mu \in \Omega$ and $\langle x_\Lambda^{-1}(\mu), 2\rho\rangle = -\langle\lambda, 2\rho\rangle$ then $\mu = x_\Lambda w_\circ(\lambda)$, so that $\mathfrak{S}^\Lambda_{\mathsf{t}(\mu)} \not\subset \mathrm{Fl}_G \smallsetminus \pi^{-1}(\overline{\mathrm{L}U_{x_\Lambda w_\circ}\cdot L_{x_\Lambda w_\circ(\lambda)}})$. Hence the same formula implies that our second module also vanishes, which finishes the proof. □

4.8.2. Monoidal structures. — Now we turn to monoidal structures. Namely, the left-hand side in (4.8.1) admits a monoidal structure, obtained by combining the constructions of §4.7.2 with (3.4.3). On the other hand, the right-hand side also admits a monoidal structure (see §1.3.4) for which we have given an alternative description in §3.3.3 (and also in §3.3.4 under some assumption on $\Bbbk$). These structures are compatible in the sense of the next statement.

Proposition 4.8.2. — *The isomorphism* (4.8.1) *is an isomorphism of monoidal functors.*

Remark 4.8.3. — Proposition 4.8.2 implies in particular that the functor $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee} \circ \mathsf{Z} : \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk) \to \mathsf{Mof}_{\Bbbk}$ is a fiber functor (i.e. a faithful $\Bbbk$-linear monoidal functor which intertwines the relevant commutativity constraints). This fact can be proved directly; see e.g. [**AB**, §§3.6.5–3.6.6] for the fact that $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee} \circ \mathsf{Z}$ intertwines the commutativity constraints. By the yoga of Tannakian categories (see in particular [**SR**, §II.3.2.3.2 and §II.3.4]), this implies that this functor is defined by a principal $G^\vee_{\Bbbk}$-bundle over $\mathrm{Spec}(\Bbbk)$. If $\Bbbk$ is an algebraically closed field then this principal bundle is automatically trivial, so that one can deduce a monoidal isomorphism between $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee} \circ \mathsf{Z}$ and F. In the general setting, however, proving Proposition 4.8.2 is the only way we know to prove the triviality of this principal bundle.

The proof of Proposition 4.8.2 will require a few preliminary steps. We begin with the following lemma, which should be compared with Remark 1.3.10.

Lemma 4.8.4. — *Let* $\mathscr{A} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ *and* $\mathscr{F} \in \mathsf{Perv}^{\Lambda}_{I}(\mathrm{Fl}_G, \Bbbk)$, *and assume either that* $\mathsf{F}(\mathscr{A})$ *is flat over* $\Bbbk$, *or that* $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}(\mathscr{F})$ *is flat over* $\Bbbk$. *Then the complex* $\mathscr{A} \boxtimes^L_{\Bbbk} \mathscr{F}$ *is perverse. As a consequence, the complex* $\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}$ *is perverse.*

Proof. — Of course it is enough to prove the claim in the case $\mathscr{F} = \mathbf{J}^{\Lambda}_{\lambda}(M)$ for some $\lambda \in \mathbf{X}^\vee$ and $M \in \mathsf{Mod}_{\Bbbk}$ (assuming that $\mathsf{F}(\mathscr{A})$ is flat in the first case, and that M is flat in the second case). In this setting, in view of Remark 4.2.2, to prove the first claim it suffices to prove that $\mathscr{A} \otimes^L_{\Bbbk} M$ is perverse. However, in the first case, identifying complexes on Gr_G with complexes on $\mathrm{Gr}_G \times \mathrm{Gr}_G$ supported on $\mathrm{Gr}_G \times \mathrm{Gr}^0_G$, this property is a special case of the statement recalled in Remark 1.3.10. And in the second case it is obvious.

The second claim follows from the first one, in view of Corollary 3.2.5. $\square$

In view of the definition of our two monoidal structures, in order to prove Proposition 4.8.2, we need to prove that for any $\mathscr{A}, \mathscr{B}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ the following diagram commutes (where, for brevity, we write Grad for $\mathrm{Grad}^{\Lambda}_{\mathbf{X}^\vee}$):

$$\begin{tikzcd}
\mathrm{Grad}(\mathsf{Z}(\mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B})) \arrow[r, "(4.5.6)"] \arrow[d, "(3.4.3)"'] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B})) \arrow[r, "(2.5.6)"] & \mathsf{F}(\mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B}) \arrow[d, "(3.3.3)"] \\
\mathrm{Grad}(\mathsf{Z}(\mathscr{A}) \star_0^{I} \mathsf{Z}(\mathscr{B})) & & \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Gr}_G, {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{A} \boxtimes^L \mathscr{B})) \\
\mathrm{Grad}(\mathsf{Z}(\mathscr{A})) \otimes \mathrm{Grad}(\mathsf{Z}(\mathscr{B})) \arrow[r, "(4.5.6)"] \arrow[u, "(4.7.8)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{B})) \arrow[r, "(2.5.6)"] & \mathsf{F}(\mathscr{A}) \otimes \mathsf{F}(\mathscr{B}). \arrow[u, "(1.3.7)"']
\end{tikzcd}$$

Here, every map is an isomorphism. By the same arguments as in the proof of Corollary 4.7.7, one can assume that $\mathscr{A}$ and $\mathscr{B}$ are such that $\mathsf{F}(\mathscr{A})$ and $\mathsf{F}(\mathscr{B})$ are $\Bbbk$-flat, which we do from now on. (In this case the bifunctors $\star_0^{\mathrm{L}^+G}$ and $\star_0^I$ can be replaced by $\star^{\mathrm{L}^+G}$ and $\star^I$ respectively, and the 0-th perverse cohomology functor can be omitted; see Remark 1.3.10, Corollary 3.3.3 and Lemma 4.8.4.)

We need two additional lemmas before we can prove the commutativity of the diagram above.

Lemma 4.8.5. — *Let $\mathscr{A} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, and assume that $\mathsf{F}(\mathscr{A})$ is flat over $\Bbbk$. For any $\mathscr{F}$ in $\mathsf{Perv}_I^{\Lambda}(\mathrm{Fl}_G, \Bbbk)$, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Grad}(\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}) \arrow[r, "(4.5.6)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}) \arrow[r, "(3.2.7)"] & \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \boxtimes^L \mathscr{F}) \\
\mathrm{Grad}(\mathsf{Z}(\mathscr{A})) \otimes \mathrm{Grad}(\mathscr{F}) \arrow[u, "(4.7.8)"] \arrow[r, "(4.5.6)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \arrow[r, "(2.5.6)"] & \mathsf{F}(\mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \arrow[u, "(3.2.7)"']
\end{tikzcd}$$

Note that both $\mathscr{A} \boxtimes_\Bbbk^L \mathscr{F}$ and $\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}$ are perverse by Lemma 4.8.4.

Proof. — Let us enlarge the diagram by inserting two terms involving the equivariant cohomology groups $\mathsf{H}_I^\bullet(\mathrm{Fl}_G, \mathscr{F})$:

$$\begin{tikzcd}
\mathrm{Grad}(\mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}) \arrow[r, "(4.5.6)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathscr{F}) \arrow[r, "(3.2.7)"] & \mathsf{H}^\bullet(\mathrm{Gr}_G \times \mathrm{Fl}_G, \mathscr{A} \boxtimes^L \mathscr{F}) \\
& \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_{\mathsf{H}_I^\bullet(\mathrm{pt};\Bbbk)} \mathsf{H}_I^\bullet(\mathrm{Fl}_G, \mathscr{F}) \arrow[u, "(3.2.11)"'] & \\
& \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_\Bbbk \mathsf{H}_I^\bullet(\mathrm{Fl}_G, \mathscr{F}) \arrow[u] \arrow[d, "(*)"'] & \\
\mathrm{Grad}(\mathsf{Z}(\mathscr{A})) \otimes \mathrm{Grad}(\mathscr{F}) \arrow[uuu, "(4.7.8)"] \arrow[r, "(4.5.6)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \arrow[r, "(2.5.6)"] & \mathsf{F}(\mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathscr{F}) \arrow[uuu, "(3.2.7) \text{ or } (3.2.10)"']
\end{tikzcd}$$

The left half of this enlarged diagram commutes by Lemma 4.7.8, and the right half commutes by Lemma 3.2.8. The arrow marked $(*)$ is surjective by Lemma 4.5.5, and all maps in the outer rectangle (i.e., all maps in the statement of the lemma) are isomorphisms. We conclude that the outer rectangle commutes. □

Lemma 4.8.6. — *Let $\mathscr{A}, \mathscr{B} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ be such that $\mathsf{F}(\mathscr{A})$ and $\mathsf{F}(\mathscr{B})$ are flat over $\Bbbk$. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B})) \arrow[r, "(3.2.7)"] \arrow[d, "(3.4.5)"'] & \mathsf{H}^\bullet(\mathscr{A} \boxtimes_\Bbbk^L \mathsf{Z}(\mathscr{B})) \arrow[r, "(3.2.7)"] & \mathsf{F}(\mathscr{A}) \otimes_\Bbbk \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{B})) \arrow[d, "(2.5.6)"] \\
\mathsf{F}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \arrow[r, "(3.3.3)"] & \mathsf{H}^\bullet(\mathscr{A} \boxtimes_\Bbbk^L \mathscr{B}) \arrow[r, "(1.3.7)"] & \mathsf{F}(\mathscr{A}) \otimes_\Bbbk \mathsf{F}(\mathscr{B}).
\end{tikzcd}$$

Proof. — We have

$$(\mathrm{id} \times \pi)_*(\mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathsf{Z}(\mathscr{B})) \cong \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \pi_* \mathsf{Z}(\mathscr{B}) \cong \mathscr{A} \overset{L}{\boxtimes}_{\Bbbk} \mathscr{B},$$

where the second isomorphism uses Lemma 2.5.1. This isomorphism allows us to define a vertical arrow in the middle column of our diagram. With this additional arrow, it is easily seen that the right part of the diagram commutes. To see that left part of the diagram commutes, apply Remark 9.1.5(1) to the sequence of maps $\mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cup \underline{0}) \to \mathbf{Gr}_{G_C}(\underline{\mathsf{y}} \cup \underline{0}) \to C$ and to the complex $\mathscr{A} \boxtimes^{L}_{\Bbbk} \mathsf{Z}(\mathscr{B})$ on $\mathrm{Gr} \times \mathrm{Fl} \cong \mathbf{Gr}_{\mathcal{G}}(\underline{\mathsf{y}} \cup \underline{0})_{|C^\circ}$. □

Proof of Proposition 4.8.2. — The desired commutative diagram can be assembled from Lemmas 3.4.3, 4.8.5, and 4.8.6, as follows:

$$\begin{tikzcd}
\mathrm{Grad}(\mathsf{Z}(\mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B})) \arrow[r, "(4.5.6)"] \arrow[d, "(3.4.3)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A} \star_0^{\mathrm{L}^+G} \mathscr{B})) \arrow[r, "(2.5.6)"] \arrow[d, "(3.4.3)"'] \arrow[dr, phantom, "\text{Lemma 3.4.3}"] & \mathsf{F}(\mathscr{A} \star^{\mathrm{L}^+G} \mathscr{B}) \arrow[d, "(3.3.3)"'] \\
\mathrm{Grad}(\mathsf{Z}(\mathscr{A}) \star_0^I \mathsf{Z}(\mathscr{B})) \arrow[r, "(4.5.6)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A}) \star^I \mathsf{Z}(\mathscr{B})) \arrow[ur, "(3.4.5)"'] \arrow[d, "(3.2.7)"] & \mathsf{H}^\bullet(\mathscr{A} \boxtimes^L \mathscr{B}) \\
{} \arrow[r, phantom, "\text{Lemma 4.8.5}"] & \mathsf{H}^\bullet(\mathscr{A} \boxtimes^L \mathsf{Z}(\mathscr{B})) \arrow[r, phantom, "\text{Lemma 4.8.6}"] & {} \\
& \mathsf{F}(\mathscr{A}) \otimes_{\Bbbk} \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{B})) \arrow[u, "(3.2.7)"'] \arrow[dr, "(2.5.6)"] & \\
\mathrm{Grad}(\mathsf{Z}(\mathscr{A})) \otimes \mathrm{Grad}(\mathsf{Z}(\mathscr{B})) \arrow[r, "(4.5.6)"] \arrow[uuu, "(4.7.8)"] & \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{A})) \otimes \mathsf{H}^\bullet(\mathrm{Fl}_G, \mathsf{Z}(\mathscr{B})) \arrow[r, "(2.5.6)"] \arrow[u, "(2.5.6)"'] & \mathsf{F}(\mathscr{A}) \otimes \mathsf{F}(\mathscr{B}). \arrow[uuu, "(1.3.7)"']
\end{tikzcd}$$

Here, the unlabeled square in the upper left-hand corner commutes by the naturality of (4.5.6), and the unlabeled triangle in the bottom right corner commutes by construction. □

CHAPTER 5

COMBINATORIAL ASPECTS AND VARIANT FOR ÉTALE SHEAVES

One of the original motivations for defining and studying central sheaves was the desire to "categorify" a description of the center of the affine Hecke algebra due to Bernstein. In this chapter we recall this description, and explain to what extent it is categorified by Z. To make full sense of this idea one needs to use mixed perverse sheaves rather than ordinary perverse sheaves; in particular one should work in the étale setting rather than the "classical" setting we have opted for so far. This variant is considered in §5.3.1; it will also play a major role in Part II.

5.1. Bernstein's description of the center of the affine Hecke algebra

5.1.1. Bernstein elements in the affine Hecke algebra. — This section builds on the notions introduced in §4.1.1. The *affine Hecke algebra* associated with G is the unique $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{H}$ with a basis $(H_w : w \in W)$ which satisfies

$$(H_s + v)(H_s - v^{-1}) = 0 \quad \text{if } s \in S$$

and

$$H_w H_y = H_{wy} \quad \text{if } \ell(wy) = \ell(w) + \ell(y).$$

(Our notational conventions are those of Soergel [**So**], which differ slightly from those of Kazhdan–Lusztig [**KL3**].) Using these rules it is not difficult to check that each element H_w ($w \in W$) is invertible in $\mathcal{H}$. It is clear also that if we consider $\mathbb{Z}$ as a $\mathbb{Z}[v, v^{-1}]$-module with v acting as the identity, then we have a canonical algebra isomorphism

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \xrightarrow{\sim} \mathbb{Z}[W]. \tag{5.1.1}$$

We will also denote by $\mathcal{H}_{\mathrm{f}}$ the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathcal{H}$ spanned by the elements $(H_w : w \in W_{\mathrm{f}})$. This subalgebra identifies with the Hecke algebra of the Coxeter system $(W_{\mathrm{f}}, S_{\mathrm{f}})$.

As in §4.1.1 we fix a closed Weyl chamber, and denote by Λ its intersection with $\mathbf{X}^\vee$. For $\lambda \in \mathbf{X}^\vee$, we choose some $\mu \in \Lambda_\lambda$, and set

$$\theta^\Lambda_\lambda := (-1)^{\langle \lambda, 2\rho \rangle} \cdot H_{\mathsf{t}(\mu)} \cdot H^{-1}_{\mathsf{t}(\mu - \lambda)}.$$

These elements were introduced (in the special case when Λ is the dominant Weyl chamber) by Bernstein. The proofs of their main properties (some of which are reproduced below) first appeared in [**Lu1**].

Lemma 5.1.1. — *For any $\lambda \in \mathbf{X}^\vee$, the element θ^Λ_λ does not depend on the choice of $\mu \in \Lambda_\lambda$. Moreover, for $\lambda, \lambda' \in \mathbf{X}^\vee$ we have*

$$\theta^\Lambda_{\lambda+\lambda'} = \theta^\Lambda_\lambda \cdot \theta^\Lambda_{\lambda'}.$$

Proof. — Let $\lambda \in \mathbf{X}^\vee$ and $\mu_1, \mu_2 \in \Lambda_\lambda$. Then in view of Lemma 4.1.2(1) we have

$$H_{\mathsf{t}(\mu_1-\lambda)} H_{\mathsf{t}(\mu_2-\lambda)} = H_{\mathsf{t}(\mu_1+\mu_2-2\lambda)} = H_{\mathsf{t}(\mu_2-\lambda)} H_{\mathsf{t}(\mu_1-\lambda)}.$$

Similarly we have

$$H_{\mathsf{t}(\mu_1)} H_{\mathsf{t}(\mu_2-\lambda)} = H_{\mathsf{t}(\mu_2)} H_{\mathsf{t}(\mu_1-\lambda)}.$$

Multiplying on the right by $H^{-1}_{\mathsf{t}(\mu_1-\lambda)} H^{-1}_{\mathsf{t}(\mu_2-\lambda)} = H^{-1}_{\mathsf{t}(\mu_2-\lambda)} H^{-1}_{\mathsf{t}(\mu_1-\lambda)}$, we deduce that θ^Λ_λ indeed does not depend on the choice of μ.

Similarly, for $\lambda, \lambda' \in \mathbf{X}^\vee$, choose $\mu \in \Lambda_\lambda$ and $\mu' \in \Lambda_{\lambda'}$, and observe that

$$H_{\mathsf{t}(\mu)} H^{-1}_{\mathsf{t}(\mu-\lambda)} H_{\mathsf{t}(\mu')} H^{-1}_{\mathsf{t}(\mu'-\lambda')} = \\ H_{\mathsf{t}(\mu)} H_{\mathsf{t}(\mu')} H^{-1}_{\mathsf{t}(\mu-\lambda)} H^{-1}_{\mathsf{t}(\mu'-\lambda')} = H_{\mathsf{t}(\mu+\mu')} \cdot H^{-1}_{\mathsf{t}(\mu+\mu'-\lambda-\lambda')},$$

which proves that $\theta^\Lambda_\lambda \theta^\Lambda_{\lambda'} = \theta^\Lambda_{\lambda+\lambda'}$ since $\mu + \mu' \in \Lambda_{\lambda+\lambda'}$. □

Example 5.1.2. — In case $G = \mathrm{PGL}(2)$, we have an identification $\mathbf{X}^\vee = \mathbb{Z}$, under which $1 \in \mathbb{Z}$ corresponds to the coweight

$$t \mapsto \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}.$$

If B is chosen as the Borel subgroup of lower triangular matrices, then under this identification we have $\mathbf{X}^\vee_+ = \mathbb{Z}_{\geq 0}$. The set S has two elements: the "finite" simple reflection s, which belongs to W_{f}, and the "affine" simple reflection $s_\circ$. These reflections satisfy $s_\circ s = \mathsf{t}(2)$. We also have $\Omega = \{e, \omega\}$, where $\omega := \mathsf{t}(1)s$, and $\omega s \omega = s_\circ$.

If Λ corresponds to the dominant Weyl chamber, then we have:

$$\theta^\Lambda_1 = -H_{\omega s}, \quad \theta^\Lambda_{-1} = -H_{s\omega} + (v^{-1} - v) H_\omega,$$
$$\theta^\Lambda_2 = H_{s_\circ s}, \quad \theta^\Lambda_{-2} = H_{ss_\circ} + (v - v^{-1}) H_s + (v - v^{-1}) H_{s_\circ} + (v^2 + v^{-2} - 2) H_e.$$

If Λ corresponds to the antidominant Weyl chamber, then we have

$$\theta^\Lambda_1 = -H_{\omega s} + (v^{-1} - v) H_\omega, \quad \theta^\Lambda_{-1} = -H_{s\omega},$$
$$\theta^\Lambda_2 = H_{s_\circ s} + (v - v^{-1}) H_s + (v - v^{-1}) H_{s_\circ} + (v^2 + v^{-2} - 2) H_e, \quad \theta^\Lambda_{-2} = H_{ss_\circ}.$$

For the proof of the following lemma, we refer to [**Lu1**, Lemma 7.1].

Lemma 5.1.3. — *In the special case $\Lambda = \mathbf{X}^\vee_+$, for any $\lambda \in \mathbf{X}^\vee$ and $s \in S \cap W_{\mathrm{f}}$ we have*

$$H_s \cdot (\theta^{\mathbf{X}^\vee_+}_\lambda + \theta^{\mathbf{X}^\vee_+}_{s(\lambda)}) = (\theta^{\mathbf{X}^\vee_+}_\lambda + \theta^{\mathbf{X}^\vee_+}_{s(\lambda)}) \cdot H_s.$$

Once this lemma is proved, one obtains that for any W_{f}-orbit $\mathbf{o} \subset \mathbf{X}^\vee$, the element $\sum_{\lambda \in \mathbf{o}} \theta_\lambda^{\mathbf{X}^\vee_+}$ commutes with all elements in $\mathcal{H}_{\mathrm{f}}$.

The elements $(\theta_\lambda^\Lambda : \lambda \in \mathbf{X}^\vee)$ depend on the choice of Λ, as shown in Example 5.1.2. However, their W_{f}-invariant linear combinations do not depend on this choice, as shown in the following lemma.

Lemma 5.1.4. — *For any W_{f}-orbit $\mathbf{o} \subset \mathbf{X}^\vee$, the element*

$$\sum_{\lambda \in \mathbf{o}} \theta_\lambda^\Lambda$$

is independent of Λ.

Proof. — We will first prove that for any simple reflection $s \in S \cap W_{\mathrm{f}}$, any Λ and any $\lambda \in \mathbf{X}^\vee$ we have

$$\theta_{s\lambda}^{s\Lambda} = \begin{cases} H_s \cdot \theta_\lambda^\Lambda \cdot H_s^{-1} & \text{if } \ell(sx_\Lambda) > \ell(x_\Lambda); \\ H_s^{-1} \cdot \theta_\lambda^\Lambda \cdot H_s & \text{otherwise.} \end{cases}$$

Here $x_{s\Lambda} = sx_\Lambda$, so the two cases are equivalent upon replacing Λ by $s\Lambda$.

So, we fix Λ and s as above, and assume that $\ell(sx_\Lambda) > \ell(x_\Lambda)$. Let α be the simple root such that $s = s_\alpha$. Then, for any $\mu \in \Lambda$, we have $\langle \mu, \alpha \rangle \geq 0$, so that in view of (4.1.1) we have

$$\ell(s\mathsf{t}(\mu)) = \ell(\mathsf{t}(\mu)) + 1. \tag{5.1.2}$$

We then consider $\lambda \in \mathbf{X}^\vee$, and $\mu \in \Lambda$ such that $\mu - \lambda \in \Lambda$, so that

$$\theta_\lambda^\Lambda = (-1)^{\langle \lambda, 2\rho \rangle} \cdot H_{\mathsf{t}(\mu)} \cdot H_{\mathsf{t}(\mu-\lambda)}^{-1}.$$

We have

$$s\mu \in s\Lambda, \quad s\mu - s\lambda \in s\Lambda, \quad (-1)^{\langle s\lambda, 2\rho \rangle} = (-1)^{\langle \lambda, 2\rho \rangle},$$

so

$$\theta_{s\lambda}^{s\Lambda} = (-1)^{\langle \lambda, 2\rho \rangle} \cdot H_{\mathsf{t}(s\mu)} \cdot H_{\mathsf{t}(s\mu - s\lambda)}^{-1}.$$

Now as observed above we have $x_{s\Lambda} = sx_\Lambda$, and using (5.1.2) we have

$$H_s H_{\mathsf{t}(\mu)} = H_{s\mathsf{t}(\mu)} = H_{\mathsf{t}(s\mu)s} = H_{\mathsf{t}(s\mu)} H_s$$

since $\ell(\mathsf{t}(s\mu)) = \ell(\mathsf{t}(\mu))$, see (4.1.5). Similarly we have

$$H_s H_{\mathsf{t}(\mu-\lambda)} = H_{\mathsf{t}(s\mu - s\lambda)} H_s,$$

from which we obtain that

$$\theta_{s\lambda}^{s\Lambda} = (-1)^{\langle \lambda, 2\rho \rangle} \cdot (H_s H_{\mathsf{t}(\mu)} H_s^{-1}) \cdot (H_s H_{\mathsf{t}(\mu-\lambda)} H_s^{-1})^{-1} = H_s \theta_\lambda^\Lambda H_s^{-1},$$

as desired.

We now prove by induction on $\ell(x_\Lambda)$ that for any W_{f}-orbit $\mathbf{o} \subset \mathbf{X}^\vee$ we have

$$\sum_{\lambda \in \mathbf{o}} \theta_\lambda^\Lambda = \sum_{\lambda \in \mathbf{o}} \theta_\lambda^{\mathbf{X}^\vee_+},$$

which will finish the proof. If $\ell(x_\Lambda) = 0$ then $\Lambda = \mathbf{X}^\vee_+$, and there is nothing to prove. Assume now that $\ell(x_\Lambda) > 0$, and let $s \in S \cap W_{\mathrm{f}}$ be such that $\ell(sx_\Lambda) < \ell(x_\Lambda)$. We have $sx_\Lambda = x_{s\Lambda}$, so that by induction we have

$$\sum_{\lambda \in \mathbf{o}} \theta^{s\Lambda}_\lambda = \sum_{\lambda \in \mathbf{o}} \theta^{\mathbf{X}^\vee_+}_\lambda.$$

We deduce that

$$\sum_{\lambda \in \mathbf{o}} \theta^{\Lambda}_\lambda = H_s \cdot \left(\sum_{\lambda \in \mathbf{o}} \theta^{s\Lambda}_{s\lambda} \right) \cdot H_s^{-1} = H_s \cdot \left(\sum_{\lambda \in \mathbf{o}} \theta^{\mathbf{X}^\vee_+}_\lambda \right) \cdot H_s^{-1} = \sum_{\lambda \in \mathbf{o}} \theta^{\mathbf{X}^\vee_+}_\lambda$$

since $\sum_{\lambda \in \mathbf{o}} \theta^{\mathbf{X}^\vee_+}_\lambda$ commutes with H_s by Lemma 5.1.3. □

5.1.2. The center of the affine Hecke algebra. — We now denote by $\mathcal{B}^\Lambda$ the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathcal{H}$ spanned by the elements $(\theta^\Lambda_\lambda : \lambda \in \mathbf{X}^\vee)$. The following statement, due to Bernstein[(1)] (see e.g. [**Lu1**]), summarizes the main properties of this subalgebra.

Theorem 5.1.5. — 1. *Multiplication induces isomorphisms of $\mathbb{Z}[v, v^{-1}]$-modules*

$$\mathcal{B}^\Lambda \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}_{\mathrm{f}} \xrightarrow{\sim} \mathcal{H} \quad \textit{and} \quad \mathcal{H}_{\mathrm{f}} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{B}^\Lambda \xrightarrow{\sim} \mathcal{H}.$$

2. *The elements $(\theta^\Lambda_\lambda : \lambda \in \mathbf{X}^\vee)$ are linearly independent; therefore the assignment $\lambda \mapsto \theta^\Lambda_\lambda$ extends to a $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism*

$$\theta^\Lambda : \mathbb{Z}[v, v^{-1}][\mathbf{X}^\vee] \xrightarrow{\sim} \mathcal{B}^\Lambda.$$

3. *The isomorphism θ^Λ restricts to an isomorphism*

$$\mathbb{Z}[v, v^{-1}][\mathbf{X}^\vee]^{W_{\mathrm{f}}} \xrightarrow{\sim} Z(\mathcal{H}),$$

where on the left-hand side W_{f} acts on $\mathbb{Z}[v, v^{-1}][\mathbf{X}^\vee]$ via its action on $\mathbf{X}^\vee$, and the right-hand side denotes the center of $\mathcal{H}$.

Consider now the trivial module triv for $\mathcal{H}_{\mathrm{f}}$, i.e. the module equal to $\mathbb{Z}[v, v^{-1}]$, with the action such that H_s acts by multiplication by v^{-1} for $s \in S_{\mathrm{f}}$. The *spherical (left) module* for $\mathcal{H}$ is the induced module

$$\mathcal{M}^{\mathrm{sph}} := \mathcal{H} \otimes_{\mathcal{H}_{\mathrm{f}}} \mathrm{triv}.$$

We have a canonical surjective morphism $\mathcal{H} \to \mathcal{M}^{\mathrm{sph}}$ induced by the action on the element $1 \otimes 1$. Theorem 5.1.5(1) implies that this morphism induces an isomorphism of $\mathcal{B}^\Lambda$-modules $\mathcal{B}^\Lambda \xrightarrow{\sim} \mathcal{M}^{\mathrm{sph}}$, and then Theorem 5.1.5(3) says in particular that the restriction of this map to $Z(\mathcal{H})$ is injective. In other words, a central element in $\mathcal{H}$ is uniquely determined by its image in $\mathcal{M}^{\mathrm{sph}}$.

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(1) As noted above, in [**Lu1**] only the case $\Lambda = \mathbf{X}^\vee_+$ is considered; the general case however follows using Lemma 5.1.4 and its proof.

5.1.3. Relation with Kazhdan–Lusztig combinatorics. — Recall that the *Kazhdan–Lusztig involution* of $\mathcal{H}$ is the unique $\mathbb{Z}$-algebra involution sending v to v^{-1} and H_w to $(H_{w^{-1}})^{-1}$ for any $w \in W$. Then, as proved in [**KL1**] (see also [**So**]), for any $w \in W$ there exists a unique element $\underline{H}_w$ stable under the Kazhdan–Lusztig involution and which belongs to

$$H_w + \sum_{y \in W} v\mathbb{Z}[v]H_y.$$

(It is a standard fact that in fact $\underline{H}_w \in \sum_{y<w} v\mathbb{Z}[v]H_y$; but this property is not needed for the characterization above.) Moreover, the collection $(\underline{H}_w : w \in W)$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}$, called the *Kazhdan–Lusztig basis*.

Let (as in §1.5.1) $w_\circ \in W_{\mathrm{f}}$ be the longest element. Then it is well known that

$$\underline{H}_{w_\circ} = \sum_{w \in W_{\mathrm{f}}} v^{\ell(w_\circ)-\ell(w)} H_w,$$

see e.g. [**So**, Proposition 2.9]. This element satisfies $H_s \cdot \underline{H}_{w_\circ} = v^{-1}\underline{H}_{w_\circ}$ for any $s \in S_{\mathrm{f}}$. Therefore, there exists a (unique) morphism of $\mathcal{H}$-modules

$$\zeta : \mathcal{M}^{\mathrm{sph}} \to \mathcal{H}$$

sending $1 \otimes 1$ to $\underline{H}_{w_\circ}$. It is not difficult to check that this morphism is injective, which allows us to regard $\mathcal{M}^{\mathrm{sph}}$ as a sub-$\mathcal{H}$-module of the regular module $\mathcal{H}$.

Now, consider the Langlands dual complex reductive group $G^\vee_{\mathbb{C}}$. For $\lambda \in \mathbf{X}^\vee_+$, one can consider the simple $G^\vee_{\mathbb{C}}$-module $\mathsf{L}_{\mathbb{C}}(\lambda)$ with highest weight λ. For $\mu \in \mathbf{X}^\vee$, the dimension of the μ-weight space of this representation will be denoted $d_\mu(\lambda)$. (These dimensions have been the subject of intense study; they can be expressed combinatorially, e.g. via the Weyl character formula or via the Kostant formula for weight multiplicities; see [**Hu1**, §24] for details.) The following result is due to Lusztig (see [**Lu1**, Theorem 6.12 and Proposition 8.6]).

Theorem 5.1.6. — *For any $\lambda \in \mathbf{X}^\vee_+$, if n_λ is the longest element in the double coset $W_{\mathrm{f}}\mathsf{t}(\lambda)W_{\mathrm{f}}$ we have*

$$\underline{H}_{n_\lambda} = (-1)^{\ell(n_\lambda)-\ell(w_\circ)} \cdot \underline{H}_{w_\circ} \cdot \left(\sum_{\mu \in \mathbf{X}^\vee} d_\mu(\lambda) \cdot \theta^\Lambda_\mu \right).$$

Example 5.1.7. — Continuing with the notation of Example 5.1.2, we have $n_1 = \omega s_\circ s$ and $n_2 = s s_\circ s$. In these cases the formula of Theorem 5.1.6 becomes

$$\underline{H}_{\omega s_\circ s} = -\underline{H}_s \cdot \big(-H_{\omega s} - H_{s\omega} + (v^{-1} - v)H_\omega\big)$$

and

$$\underline{H}_{s s_\circ s} = \underline{H}_s \cdot \big(H_{s_\circ s} + H_e + H_{s s_\circ} + (v - v^{-1})H_s + (v - v^{-1})H_{s_\circ} + (v^2 + v^{-2} - 2)H_e\big).$$

These formulas can be checked directly using the fact that

$$\underline{H}_{\omega s_\circ s} = H_{\omega s_\circ s} + vH_{\omega s_\circ} + vH_{\omega s} + v^2 H_\omega$$

and

$$\underline{H}_{s s_\circ s} = H_{s s_\circ s} + vH_{s s_\circ} + vH_{s_\circ s} + v^2 H_s + v^2 H_{s_\circ} + v^3 H_e.$$

5.2. Combinatorics of central sheaves

In this section we assume that $\Bbbk$ is a field.

5.2.1. Description of the Grothendieck group of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$. — Recall that the category $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ has a natural monoidal structure, with monoidal product $\star^I$. As a consequence, its Grothendieck group $[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]$ acquires a ring structure. Recall also that all simple perverse sheaves in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ are fixed under Verdier duality; it follows that this functor induces the identity on $[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]$, and hence in particular that

$$[\Delta^I_w(\Bbbk)] = [\nabla^I_w(\Bbbk)] \tag{5.2.1}$$

for any $w \in W$.

The reason why the combinatorics of the affine Hecke algebra can be used to describe some aspects of the study of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ is provided by the following standard result.

Lemma 5.2.1. — *There exists a canonical algebra isomorphism*

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \xrightarrow{\sim} [D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)] \tag{5.2.2}$$

such that the image of $(-1)^{\ell(w)} \cdot (1 \otimes H_w)$ is the class of $\nabla^I_w(\Bbbk)$.

Proof. — Recall that the left-hand side identifies with $\mathbb{Z}[W]$, see (5.1.1). We define a map from $\mathbb{Z}[W]$ to $[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]$ by sending w to $(-1)^{\ell(w)} \cdot [\nabla^I_w(\Bbbk)]$. It is well known (and easy to check) that the objects $(\nabla^I_w(\Bbbk) : w \in W)$ generate $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ as a triangulated category, so that this map is surjective. On the other hand, consider the set of maps

$$\varphi_w : [D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)] \to \mathbb{Z} \qquad \text{given by} \qquad \varphi_w([\mathscr{F}]) = \sum_{n \in \mathbb{Z}} (-1)^n \dim_{\Bbbk} \mathsf{H}^n(\mathrm{Fl}_{G,w}, j_w^! \mathscr{F})$$

(where $j_w : \mathrm{Fl}_{G,w} \hookrightarrow \mathrm{Fl}_G$ is the embedding, as in §4.1.2). These maps satisfy

$$\varphi_w([\nabla^I_x(\Bbbk)]) = (-1)^{\ell(w)} \cdot \delta_{x,w},$$

which implies that the classes $([\nabla^I_w(\Bbbk)] : w \in W)$ are linearly independent. Hence our map is injective, and thus an isomorphism of $\mathbb{Z}$-modules.

It remains to check that our map is an algebra homomorphism, i.e., that

$$(-1)^{\ell(w_1)+\ell(w_2)}[\nabla^I_{w_1}(\Bbbk)][\nabla^I_{w_2}(\Bbbk)] = (-1)^{\ell(w_1 w_2)}[\nabla^I_{w_1 w_2}(\Bbbk)]$$

for any $w_1, w_2 \in W$. By induction on the length of w_2, one can reduce to the cases where $w_2 \in \Omega$ or $w_2 \in S$. If $w_2 \in \Omega$, or if $w_2 \in S$ and $\ell(w_1 w_2) = \ell(w_1) + 1$, then the claim follows from Lemma 4.1.4. The remaining case is that in which $w_2 \in S$ and $\ell(w_1 w_2) = \ell(w_1) - 1$. In this case, by the case treated above we have $(-1)^{\ell(w_1 w_2)+1}[\nabla^I_{w_1 w_2}(\Bbbk)][\nabla^I_{w_2}(\Bbbk)] = (-1)^{\ell(w_1)}[\nabla^I_{w_1}(\Bbbk)]$, so using (5.2.1), we obtain

$$\begin{aligned}(-1)^{\ell(w_1)+\ell(w_2)}[\nabla^I_{w_1}(\Bbbk)][\nabla^I_{w_2}(\Bbbk)] &= (-1)^{\ell(w_1 w_2)}[\nabla^I_{w_1 w_2}(\Bbbk)][\nabla^I_{w_2}(\Bbbk)][\Delta^I_{w_2}(\Bbbk)] \\ &= (-1)^{\ell(w_1 w_2)}[\nabla^I_{w_1 w_2}(\Bbbk)],\end{aligned}$$

where the last equality again follows from Lemma 4.1.4. $\square$

In §5.1.2 we introduced the $\mathcal{H}$-module $\mathcal{M}^{\mathrm{sph}}$. The same arguments as for Lemma 5.2.1 prove the following.

Lemma 5.2.2. — *There exists a unique isomorphism of* $\mathbb{Z}$*-modules*

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{sph}} \xrightarrow{\sim} [D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)] \tag{5.2.3}$$

such that the following diagram commutes (where the left vertical arrow is induced by the canonical map $\mathcal{H} \to \mathcal{M}^{\mathrm{sph}}$*):*

$$\begin{tikzcd} \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \arrow[r, "(5.2.2)"] \arrow[d] & {[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]} \arrow[d, "{[\pi_*]}"] \\ \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{sph}} \arrow[r, "(5.2.3)"] & {[D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)].} \end{tikzcd}$$

Moreover this isomorphism intertwines the action of $\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}$ *on* $\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{sph}}$ *and the action of* $[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]$ *on* $[D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)]$ *by convolution, via the isomorphism* (5.2.2)*.*

The functor

$$\pi^* : D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$$

intertwines the actions of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ on $D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)$ and $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ by convolution on the left. Therefore, the induced map

$$[\pi^*] : [D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)] \to [D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]$$

is a homomorphism of $[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]$-modules. Since this map sends the skyscraper sheaf at the origin of Gr_G to $\underline{\Bbbk}_{\overline{\mathrm{Fl}_{G,w_\circ}}}$, whose class has preimage $1 \otimes \underline{H}_{w_\circ}$ under (5.2.2), it follows that the following diagram commutes, where ζ is as in §5.1.3:

$$\begin{tikzcd} \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \arrow[r, "(5.2.2)"] & {[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)]} \\ \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{sph}} \arrow[r, "(5.2.3)"] \arrow[u, "\mathbb{Z}\otimes\zeta"] & {[D^{\mathrm{b}}_I(\mathrm{Gr}_G, \Bbbk)].} \arrow[u, "{[\pi^*]}"'] \end{tikzcd} \tag{5.2.4}$$

Recall from §4.1.2 that for $w \in W$ we denote by $\mathscr{IC}^I_w$ the intermediate extension of the constant sheaf along j_w, i.e. the image of the unique (up to scalar) nonzero morphism $\Delta^I_w(\Bbbk) \to \nabla^I_w(\Bbbk)$. Below we will also use the following result, due to Kazhdan–Lusztig [**KL2**] (see also [**Sp1**]).

Theorem 5.2.3. — *Assume that* $\mathrm{char}(\Bbbk) = 0$*. Then, for any* $w \in W$*, the isomorphism* (5.2.2) *sends* $(-1)^{\ell(w)} \cdot (1 \otimes \underline{H}_w)$ *to* $[\mathscr{IC}^I_w]$*.*

Remark 5.2.4. — In the notation of [**KL1**], the basis $(\underline{H}_w : w \in W)$ considered above is denoted by $(C'_w : w \in W)$. In [**KL1**] the authors also consider another "canonical" basis, denoted $(C_w : w \in W)$, which differs from the previous one by the replacement of v by v^{-1} (see also [**So**, Theorem 2.7 and Remark 2.8]). The elements C_w can also be related to perverse sheaves via a variant of the isomorphism (5.2.2):

they correspond up to sign to the *tilting* objects in the category of I-constructible perverse sheaves on Fl_G. For more on this description, see [**Yun**].

5.2.2. Classes of central sheaves. — We can now identify the classes of central sheaves in the Grothendieck group.

Proposition 5.2.5. — *For any $\mathscr{F}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, and for any choice of Λ, the preimage under the isomorphism* (5.2.2) *of the class $[\mathsf{Z}(\mathscr{F})]$ is*

$$1 \otimes \left(\sum_{\lambda \in \mathbf{X}^\vee} \dim(\mathcal{S}(\mathscr{F})_\lambda) \cdot \theta^\Lambda_\lambda \right),$$

and this element lies in the center of $\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}$.

Proof. — From the definition and (4.5.3) we see that the isomorphism of Lemma 5.2.1 sends $1 \otimes \theta^\Lambda_\lambda$ to $[\mathbf{J}^\Lambda_\lambda(\Bbbk)]$, for any $\lambda \in \mathbf{X}^\vee$. (Note that here we have $(-1)^{\langle x_\Lambda^{-1}(\lambda), 2\rho \rangle} = (-1)^{\langle \lambda, 2\rho \rangle}$ for any λ.) The formula for $[\mathsf{Z}(\mathscr{F})]$ then follows from Theorem 4.4.5 and Lemma 4.8.1, since $\dim(\mathcal{S}(\mathscr{F})_{w_\circ x_\Lambda^{-1}(\lambda)}) = \dim(\mathcal{S}(\mathscr{F})_\lambda)$ for any $\lambda \in \mathbf{X}^\vee$. The last assertion in the proposition follows from Theorem 5.1.5. □

Remark 5.2.6. — Assume that $\mathscr{F}$ is supported on a connected component of Gr_G. Then all the Wakimoto sheaves $\mathbf{J}^\Lambda_\lambda(\Bbbk)$ appearing in a Wakimoto filtration of $\mathsf{Z}(\mathscr{F})$ are supported on the same connected component of Fl_G. If $\omega \in \Omega$ is the unique element such that $\mathrm{Fl}_{G,\omega}$ is in this component, then in view of Remark 4.2.4 the simple perverse sheaf $\mathscr{IC}^I_\omega$ is a composition factor of $\mathbf{J}^\Lambda_\lambda(\Bbbk)$ with multiplicity 1 for any $\lambda \in \mathbf{X}^\vee$ such that $\mathcal{S}(\mathscr{F})_\lambda \neq 0$. Hence Proposition 5.2.5 implies that

$$[\mathsf{Z}(\mathscr{F}) : \mathscr{IC}^I_\omega] = \dim\bigl(\mathcal{S}(\mathscr{F})\bigr).$$

This provides an extension for general field coefficients of [**GH**, Corollary 1.2].

Since $\pi_* \circ \mathsf{Z} = \mathrm{id}$ (see Lemma 2.5.1), Proposition 5.2.5 also allows us to describe the classes of the objects of $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ in terms of the isomorphism of Lemma 5.2.2.

Corollary 5.2.7. — *For any $\mathscr{F}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$, and for any choice of Λ, the preimage under the isomorphism* (5.2.3) *of the class $[\mathscr{F}]$ is*

$$1 \otimes \left(\sum_{\lambda \in \mathbf{X}^\vee} \dim(\mathcal{S}(\mathscr{F})_\lambda) \cdot \theta^\Lambda_\lambda \otimes 1 \right).$$

Using the commutative diagram (5.2.4), from this corollary we obtain that for any $\mathscr{F}$ in $\mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \Bbbk)$ the preimage under (5.2.2) of the class $[\pi^*\mathscr{F}]$ is

$$1 \otimes \left(\sum_{\lambda \in \mathbf{X}^\vee} \dim(\mathcal{S}(\mathscr{F})_\lambda) \cdot \theta^\Lambda_\lambda \right) \cdot \underline{H}_{w_\circ} = 1 \otimes \underline{H}_{w_\circ} \cdot \left(\sum_{\lambda \in \mathbf{X}^\vee} \dim(\mathcal{S}(\mathscr{F})_\lambda) \cdot \theta^\Lambda_\lambda \right). \tag{5.2.5}$$

Assume now that $\Bbbk = \mathbb{C}$ and that $\mathscr{F}$ is simple, i.e. that $\mathscr{F} = \mathcal{J}_{!*}(\lambda, \mathbb{C})$ for some $\lambda \in \mathbf{X}^\vee_+$. Then $\mathcal{S}(\mathscr{F}) = \mathsf{L}_\mathbb{C}(\lambda)$, and since π is a smooth morphism we have $\pi^*\mathscr{F} \cong \mathscr{IC}^I_{n_\lambda}[-\ell(w_\circ)]$, see [**BBDG**, §§4.2.5–4.2.6] or [**Ac3**, Corollary 3.6.9]. In view of Theorem 5.2.3, this implies that the preimage under (5.2.2) of $[\pi^*\mathscr{F}]$ is $(-1)^{\ell(n_\lambda)-\ell(w_\circ)}$.

$\underline{H}_{n_\lambda}$. Comparing with (5.2.5), this provides a geometric proof of the specialization at $v = 1$ of the formula in Theorem 5.1.6.

5.3. Combinatorics of mixed central sheaves

The formulas considered in §5.2.2 have the drawback of only involving the specializations of $\mathcal{H}$ and $\mathcal{M}^{\mathrm{sph}}$ at $v = 1$. In order to obtain formulas in $\mathcal{H}$ or $\mathcal{M}^{\mathrm{sph}}$ one needs to use mixed $\overline{\mathbb{Q}}_\ell$-sheaves over versions of Gr_G and Fl_G over a finite field. One of our goals in this chapter is to explain these formulas. (These results will be required in a technical proof in Chapter 6.) As preparation, we first explain how to make sense of variants of the results of Chapters 2–3–4 for étale sheaves (for various choices of coefficients). These variants will play an important role in Chapters 6 and 8.

5.3.1. Étale central sheaves. — We consider an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$, and a connected reductive algebraic group G over $\mathbb{F}$, with a choice of Borel subgroup $B \subset G$ and of maximal torus T contained in B. Then one can consider the affine Grassmannian Gr_G and the affine flag variety Fl_G associated with G. All the ind-schemes considered in Chapter 2 have obvious counterparts over $\mathbb{F}$, which will be denoted similarly, and all the properties proved in that chapter remain true in this setting. Thus, if $\Bbbk$ is one of the coefficient rings considered in §9.5.1 (for a fixed prime number $\ell \neq p$), one can consider the constructible derived categories $D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk)$ and $D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$ of étale $\Bbbk$-sheaves, and the "central" functor

$$\mathsf{Z} : D^{\mathrm{b}}_c(\mathrm{Gr}_G, \Bbbk) \to D^{\mathrm{b}}_c(\mathrm{Fl}_G, \Bbbk)$$

defined in terms of nearby cycles (in the sense recalled in Section 9.5) as in the complex setting. The proofs in Chapters 3 and 4 rely only on formal properties of the nearby cycles functor (see Chapter 9) and of the various ind-schemes considered in Chapter 2. Since the étale nearby cycles functor has the same formal properties as its complex counterpart (see §9.5.2 for details), these proofs go through with no change in the étale setting. (The geometric Satake equivalence also holds in this setting, see [**BR**, §1.1.4] for some remarks.)

Remark 5.3.1. — In the étale setting, the statement that the monodromy action is unipotent (see Proposition 2.4.6(1)) should be modified as follows. First, one should assume that the complexes under consideration are perverse sheaves. In this case, using Corollary 9.5.3 we obtain that the monodromy action factors through the morphism t of §9.5.2, and is given by the inverse of the monodromy morphism (9.5.4) of the perverse sheaf $\mathsf{Z}(\mathscr{F})$. Then the same arguments as in the proof of Proposition 2.4.6(1) show that this action further factors through the morphism $t_\ell : I \to \mathbb{Z}_\ell(1)$ of §9.5.2, and is given by unipotent endomorphisms. (Note that the kernel of the morphism $\mathbb{Z}_{(p')}(1) \to \mathbb{Z}_\ell(1)$ is a projective limit of finite groups of order invertible in $\Bbbk$; hence the property that a continuous action of $\mathbb{Z}_{(p')}(1)$ factors through a unipotent action of $\mathbb{Z}_\ell(1)$ is stable under extensions.)

In case $\Bbbk = \overline{\mathbb{Q}}_\ell$, an alternative proof of Proposition 2.4.6(1) was given by Gaitsgory [**G1**], see also [**GH**, Theorem 5.7]. This proof uses the comparison with the affine Hecke algebra and Bernstein's results (see Sections 5.1–5.2) in a crucial way.

5.3.2. Mixed complexes. — From now on we assume that $\mathbb{F}$ is the algebraic closure of a finite field $\mathbb{F}_p$. We choose a prime number ℓ different from p, and restrict to the case when $\Bbbk = \overline{\mathbb{Q}}_\ell$. We also fix once and for all a square root $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$ of the Tate sheaf $\overline{\mathbb{Q}}_\ell(1)$.

We will assume (as we may) that G (resp. B, resp. T) is obtained by base change from a split reductive group scheme $G_\circ$ over $\mathbb{F}_p$ (resp. a Borel subgroup $B_\circ$ of $G_\circ$, resp. a split maximal torus $T_\circ$ contained in $B_\circ$). Then the affine Grassmannian Gr_G and the affine flag variety Fl_G over $\mathbb{F}$ are obtained by base change from ind-schemes $\mathrm{Gr}_{G,\circ}$ and $\mathrm{Fl}_{G,\circ}$ over $\mathbb{F}_p$. We can therefore consider the Iwahori-equivariant derived category

$$D^{\mathrm{mix}}_{I,\circ}(\mathrm{Gr}_{G,\circ}, \overline{\mathbb{Q}}_\ell)$$

of mixed $\overline{\mathbb{Q}}_\ell$-complexes on $\mathrm{Gr}_{G,\circ}$ in the sense of [**BBDG**, §5.1.5], and its counterpart

$$D^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ}, \overline{\mathbb{Q}}_\ell)$$

for $\mathrm{Fl}_{G,\circ}$. The Schubert cells $\mathrm{Fl}_{G,w}$ ($w \in W$) also have counterparts over $\mathbb{F}_p$, denoted $\mathrm{Fl}_{G,\circ,w}$. We will again denote by $j_w : \mathrm{Fl}_{G,\circ,w} \to \mathrm{Fl}_{G,\circ}$ the embedding, and set

$$\nabla^{\mathrm{mix}}_w := (j_w)_* \underline{\overline{\mathbb{Q}}_\ell}_{\mathrm{Fl}_{G,w}}[\ell(w)](\tfrac{\ell(w)}{2}), \quad \Delta^{\mathrm{mix}}_w := (j_w)_! \underline{\overline{\mathbb{Q}}_\ell}_{\mathrm{Fl}_{G,w}}[\ell(w)](\tfrac{\ell(w)}{2}).$$

Then ∇^{mix}_w and Δ^{mix}_w are mixed perverse sheaves, and the image $\mathscr{IC}^{\mathrm{mix}}_w$ of the unique (up to scalar) nonzero morphim $\Delta^{\mathrm{mix}}_w \to \nabla^{\mathrm{mix}}_w$ is simple and pure of weight 0.

Let us denote by R the Grothendieck group of the category of mixed complexes over $\mathrm{Spec}(\mathbb{F}_p)$. Then R has a natural ring structure, induced by tensor product of complexes. Moreover, there exists a natural algebra morphism $\mathbb{Z}[v, v^{-1}] \to R$, where v is sent to the class of $\overline{\mathbb{Q}}_\ell(-\frac{1}{2})$.

The following lemma provides analogues of Lemmas 5.2.1 and 5.2.2. Its proof is similar to that of those statements.

Lemma 5.3.2. — 1. *There exists a canonical algebra isomorphism*

$$R \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \xrightarrow{\sim} [D^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ}, \overline{\mathbb{Q}}_\ell)] \tag{5.3.1}$$

such that the image of $(-1)^{\ell(w)} \cdot (1 \otimes H_w)$ is the class of ∇^{mix}_w.

2. *There exists a unique isomorphism of R-modules*

$$R \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{sph}} \xrightarrow{\sim} [D^{\mathrm{mix}}_{I,\circ}(\mathrm{Gr}_{G,\circ}, \overline{\mathbb{Q}}_\ell)] \tag{5.3.2}$$

such that the following diagram commutes (where the left vertical arrow is induced by the canonical morphism $\mathcal{H} \to \mathcal{M}^{\mathrm{sph}}$):

$$\begin{tikzcd}
R \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \arrow[r, "(5.2.2)"] \arrow[d] & {[D^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ}, \overline{\mathbb{Q}}_\ell)]} \arrow[d, "{[\pi_*]}"] \\
R \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{sph}} \arrow[r, "(5.2.3)"] & {[D^{\mathrm{mix}}_{I,\circ}(\mathrm{Gr}_{G,\circ}, \overline{\mathbb{Q}}_\ell)]}.
\end{tikzcd}$$

Moreover this isomorphism intertwines the action of $R\otimes_{\mathbb{Z}[v,v^{-1}]}\mathcal{H}$ *on* $R\otimes_{\mathbb{Z}[v,v^{-1}]}\mathcal{M}^{\mathrm{sph}}$ *and the action of* $[D^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ},\overline{\mathbb{Q}}_\ell)]$ *on* $[D^{\mathrm{mix}}_{I,\circ}(\mathrm{Gr}_{G,\circ},\overline{\mathbb{Q}}_\ell)]$ *by convolution, via the isomorphism* (5.3.1).

Here the inverse of the isomorphism (5.3.1) sends the class of a complex $\mathscr{F}$ to

$$\sum_{w\in W}[\mathrm{j}_w^!\mathscr{F}(-\tfrac{\ell(w)}{2})]\cdot H_w,$$

where, for any $w\in W$, $\mathrm{j}_w:\mathrm{Spec}(\mathbb{F}_p)\to\mathrm{Fl}_{G,\circ,w}$ is the inclusion of a chosen point.

Remark 5.3.3. — In the mixed setting, it is no longer true that standard and costandard perverse sheaves have the same image in the Grothendieck group. In fact, since $[\Delta^{\mathrm{mix}}_w]$ is inverse to $[\nabla^{\mathrm{mix}}_{w^{-1}}]$ (by a mixed analogue of Lemma 4.1.4), it must be the image of $(-1)^{\ell(w)}\cdot(1\otimes(H_{w^{-1}})^{-1})$ under (5.3.1).

We also have the following analogue of Theorem 5.2.3, which follows from the results of [**KL2**] together with [**BGS**, Corollary 4.4.3].

Theorem 5.3.4. — *For any* $w\in W$*, the preimage of* $[\mathscr{IC}^{\mathrm{mix}}_w]$ *under* (5.3.1) *is* $(-1)^{\ell(w)}\cdot(1\otimes\underline{H}_w)$.

5.3.3. Associated graded of the weight filtration. — Recall from [**BBDG**, Théorème 5.3.5] that any mixed perverse sheaf on $\mathrm{Spec}(\mathbb{F}_p)$ has a canonical weight filtration. Taking the associated graded of the weight filtration, and then extending scalars from $\mathbb{F}_p$ to $\mathbb{F}$, we obtain an exact functor from mixed perverse sheaves on $\mathrm{Spec}(\mathbb{F}_p)$ to graded perverse sheaves on $\mathrm{Spec}(\mathbb{F})$. At the level of Grothendieck groups, this functor gives rise to a ring homomorphism

$$R\to\mathbb{Z}[v,v^{-1}]$$

that is a left inverse to the ring homomorphism $\mathbb{Z}[v,v^{-1}]\to R$ considered above. More concretely, if we denote by $\varkappa$ the functor of base change to $\mathrm{Spec}(\mathbb{F})$, and identifying perverse sheaves on $\mathrm{Spec}(\mathbb{F})$ with finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces, then our morphism sends a perverse sheaf $\mathscr{F}$ to

$$\sum_{i\in\mathbb{Z}}v^i\dim(\varkappa(\mathrm{gr}^{\mathrm{W}}_i(\mathscr{F}))),$$

where $\mathrm{gr}^{\mathrm{W}}_i(\mathscr{F})$ is the i-th part of the associated graded of the weight filtration on $\mathscr{F}$.

In particular, in view of (5.3.1) and (5.3.2) we obtain a canonical surjective algebra homomorphism

$$[D^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ},\overline{\mathbb{Q}}_\ell)]\to\mathcal{H} \tag{5.3.3}$$

and a canonical surjective map

$$[D^{\mathrm{mix}}_{I,\circ}(\mathrm{Gr}_{G,\circ},\overline{\mathbb{Q}}_\ell)]\to\mathcal{M}^{\mathrm{sph}} \tag{5.3.4}$$

which intertwines the actions of $[D^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ}, \overline{\mathbb{Q}}_\ell)]$ and $\mathcal{H}$ via (5.3.3). Concretely, denoting again by $\varkappa$ the functor of base change to $\mathbb{F}$, the morphism (5.3.3) sends the class of a mixed perverse sheaf $\mathscr{F}$ to

$$\sum_{w\in W} (-1)^{\ell(w)} \cdot \left(\sum_{i\in\mathbb{Z}} v^i [\varkappa(\mathrm{gr}^{\mathrm{W}}_i(\mathscr{F})) : \mathscr{IC}^I_w] \cdot \underline{H}_w \right).$$

5.3.4. Mixed central sheaves. — The ind-schemes considered in Chapter 2 also have natural analogues over $\mathbb{F}_p$. Using the nearby cycles functor for étale sheaves in this setting (see in particular §9.5.4) we obtain a functor

$$\mathsf{Z}^{\mathrm{mix}} : \mathsf{Perv}^{\mathrm{mix}}_{\mathrm{L}^+G,\circ}(\mathrm{Gr}_{G,\circ}, \overline{\mathbb{Q}}_\ell) \to \mathsf{Perv}^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ}, \overline{\mathbb{Q}}_\ell),$$

where $\mathsf{Perv}^{\mathrm{mix}}_{\mathrm{L}^+G,\circ}(\mathrm{Gr}_{G,\circ}, \overline{\mathbb{Q}}_\ell)$ is the category of mixed $\mathrm{L}^+(G_\circ)$-equivariant perverse sheaves on $\mathrm{Gr}_{G,\circ}$ and $\mathsf{Perv}^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ}, \overline{\mathbb{Q}}_\ell)$ is the heart of the perverse t-structure on $D^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ}, \overline{\mathbb{Q}}_\ell)$.

For $\lambda \in \mathbf{X}^\vee_+$, we denote by $\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell)$ the canonical lift of the perverse sheaf $\mathcal{J}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell)$ to a pure perverse sheaf of weight 0. The "mixed" version of Proposition 5.2.5 is the following claim.

Proposition 5.3.5. — *For any $\lambda \in \mathbf{X}^\vee_+$, the image of the class of the mixed perverse sheaf $\mathsf{Z}^{\mathrm{mix}}(\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell))$ under* (5.3.3) *is*

$$\sum_{\mu\in\mathbf{X}^\vee} d_\mu(\lambda) \cdot \theta^\Lambda_\mu.$$

Proof. — All the constructions considered in Chapter 3 can be done over $\mathbb{F}_p$, and from this we deduce that the class $[\mathsf{Z}^{\mathrm{mix}}(\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell))]$ belongs to the center of $[D^{\mathrm{mix}}_{I,\circ}(\mathrm{Fl}_{G,\circ}, \overline{\mathbb{Q}}_\ell)]$, and then that its image c_λ under (5.3.3) belongs to the center of $\mathcal{H}$. Now Lemma 2.5.1 also holds over $\mathbb{F}_p$, which implies that the image of c_λ in $\mathcal{M}^{\mathrm{sph}}$ is the image under (5.3.4) of the class of $\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell)$.

If we set $Q := \sum_{w\in W_{\mathrm{f}}} v^{2\ell(w)}$, by the projection formula we have

$$[\pi_*\pi^*\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell)] = Q \cdot [\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell)].$$

On the other hand, by the same arguments as in the non-mixed setting (see §5.2.2) we have $\pi^*\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell)[\ell(w_\circ)](\frac{\ell(w_\circ)}{2}) \cong \mathscr{IC}^{\mathrm{mix}}_{n_\lambda}$ (where n_λ is as in Theorem 5.1.6), so that

$$[\pi_*\pi^*\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell)] = (-v)^{\ell(w_\circ)} \cdot [\pi_*\mathscr{IC}^{\mathrm{mix}}_{n_\lambda}].$$

In view of Theorem 5.3.4, comparing these equalities we obtain that the image of $[\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell)]$ under (5.3.4) is

$$\frac{1}{Q} \cdot (-1)^{\ell(n_\lambda)-\ell(w_\circ)} \cdot v^{\ell(w_\circ)} \cdot (\underline{H}_{n_\lambda} \otimes 1).$$

Using Theorem 5.1.6 and the observation that Q is equal to the image of $v^{\ell(w_\circ)}\underline{H}_{w_\circ}$ under the natural map $\mathcal{H} \to \mathcal{M}^{\mathrm{sph}}$, we rewrite the expression above as

$$\left(\sum_{\mu \in \mathbf{X}^\vee} d_\mu(\lambda) \cdot \theta^\Lambda_\mu \right) \otimes 1.$$

Since a central element in $\mathcal{H}$ is uniquely determined by its image in $\mathcal{M}^{\mathrm{sph}}$ (see §5.1.2), the desired equality follows. □

Proposition 5.3.5 should be seen as a combinatorial recipe for computing the composition factors of each piece of the associated graded of the weight filtration on $\mathsf{Z}^{\mathrm{mix}}(\mathcal{J}^{\mathrm{mix}}_{!*}(\lambda, \overline{\mathbb{Q}}_\ell))$. Namely, in view of the formula for (5.3.3) given at the end of §5.3.3, to determine these data one should consider the element $\sum_{\mu \in \mathbf{X}^\vee} d_\mu(\lambda) \cdot \theta^\Lambda_\mu \in \mathcal{H}$, and decompose it in the Kazhdan–Lusztig basis $(\underline{H}_w : w \in W)$; the required multiplicities are then obtained from the coefficients in this decomposition.

Example 5.3.6. — In the setting of Examples 5.1.2 and 5.1.7, Proposition 5.3.5 says that $[\mathsf{Z}^{\mathrm{mix}}(\mathcal{J}^{\mathrm{mix}}_{!*}(1, \overline{\mathbb{Q}}_\ell))] = -H_{\omega s} - H_{s\omega} + (v^{-1} - v)H_\omega$. Using the fact that

$$\underline{H}_{\omega s} = H_{\omega s} + vH_\omega, \qquad \underline{H}_{s\omega} = H_{s\omega} + vH_\omega, \qquad \underline{H}_\omega = H_\omega,$$

along with Theorem 5.3.4, we find that

$$[\mathsf{Z}^{\mathrm{mix}}(\mathcal{J}^{\mathrm{mix}}_{!*}(1, \overline{\mathbb{Q}}_\ell))] = [\mathscr{IC}^{\mathrm{mix}}_\omega(-\tfrac{1}{2})] + [\mathscr{IC}^{\mathrm{mix}}_{\omega s}] + [\mathscr{IC}^{\mathrm{mix}}_{s\omega}] + [\mathscr{IC}^{\mathrm{mix}}_\omega(\tfrac{1}{2})].$$

The geometric setting in this case is that described in §2.2.4 (in the special case $n = 2$). The computation of nearby cycles in this case is explained in [**Ac3**, Exercise 4.6.8] (or in [**G1**, §1.2.3]), and is consistent with this formula.

PART II

APPLICATION TO THE ARKHIPOV–BEZRUKAVNIKOV EQUIVALENCE

CHAPTER 6

THE CHARACTERISTIC-0 ARKHIPOV–BEZRUKAVNIKOV EQUIVALENCE

The goal of this chapter is to review the construction of an equivalence of categories (due to Arkhipov–Bezrukavnikov [**AB**]) relating the category of Iwahori-equivariant $\overline{\mathbb{Q}}_\ell$-perverse sheaves on Fl_G and the category of $G^\vee$-equivariant coherent sheaves on the Springer resolution of $G^\vee$ (taken with respect to $\Bbbk = \overline{\mathbb{Q}}_\ell$). The construction of this equivalence is based in an essential way on the results from Part I.

This equivalence can be seen as an important advance in the local geometric Langlands program, and is a key step in the proof of Bezrukavnikov's equivalence [**Be5**], which gives a categorical upgrade of an isomorphism between two different geometric realizations of the group ring $\mathbb{Z}[W]$ (see §6.1.1 for more context). It also has important applications in representation theory (e.g. in [**Be3**] and [**BM**]), which will not be reviewed here.

6.1. Overview of the chapter

In this section, we explain the ideas behind the construction of the equivalence, and outline the main steps that will be carried out in this chapter.

6.1.1. Categorifying the affine Weyl group and its antispherical module. — Let $\widetilde{\mathcal{N}}$ denote the Springer resolution for $G^\vee$ (see §6.2.2 for more discussion), and let

$$\mathrm{St} := \widetilde{\mathcal{N}} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}$$

be the Steinberg variety. The group $G^\vee$ acts on both $\widetilde{\mathcal{N}}$ and St, and we can consider their respective derived categories of equivariant coherent sheaves, $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ and $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathrm{St})$. It is a classical fact that the Grothendieck groups of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ and $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathrm{St})$ are described by

$$[D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)] \cong \mathbb{Z}[W] \cong [D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathrm{St})]. \tag{6.1.1}$$

The first isomorphism above goes back to work of Iwahori–Matsumoto [**IM**] (and has been treated above in Lemma 5.2.1); the second is due to Kazhdan–Lusztig [**KL3**] and Ginzburg [**CG**].

There is a similar story involving the *(right) antispherical module* M^{asph}, defined as follows: let $\mathbb{Z}_{\mathrm{sgn}}$ denote the group $\mathbb{Z}$ made into a right $\mathbb{Z}[W_{\mathrm{f}}]$-module by letting $w \in W_{\mathrm{f}}$ act by multiplication by $(-1)^{\ell(w)}$, and then set

$$M^{\mathrm{asph}} := \mathbb{Z}_{\mathrm{sgn}} \otimes_{\mathbb{Z}[W_{\mathrm{f}}]} \mathbb{Z}[W].$$

This module also arises as the Grothendieck group of various geometric categories. Three such incarnations will be relevant to us.

The first involves coherent sheaves on the Springer resolution: we have

$$[D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}})] \cong M^{\mathrm{asph}}. \tag{6.1.2}$$

(This identification is one of the key ingredients of the proof in [**CG**]. See [**BeR1**] for an "upgrade" of this identification at the categorical level.)

The second incarnation involves perverse sheaves on Fl_G. The kernel of the natural map $\mathbb{Z}[W] \to M^{\mathrm{asph}}$ is spanned by Kazhdan–Lusztig basis elements whose label $w \in W$ is not of minimal length in $W_{\mathrm{f}}w$. In view of the first isomorphism in (6.1.1), this suggests considering the Serre quotient

$$\mathsf{P}_I^{\mathrm{asph}} := \mathsf{Perv}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)/(\text{the Serre subcategory generated by } \{\mathscr{IC}_w^I : w \text{ not minimal in } W_{\mathrm{f}}w\}).$$

We then have

$$[\mathsf{P}_I^{\mathrm{asph}}] \cong [D^{\mathrm{b}}\mathsf{P}_I^{\mathrm{asph}}] \cong M^{\mathrm{asph}}. \tag{6.1.3}$$

Finally, the third incarnation originates in the theory of p-adic groups.(1) Let q be a power of p, and let $\mathbb{F}_q$ be a finite field with q elements. Assume (as we may) that our group G, resp. the Borel subgroups B and B^+, is obtained by base change from a split reductive group $G_\circ$ over $\mathbb{F}_q$, resp. Borel subgroups $B_\circ$ and $B_\circ^+$ of $G_\circ$. Taking the preimage of $B_\circ(\mathbb{F}_q)$ (resp. $B_\circ^+(\mathbb{F}_q)$) in $G_\circ(\mathbb{F}_q[[x]])$ provides "Iwahori subgroups" $I_\circ$ and $I_\circ^+$ in $G_\circ(\mathbb{F}_q((x)))$. In terms of these data, the algebra $\mathcal{H}_q := \mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \overline{\mathbb{Q}}_\ell$ (where $\overline{\mathbb{Q}}_\ell$ is viewed as a $\mathbb{Z}[v,v^{-1}]$-algebra with v acting as multiplication by a fixed square root of q^{-1}) can be realized as the convolution algebra of $I_\circ$-biinvariant functions $G_\circ(\mathbb{F}_q((x))) \to \overline{\mathbb{Q}}_\ell$ supported on finitely many double cosets.

Fix a nontrivial homomorphism $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. Denoting by $U_\circ^+$ the unipotent radical of $B_\circ^+$, and choosing a trivialization of each root subgroup of $U_\circ^+$ corresponding to a simple root, we obtain a group homomorphism

$$U_\circ^+(\mathbb{F}_q((x))) \to \prod_{\alpha \text{ simple root}} \mathbb{F}_q((x)).$$

Then, composing with the morphism $(f_\alpha)_\alpha \mapsto \psi(\mathrm{Res}(\sum_\alpha f_\alpha))$ (where the residue is defined by $\mathrm{Res}(\sum_{i\in\mathbb{Z}} a_i x^i) = a_{-1}$) we obtain a homomorphism $\chi : U_\circ^+(\mathbb{F}_q((x))) \to \overline{\mathbb{Q}}_\ell^\times$.

(1) This incarnation, which easily follows from the Iwasawa decomposition, seems to have been well known to experts for a long time, but we were not able to attribute it in a precise way. See [**BBL**] for more details regarding the description of the antispherical module in terms of Whittaker models, a discussion of some references on this subject, and an application.

The $\mathcal{H}_q$-module of functions $f : G_\circ\big(\mathbb{F}_q((x))\big) \to \overline{\mathbb{Q}}_\ell$ satisfying

$$f(ngh) = \chi(n)f(g) \quad \text{for all } n \in U_\circ^+\big(\mathbb{F}_q((x))\big),\ g \in G_\circ\big(\mathbb{F}_q((x))\big) \text{ and } h \in I_\circ$$

(and supported on finitely many cosets) provides a "q-version" M_q^{asph} of M^{asph}. Modules constructed in such a way are called "Whittaker modules."

Translating this construction into the language of perverse sheaves turns out to be difficult, because the $U_\circ^+\big(\mathbb{F}_q((x))\big)$-orbits on $G_\circ\big(\mathbb{F}_q((x))\big)/I_\circ$ are ind-schemes rather rather schemes.(2) But this construction has a "baby" (or "Iwahori–Whittaker") version, as follows. Instead of considering the group $U_\circ^+\big(\mathbb{F}_q((x))\big)$, one considers the pro-unipotent radical $I_{\mathrm{u},\circ}^+$ of $I_\circ^+$. A similar construction as above (involving the surjection $I_{\mathrm{u},\circ}^+ \to U_\circ^+(\mathbb{F}_q)$ and the map $U_\circ^+(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ induced by ψ) provides a group homomorphism $\chi' : I_{\mathrm{u},\circ}^+ \to \overline{\mathbb{Q}}_\ell$. The $\mathcal{H}_q$-module of functions $f : G_\circ\big(\mathbb{F}_q((x))\big) \to \overline{\mathbb{Q}}_\ell$ satisfying

$$f(ngh) = \chi'(n)f(g) \quad \text{for all } n \in I_{\mathrm{u},\circ}^+,\ g \in G_\circ\big(\mathbb{F}_q((x))\big) \text{ and } h \in I_\circ$$

is also isomorphic to M_q^{asph}. Now the $I_{\mathrm{u},\circ}^+$-orbits on $G_\circ\big(\mathbb{F}_q((x))\big)/I_\circ$ have much better geometric properties, and this space of "Iwahori–Whittaker" functions has a geometric counterpart which will be called the category of Iwahori–Whittaker perverse sheaves on Fl_G. This category, denoted by $\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$, satisfies

$$[\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)] \cong [D^{\mathrm{b}}\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)] \cong M^{\mathrm{asph}}. \tag{6.1.4}$$

The main result of [**AB**] (and therefore of this chapter) gives a categorical upgrade of the isomorphisms in (6.1.2), (6.1.3), and (6.1.4): it asserts that we have equivalences of categories

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \cong D^{\mathrm{b}}\mathsf{P}_I^{\mathrm{asph}} \cong D^{\mathrm{b}}\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell).$$

See Theorem 6.6.1 and Corollary 6.6.2 for precise statements. These equivalences are an easier counterpart of (and a preliminary step towards) Bezrukavikov's categorical upgrade of (6.1.1) in [**Be5**].

6.1.2. Outline of the chapter. — We begin in Section 6.2 by trying to reconstruct the derived category $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ in terms of the following objects:

$$\begin{cases} \text{sheaves of the form } V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}} \text{ with } V \in \mathsf{Rep}(G^\vee), \text{ and} \\ \text{line bundles } \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda), \text{ where } \lambda \in \mathbf{X}^\vee. \end{cases} \tag{6.1.5}$$

It is straightforward to show that these objects generate $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ as a triangulated category, but our goal of "reconstructing" the category entails also "describing the relations"—which is much more subtle! The most important statement in this section is Proposition 6.2.10, which describes $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ as a (Verdier) quotient of a triangulated category built in terms of the objects considered above.

(2) A solution to this difficulty has now been found, see e.g. [**Ras1**]; this is however not relevant for our considerations below. See also [**Ras1**, Remark 1.6.2] for a conceptual explanation why Whittaker sheaves can be replaced by Iwahori–Whittaker sheaves in this construction.

Next, Section 6.3 gives the construction of a functor

$$F : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{Perv}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell). \tag{6.1.6}$$

The rough idea is first define F on the generating objects (6.1.5) by

$$\text{sheaves of the form } V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}} \mapsto \text{central sheaves,}$$
$$\text{line bundles} \mapsto \text{Wakimoto sheaves.}$$

To show that this assignment determines a well-defined functor as in (6.1.6), we will use the theory developed in Section 6.2, along with results from previous chapters on central and Wakimoto sheaves.

In Section 6.4 we define the categories $\mathsf{P}_I^{\mathrm{asph}}$ and $\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ that were mentioned above. We also construct an exact functor

$$\mathsf{P}_I^{\mathrm{asph}} \to \mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell), \tag{6.1.7}$$

and prove that this functor is fully faithful.

Next, in Section 6.5 we prove a statement that is vital for the rest of the proofs in this chapter, but which is also interesting in its own right, namely that the images in $\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ of central sheaves are *tilting objects* for the natural structure of highest weight category on $\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. The idea of this proof is this: one proves the claim for central sheaves corresponding to certain "small" orbits in Gr_G, and then checks that it can be propagated to all central sheaves using convolution. The latter fact is not difficult to establish. The "small" orbits one has to consider are those corresponding to minuscule and quasi-minuscule coweights. The claim in the minuscule case is not so difficult to prove (see §6.5.5); the case of quasi-minuscule coweights is however much more delicate, and requires investigating the Serre quotient of P_I by the subcategory generated by simple objects corresponding to all elements in W of positive length.

Finally, in Section 6.6 we conclude by proving that the composition

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \xrightarrow{F} D^{\mathrm{b}}\mathsf{Perv}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}\mathsf{P}_I^{\mathrm{asph}} \xrightarrow{(6.1.7)} D^{\mathrm{b}}\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

is an equivalence of categories. This proof relies in particular on the tilting property of the images of central sheaves explained above. From this one deduces that the functor (6.1.7) is essentially surjective, and hence an equivalence of categories, which completes our program.

The construction presented in this chapter closely follows that of [**AB**], and most of the proofs are taken from that reference. We have however made a few simplifications and clarifications; see in particular Remarks 6.2.11 and 6.4.9, and §6.5.8.

6.1.3. Conventions. — Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. In this chapter, we always assume that G is a connected reductive group over $\mathbb{F}$. We then have the affine Grassmannian Gr_G and the affine flag variety Fl_G defined over $\mathbb{F}$, and given $\ell \neq p$ one can consider the derived categories of étale $\overline{\mathbb{Q}}_\ell$-sheaves on Gr_G and Fl_G. As explained in §5.3.1, all the constructions of the previous chapters have obvious counterparts in this setting. For simplicity, we will use the same notation for these categories or functors as for their classical counterparts. (The reason for working in the étale setting is that the theory of Iwahori–Whittaker perverse sheaves has no convenient counterpart in the classical topology.)

All constructible complexes or perverse sheaves will have coefficients in $\overline{\mathbb{Q}}_\ell$, and so from Section 6.3 on, the symbol "$\Bbbk$" will be almost never be used. Varieties that were labeled with a subscript "$\Bbbk$" in previous chapters will usually have the subscript omitted in this chapter; such varieties should be understood to be over $\overline{\mathbb{Q}}_\ell$. In particular, as we have already done above, we denote various groups by $G^\vee$, $B^\vee$, etc., instead of by $G^\vee_{\overline{\mathbb{Q}}_\ell}$, $B^\vee_{\overline{\mathbb{Q}}_\ell}$, etc.

6.2. Coherent sheaves on the Springer resolution

In this section we denote by $\Bbbk$ an algebraically closed field of characteristic 0. (The results presented below will only be used in the case $\Bbbk = \overline{\mathbb{Q}}_\ell$. However, since they do not require any special property of the field $\overline{\mathbb{Q}}_\ell$, we prefer presenting them in a more general setting.)

6.2.1. The basic affine space and its affine completion. — Before proceding, we note that in this chapter we will consider various categories of equivariant coherent sheaves (for an affine algebraic $\Bbbk$-group H) on some $\Bbbk$-varieties; for this notion we refer e.g. to [**Bri1**, §2] (where these objects are called "linearized sheaves" instead). An important property we will use is that for a $\Bbbk$-algebraic group H acting on a $\Bbbk$-variety X, for any $\mathscr{F},\mathscr{G}$ in $D^{\mathrm{b}}\mathsf{Coh}^H(X)$ the $\Bbbk$-vector space $\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F},\mathscr{G})$ admits a natural structure of a (not necessarily finite-dimensional) algebraic H-module. Moreover, if H is reductive then we have

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^H(X)}(\mathscr{F},\mathscr{G}) = \left(\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F},\mathscr{G})\right)^H. \tag{6.2.1}$$

(This fact uses the property that the functor of H-fixed points is exact on algebraic H-modules. For a study of equivariant quasi-coherent sheaves in a much more general setting, and further references, we refer the reader to [**MR1**, Appendix A].)

Recall the (negative) Borel subgroup $B^\vee \subset G^\vee$; its unipotent radical will be denoted $U^\vee$. The *basic affine space* is the quotient $G^\vee/U^\vee$, with its natural action of $G^\vee$. This variety is also equipped with a (right) action of $T^\vee$, given by

$$t \cdot gU^\vee = gt^{-1}U^\vee.$$

We then obtain an isomorphism of $G^\vee \times T^\vee$-varieties

$$(G^\vee \times T^\vee)/B^\vee \xrightarrow{\sim} G^\vee/U^\vee \tag{6.2.2}$$

where the embedding $B^\vee \subset G^\vee \times T^\vee$ is the product of the embedding $B^\vee \subset G^\vee$ and the projection $B^\vee \to B^\vee/U^\vee \cong T^\vee$.

The $T^\vee$-action on $G^\vee/U^\vee$ induces a $T^\vee$-action on $\mathscr{O}(G^\vee/U^\vee)$, for which the λ-weight space identifies with the representation denoted $\mathsf{N}(-\lambda)$ in §1.5.1; we therefore obtain a canonical isomorphism of $G^\vee \times T^\vee$-modules

$$\mathscr{O}(G^\vee/U^\vee) \cong \bigoplus_{\lambda \in \mathbf{X}^\vee} \mathsf{N}(\lambda) \otimes \Bbbk_{T^\vee}(-\lambda) \tag{6.2.3}$$

(where $\Bbbk_{T^\vee}(\eta)$ denotes the 1-dimensional $T^\vee$-module associated with η, or in other words the restriction of $\Bbbk_{B^\vee}(\eta)$ to $T^\vee$). Here the module $\mathsf{N}(\lambda)$ vanishes unless $\lambda \in \mathbf{X}^\vee_+$, and in this case it has highest weight λ (see [**J1**, Proposition II.2.6]).

In view of (6.2.2) and [**Bri1**, Lemma 2], we have a canonical equivalence of categories

$$\mathsf{Coh}^{G^\vee \times T^\vee}(G^\vee/U^\vee) \cong \mathsf{Rep}(B^\vee), \tag{6.2.4}$$

where the right-hand side denotes the category of finite-dimensional $B^\vee$-modules. Under this equivalence, the $B^\vee$-module $\Bbbk_{B^\vee}(\lambda)$ corresponds to the equivariant coherent sheaf $\mathscr{O}_{G^\vee/U^\vee} \otimes \Bbbk_{T^\vee}(\lambda)$. In more geometric terms, one can consider the flag variety $\mathcal{B} := G^\vee/B^\vee$ of $G^\vee$. Then the pullback under the Zariski locally trivial principal $T^\vee$-bundle $G^\vee/U^\vee \to G^\vee/B^\vee$ induces a canonical equivalence of categories

$$\mathsf{Coh}^{G^\vee}(\mathcal{B}) \xrightarrow{\sim} \mathsf{Coh}^{G^\vee \times T^\vee}(G^\vee/U^\vee)$$

(whose quasi-inverse is the $T^\vee$-invariant pushforward). Under this equivalence, the object $\mathscr{O}_{G^\vee/U^\vee} \otimes \Bbbk_{T^\vee}(\lambda)$ corresponds to the line bundle $\mathscr{O}_{\mathcal{B}}(\lambda)$ naturally associated with λ, for any $\lambda \in \mathbf{X}^\vee$. (In our conventions we have $\mathsf{N}(\lambda) = \Gamma(\mathcal{B}, \mathscr{O}_{\mathcal{B}}(\lambda))$.)

Recall the morphisms $\mathbf{a}_{\lambda,\mu}$ defined in §1.5.2. We will denote similarly the induced morphism of $G^\vee \times T^\vee$-modules

$$\big(\mathsf{N}(\lambda) \otimes \Bbbk_{T^\vee}(-\lambda)\big) \otimes \big(\mathsf{N}(\mu) \otimes \Bbbk_{T^\vee}(-\mu)\big) \to \mathsf{N}(\lambda+\mu) \otimes \Bbbk_{T^\vee}(-\lambda-\mu).$$

Lemma 6.2.1. — *The multiplication morphism*

$$\mathscr{O}(G^\vee/U^\vee) \otimes_\Bbbk \mathscr{O}(G^\vee/U^\vee) \to \mathscr{O}(G^\vee/U^\vee)$$

identifies, via (6.2.3), *with* $\bigoplus_{\lambda,\mu} \mathbf{a}_{\lambda,\mu}$.

Proof. — The multiplication morphism $\mathscr{O}(G^\vee/U^\vee) \otimes_\Bbbk \mathscr{O}(G^\vee/U^\vee) \to \mathscr{O}(G^\vee/U^\vee)$ is $T^\vee$-equivariant, so it must decompose as a direct sum of morphisms of $G^\vee \times T^\vee$-modules

$$\big(\mathsf{N}(\lambda) \otimes \Bbbk_{T^\vee}(-\lambda)\big) \otimes \big(\mathsf{N}(\mu) \otimes \Bbbk_{T^\vee}(-\mu)\big) \to \mathsf{N}(\lambda+\mu) \otimes \Bbbk_{T^\vee}(-\lambda-\mu).$$

By Frobenius reciprocity such a morphism is uniquely determined by its action on $T^\vee$-weight spaces (for the action of $T^\vee \subset G^\vee$) of weight $\lambda+\mu$. Now if $v_\lambda \in \mathsf{N}(\lambda)_\lambda$, resp. $v_\mu \in \mathsf{N}(\mu)_\mu$, is the unique vector of weight λ such that $\mathbf{f}_\lambda(v_\lambda) = 1$, resp. the unique vector of weight μ such that $\mathbf{f}_\mu(v_\mu) = 1$, then v_λ and v_μ can be considered as certain functions on $G^\vee$ whose value at $1 \in G^\vee$ is $1 \in \Bbbk$. Hence the image of $(v_\lambda \otimes 1) \otimes (v_\mu \otimes 1)$ must be a function of weight $\lambda+\mu$ in $\mathsf{N}(\lambda+\mu)$ whose value at 1 is 1, i.e. it must equal $\mathbf{a}_{\lambda,\mu}(v_\lambda \otimes v_\mu) \otimes 1$. □

In particular, it follows from Lemma 6.2.1 (and the surjectivity of the maps $\mathbf{a}_{\lambda,\mu}$, see Remark 1.5.6(2)) that the $\Bbbk$-algebra $\mathscr{O}(G^\vee/U^\vee)$ is finitely generated (in fact, generated by the weight spaces corresponding to any family of generators of the monoid $\mathbf{X}^+$) so that we can consider the "affine completion"

$$\mathcal{X} := \mathrm{Spec}(\mathscr{O}(G^\vee/U^\vee)),$$

an affine $\Bbbk$-variety endowed with a natural action of $G^\vee \times T^\vee$.

Evaluation at the "base point" $U^\vee \in G^\vee/U^\vee$ defines an algebra homomorphism $\mathscr{O}(G^\vee/U^\vee) \to \Bbbk$, and hence a point in $\mathcal{X}$. Since $G^\vee/U^\vee$ is quasi-affine (see e.g. [**Sp2**, Exercise 5.5.9(2)]), the stabilizer of this point in $G^\vee$ is $U^\vee$; we deduce a canonical map

$$G^\vee/U^\vee \to \mathcal{X}. \tag{6.2.5}$$

Since the induced morphism $\mathscr{O}(\mathcal{X}) \to \mathscr{O}(G^\vee/U^\vee)$ is injective (in fact, an isomorphism), this morphism is dominant. And since $G^\vee/U^\vee$ is open in its closure (as an orbit of an action of $G^\vee$), we deduce that (6.2.5) is an open embedding. The complement of its image (a closed subset of $\mathcal{X}$) will be denoted by $\partial\mathcal{X}$.

The proof of the following claim was explained to one of us by R. Bezrukavnikov.

Lemma 6.2.2. — 1. *For any $x \in \partial\mathcal{X}$, there exists a simple root α of $G^\vee$ such that x is fixed by the action of the cocharacter $\alpha^\vee$ (via the action of $T^\vee = \{1\} \times T^\vee \subset G^\vee \times T^\vee$).*
2. *If $\lambda \in \mathbf{X}_+^\vee$ is regular dominant (i.e. if $\langle \lambda, \alpha\rangle > 0$ for any simple root α of G), any function in $\mathsf{N}(\lambda) \otimes \Bbbk_{T^\vee}(-\lambda) \subset \mathcal{O}(\mathcal{X})$ vanishes on $\partial\mathcal{X}$.*

Proof. — (1) Choose a finite central isogeny $p : H \to G^\vee$ where H is a product of a semisimple, simply-connected algebraic groups and a torus. Then if $B_H = p^{-1}(B^\vee)$, B_H is a Borel subgroup of H, whose unipotent radical will be denoted U_H. We have a natural map $H/U_H \to G^\vee/U^\vee$ which is a principal bundle for the finite abelian group $Z = \ker(p)$. This morphism induces an embedding $\mathscr{O}(G^\vee/U^\vee) \hookrightarrow \mathscr{O}(H/U_H)$ which identifies $\mathscr{O}(G^\vee/U^\vee)$ with the subspace of Z-fixed points in $\mathscr{O}(H/U_H)$. We deduce a morphism $\mathrm{Spec}(\mathscr{O}(H/U_H)) \to \mathcal{X}$ which identifies $\mathcal{X}$ with the quotient of the affine variety $\mathrm{Spec}(\mathscr{O}(H/U_H))$ by the action of the finite group Z. In particular this morphism is surjective, and sends H/U_H into $G^\vee/U^\vee$. This construction reduces the proof of the claim for $G^\vee$ to the proof of the analogous claim for H. Of course the torus factor does not play any role in this story, so we can assume that $G^\vee$ is semisimple and simply-connected.

In this case, let V be the $B^\vee$-module which is the direct sum of the 1-dimensional modules associated with the negatives of the fundamental weights of $G^\vee$, and set $\mathcal{Y} := G^\vee \times^{B^\vee} V$. We will consider V as a $B^\vee \times T^\vee$-module, where $T^\vee$ acts via the composition of the restriction of the action of $B^\vee$ to $T^\vee$ with the map $t \mapsto t^{-1}$; in this way $\mathcal{Y}$ becomes a $G^\vee \times T^\vee$-variety. The action on the vector $(1, \cdots, 1)$ defines a $B^\vee \times T^\vee$-equivariant open embedding $T^\vee = B^\vee/U^\vee \hookrightarrow V$, which then provides a $G^\vee \times T^\vee$-equivariant open embedding $G^\vee/U^\vee = G^\vee \times^{B^\vee} B^\vee/U^\vee \hookrightarrow \mathcal{Y}$. Moreover we have a canonical isomorphism

$$\mathcal{O}(\mathcal{Y}) = \mathrm{Ind}_{B^\vee}^{G^\vee}(\mathcal{O}(V)) = \mathrm{Ind}_{B^\vee}^{G^\vee}\left(\bigoplus_{\lambda \in \mathbf{X}_+^\vee} \Bbbk_\lambda\right) = \mathcal{O}(\mathcal{X}).$$

Hence the open embedding $G^\vee/U^\vee \hookrightarrow \mathcal{X}$ factors through a $G^\vee \times T^\vee$-equivariant morphism $f : \mathcal{Y} \to \mathcal{X}$.

We claim that f is surjective. In fact, since this morphism restricts to an isomorphism on $G^\vee/U^\vee$ (and hence, in particular, is birational), it suffices to prove that it is proper. This morphism can be written as a composition

$$\mathcal{Y} \to \mathcal{X} \times \mathcal{B} \to \mathcal{X}, \tag{6.2.6}$$

where the first morphism is the product of f with the canonical projection $\mathcal{Y} \to \mathcal{B}$, and the second one is the projection on the first factor. This second morphism is clearly proper, and the first one can be written, via the canonical isomorphism $G^\vee \times^{B^\vee} \mathcal{X} \xrightarrow{\sim} \mathcal{X} \times \mathcal{B}$, as $G^\vee \times^{B^\vee} f_{|V}$. Now, at the level of coordinate rings, the restriction of f to V corresponds to the surjection

$$\bigoplus_{\lambda \in \mathbf{X}^\vee_+} \mathbf{f}_\lambda : \mathcal{O}(\mathcal{X}) \to \bigoplus_{\lambda \in \mathbf{X}^\vee_+} \Bbbk_\lambda = \mathcal{O}(V).$$

Thus, $f_{|V}$ is a closed embedding, and hence so is the first map in (6.2.6). This completes the proof that f is proper.

Now we can conclude: any point in $\partial\mathcal{X}$ is the image of a point $[g, v] \in G^\vee \times^{B^\vee} V$ where at least one of the coordinates of v vanishes. If α is a simple root corresponding to one of these vanishing coordinates, then $[g, v]$ is stable under the action of $\alpha^\vee$, and hence so is our initial point.

(2) If $\lambda \in \mathbf{X}^\vee$ is regular dominant and α is a simple root of G (i.e. a simple coroot of $G^\vee$), any function $g \in \mathsf{N}(\lambda) \otimes \Bbbk_{T^\vee}(-\lambda)$ satisfies $g(\alpha(z) \cdot x) = z^{-\langle \lambda, \alpha \rangle} x$, where $-\langle \lambda, \alpha \rangle \neq 0$. Thus g vanishes on any point fixed by α, and in particular (in view of (1)) on all of $\partial\mathcal{X}$. □

Remark 6.2.3. — Let $\mathcal{O}(\mathcal{X})^+ \subset \mathcal{O}(\mathcal{X})$ be the ideal consisting of the direct sum of the subspaces $\mathsf{N}(\lambda) \otimes \Bbbk_{T^\vee}(-\lambda)$ where λ runs over the regular dominant coweights. Lemma 6.2.2(2) implies that the closed subset $\partial\mathcal{X} \subset \mathcal{X}$ consists of the vanishing locus of $\mathcal{O}(\mathcal{X})^+$. In fact this statement shows that $\partial\mathcal{X}$ is contained in this vanishing locus. Since this locus is stable under the action of $G^\vee$ it cannot intersect the open orbit $G^\vee/U^\vee$, which implies our claim. This claim can be used to define a scheme structure on $\partial\mathcal{X}$.

6.2.2. The Springer resolution and some variants. — We will be interested in the "Springer resolution"

$$\widetilde{\mathcal{N}} := G^\vee \times^{B^\vee} \mathfrak{n}^\vee,$$

where $\mathfrak{n}^\vee$ is the Lie algebra of $U^\vee$. This variety is (in a natural way) a vector bundle over $\mathcal{B}$; indeed, the morphism $[g, x] \mapsto (g \cdot x, gB^\vee)$ defines an embedding $\widetilde{\mathcal{N}} \hookrightarrow \mathfrak{g}^\vee \times \mathcal{B}$ as a sub-vector bundle. (Here $\mathfrak{g}^\vee$ is the Lie algebra of $G^\vee$, with the natural adjoint action of $G^\vee$.) For $\lambda \in \mathbf{X}^\vee$, the pullback of $\mathscr{O}_{\mathcal{B}}(\lambda)$ to $\widetilde{\mathcal{N}}$ will be denoted $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$. By [**Bri1**, Lemma 2] we have a canonical equivalence of categories

$$\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} \mathsf{Coh}^{B^\vee}(\mathfrak{n}^\vee);$$

under this equivalence the line bundle $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ corresponds to $\mathscr{O}_{\mathfrak{n}^\vee} \otimes \Bbbk_{B^\vee}(\lambda)$.

We will denote by $\widehat{\mathcal{N}}$ the preimage of $\widetilde{\mathcal{N}}$ under the map $\mathfrak{g}^\vee \times G^\vee/U^\vee \to \mathfrak{g}^\vee \times \mathcal{B}$ induced by the projection $G^\vee/U^\vee \to \mathcal{B}$; then $\widehat{\mathcal{N}}$ is a vector bundle over $G^\vee/U^\vee$, and identifies with the induced variety $G^\vee \times^{U^\vee} \mathfrak{n}^\vee$. We will denote by $e_{\widehat{\mathcal{N}}}$ the "base point" of $\widehat{\mathcal{N}}$, i.e. the point $[1 : 0] \in G^\vee \times^{U^\vee} \mathfrak{n}^\vee$.

We have a natural free action of $T^\vee$ on $\widehat{\mathcal{N}}$ (induced by the action on $G^\vee/U^\vee$) such that $\widehat{\mathcal{N}}/T^\vee = \widetilde{\mathcal{N}}$; as for $G^\vee/U^\vee$ in §6.2.1 we deduce an equivalence of categories

$$\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} \mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}). \tag{6.2.7}$$

Note that for any $\lambda \in \mathbf{X}^\vee$ we have

$$\mathscr{O}(\widehat{\mathcal{N}})_{-\lambda} = \Gamma(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)), \tag{6.2.8}$$

where the left-hand side means the $(-\lambda)$-weight space (for the action of $T^\vee = \{1\} \times T^\vee$). In other words, under the equivalence (6.2.7) the object $\mathscr{O}_{\widehat{\mathcal{N}}} \otimes \Bbbk_{T^\vee}(\lambda)$ is sent to $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$, for any $\lambda \in \mathbf{X}^\vee$.

The variety $\widehat{\mathcal{N}}$ can also be described as a special case of the following construction. Consider an algebraic $\Bbbk$-group H, and a smooth $\Bbbk$-variety X endowed with an action of H. Then differentiating the H-action morphism we obtain a canonical morphism of Lie algebras $\mathfrak{h} \to \Gamma(X, \mathscr{T}_X)$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathscr{T}_X$ is the tangent bundle of X. Any vector field on X defines a derivation of the sheaf of algebras $\mathscr{O}_X$; therefore this morphism provides an action of $\mathfrak{h}$ on $\mathscr{O}_X$ by derivations. We then consider the derivation of the sheaf of rings $\mathscr{O}(\mathfrak{h}) \otimes_\Bbbk \mathscr{O}_X$ defined as the composition

$$\mathscr{O}(\mathfrak{h}) \otimes_\Bbbk \mathscr{O}_X \to \mathscr{O}(\mathfrak{h}) \otimes_\Bbbk \mathfrak{h}^* \otimes_\Bbbk \mathfrak{h} \otimes_\Bbbk \mathscr{O}_X \to \mathscr{O}(\mathfrak{h}) \otimes_\Bbbk \mathscr{O}_X, \tag{6.2.9}$$

where the first map sends $f \otimes g$ to $f \otimes \mathrm{id}_\mathfrak{h} \otimes g$ (here $\mathrm{id}_\mathfrak{h}$ is seen as a vector in $\mathrm{End}(\mathfrak{h}) \cong \mathfrak{h}^* \otimes \mathfrak{h}$) and the second one sends $f \otimes \xi \otimes x \otimes g$ to $(f\xi) \otimes (x \cdot g)$. We denote by $\mathscr{J}_{H,X} \subset \mathscr{O}_{\mathfrak{h} \times X}$ the sheaf of ideals whose pushforward under the (affine) projection $\mathfrak{h} \times X \to X$ is the sheaf of ideals generated by the image of (6.2.9). We can then define the *infinitesimal universal stabilizer* for this action as the closed subscheme of $\mathfrak{h} \times X$ defined by the ideal $\mathscr{J}_{H,X}$. The same procedure can also be performed in case X is affine but not necessarily smooth: in this case we consider the $\mathfrak{h}$-action on $\mathscr{O}(X)$ obtained by differentiating the H-action induced by the action on X.

It can easily be seen from the definition that the infinitesimal universal stabilizer for the $G^\vee$-action on $G^\vee/U^\vee$ is the subvariety $\widehat{\mathcal{N}} \subset \mathfrak{g}^\vee \times G^\vee/U^\vee$. We define $\widehat{\mathcal{N}}_\mathcal{X}$ as the infinitesimal universal stabilizer for the $G^\vee$-action on $\mathcal{X}$. Then $\widehat{\mathcal{N}}_\mathcal{X}$ is a closed subscheme of $\mathfrak{g}^\vee \times \mathcal{X}$, and we have

$$\widehat{\mathcal{N}}_\mathcal{X} \cap (\mathfrak{g}^\vee \times G^\vee/U^\vee) = \widehat{\mathcal{N}}; \tag{6.2.10}$$

in particular, $\widehat{\mathcal{N}}$ is an open (but not dense) subvariety in $\widehat{\mathcal{N}}_\mathcal{X}$. Concretely, we have

$$\mathscr{O}(\mathfrak{g}^\vee \times \mathcal{X}) = \bigoplus_{\lambda \in \mathbf{X}^\vee} \mathscr{O}(\mathfrak{g}^\vee) \otimes \mathsf{N}(\lambda),$$

see (6.2.3). For $\lambda \in \mathbf{X}^\vee_+$, differentiating the $G^\vee$-action on $\mathsf{N}(\lambda)$ we obtain an action of $\mathfrak{g}^\vee$. Then the ideal of definition of $\widehat{\mathcal{N}}_\mathcal{X}$ is generated by the images of the compositions

$$\mathsf{N}(\lambda) \xrightarrow{v \mapsto 1 \otimes v} \mathscr{O}(\mathfrak{g}^\vee) \otimes \mathsf{N}(\lambda) \to \mathscr{O}(\mathfrak{g}^\vee) \otimes \mathsf{N}(\lambda) \hookrightarrow \mathscr{O}(\mathfrak{g}^\vee \times \mathcal{X}) \tag{6.2.11}$$

(where the second map is defined similarly to (6.2.9)) for all $\lambda \in \mathbf{X}^\vee_+$.

The proof of the following lemma is a corrected version of an earlier wrong proof. This error was pointed to us by J. Lourenço, who also provided the main ingredients to correct it.

Lemma 6.2.4. — *There exists $N \in \mathbb{Z}_{\geq 0}$ such that, for any $\lambda \in \mathbf{X}^\vee$ such that $\langle \lambda, \alpha^\vee \rangle \geq N$ for any simple root α, the morphism*

$$\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})_{-\lambda} \to \mathscr{O}(\widehat{\mathcal{N}})_{-\lambda}$$

induced by restriction is an isomorphism.

Proof. — We first establish surjectivity of the morphism under consideration, for any $\lambda \in \mathbf{X}^\vee$ which satisfies the following condition: for any distinct negative roots $\alpha_1, \dots, \alpha_r$ (with $r \geq 0$), the weight $\lambda + \sum_i \alpha_i$ is dominant. For that it suffices to prove that, for such λ, the morphism

$$\mathscr{O}(\mathfrak{g}^\vee \times \mathcal{X})_{-\lambda} \to \mathscr{O}(\widehat{\mathcal{N}})_{-\lambda} \tag{6.2.12}$$

induced by restriction is surjective.

Consider the Koszul resolution $\wedge^\bullet((\mathfrak{g}^\vee/\mathfrak{n}^\vee)^*) \otimes \mathscr{O}(\mathfrak{g}^\vee)$ for $\mathscr{O}(\mathfrak{n}^\vee)$ as a module over $\mathscr{O}(\mathfrak{g}^\vee)$. Tensoring with $\Bbbk_{B^\vee}(\lambda)$ and using the natural equivalence

$$\mathsf{Coh}^{B^\vee}(\mathfrak{g}^\vee) \cong \mathsf{Coh}^{G^\vee}(\mathfrak{g}^\vee \times \mathcal{B})$$

(which is again an application of [**Bri1**, Lemma 2]) we obtain a complex $(\mathscr{F}^\bullet, d^\bullet)$ of coherent sheaves on $\mathfrak{g}^\vee \times \mathcal{B}$, concentrated in degrees between $-\dim(\mathfrak{g}^\vee/\mathfrak{n}^\vee)$ and 0, and a quasi-isomorphism $f^\bullet : \mathscr{F}^\bullet \to \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$, such that $\mathscr{F}^0 = \mathscr{O}_{\mathfrak{g}^\vee \times \mathcal{B}}(\lambda)$ and each $\mathscr{F}^i$ admits a filtration with subquotients of the form $\mathscr{O}_{\mathfrak{g}^\vee \times \mathcal{B}}(\lambda + \mu)$ where μ is a sum of distinct negative roots. (Here, for $\nu \in \mathbf{X}^\vee$ we denote by $\mathscr{O}_{\mathfrak{g}^\vee \times \mathcal{B}}(\nu)$ the pullback of $\mathscr{O}_{\mathcal{B}}(\nu)$ to $\mathfrak{g}^\vee \times \mathcal{B}$, and still write $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ for the pushforward of the sheaf denoted in this way to $\mathfrak{g}^\vee \times \mathcal{B}$.)

Our assumption ensures that $\mathsf{H}^{>0}(\mathfrak{g}^\vee \times \mathcal{B}, \mathscr{F}^i) = 0$ for any i by Kempf's vanishing theorem (see [**J1**, Proposition II.4.5]). Starting from $\ker(d^{-\dim(\mathfrak{g}^\vee/\mathfrak{n}^\vee)}) = 0$ and using the short exact sequences $\ker(d^i) \hookrightarrow \mathscr{F}^i \twoheadrightarrow \ker(d^{i+1})$ (valid for $i \leq -2$), we deduce that $\mathsf{H}^{>0}(\mathfrak{g}^\vee \times \mathcal{B}, \ker(d^i)) = 0$ for $i \leq -1$. Next, using the exact sequence $\ker(d^{-1}) \hookrightarrow \mathscr{F}^0 \twoheadrightarrow \ker(f^0)$, we find that $\mathsf{H}^{>0}(\mathfrak{g}^\vee \times \mathcal{B}, \ker(f^0)) = 0$. Finally, from the exact sequence $\ker(f^0) \hookrightarrow \mathscr{F}^0 \twoheadrightarrow \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ we obtain a short exact sequence

$$\Gamma(\mathfrak{g}^\vee \times \mathcal{B}, \ker(f^0)) \hookrightarrow \Gamma(\mathfrak{g}^\vee \times \mathcal{B}, \mathscr{F}^0) \twoheadrightarrow \Gamma(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)).$$

Now we have $\Gamma(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) \cong \mathscr{O}(\widehat{\mathcal{N}})_{-\lambda}$ (see (6.2.8)) and

$$\Gamma(\mathfrak{g}^\vee \times \mathcal{B}, \mathscr{F}^0) = \mathscr{O}(\mathfrak{g}^\vee) \otimes \Gamma(\mathcal{B}, \mathscr{O}_{\mathcal{B}}(\lambda)) = \mathscr{O}(\mathfrak{g}^\vee \times \mathcal{X})_{-\lambda}$$

(see §6.2.1), and the morphism in our exact sequence identifies with the morphism (6.2.12), so that our claim is proved.

Now we prove that there exists $N \in \mathbb{Z}_{\geq 0}$ such that our map is injective for any λ which satisfies $\langle \lambda, \alpha \rangle \geq N$ for any simple root α. As in Lemma 6.2.2(1) we can assume that $G^\vee$ is semisimple and simply connected. In this case, the condition that $\langle \lambda, \alpha \rangle \geq N$ for any simple root α is equivalent to the condition that λ is a sum of N regular dominant weights.

Denote by $\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})^+ \subset \mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})$ the ideal consisting of the direct sum of the subspaces $\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})_{-\lambda}$ where λ runs over the regular dominant coweights. By Remark 6.2.3, the underlying closed subspace of the closed subscheme of $\widehat{\mathcal{N}}_{\mathcal{X}}$ defined by $\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})^+$ is the complement of $\widehat{\mathcal{N}}$. Consider now the restriction morphism

$$\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to \mathscr{O}(\widehat{\mathcal{N}}).$$

Its kernel K is an ideal of the noetherian algebra $\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})$, and hence is generated by finitely many homogeneous elements $(m_i : i \in I)$, of degrees (for the action of $T^\vee$) denoted $(\mu_i : i \in I)$. Each of these elements is annihilated by a power of $\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})^+$; we can therefore choose $M \in \mathbb{Z}_{\geq 0}$ such that each m_i is annihilated by $(\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})^+)^M$.

With this notation, we claim that injectivity holds for any λ which satisfies

$$\langle \lambda, \alpha \rangle \geq -\langle \mu_i, \alpha \rangle + M$$

for any i and any α. In fact, to check this it suffices to check that K has no nonzero element of degree $-\lambda$ for such a λ. Now if $m \in K$ has degree $-\lambda$, we can write

$$m = \sum_i a_i m_i$$

where $a_i \in \mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})$ has degree $-\lambda - \mu_i$ for any i. For any $i \in I$, by assumption we have $\langle \lambda + \mu_i, \alpha \rangle \geq M$ for any simple root α, hence $\lambda + \mu_i$ can be written as a sum of M regular dominant weights. Using Remark 1.5.6(2), we deduce that a_i belongs to $(\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})^+)^M$. Hence $a_i m_i = 0$, which implies that $m = 0$ and finishes the proof. □

Remark 6.2.5. — The considerations in the first step of the proof of Lemma 6.2.4 show that, if $\lambda \in \mathbf{X}^\vee$ is such that $\lambda + \mu$ is dominant for any sum μ of distinct negative roots, we have $\mathsf{H}^{>0}(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) = 0$. For a more general (and much more subtle) study of this question, the reader might consult [**Bro**].

6.2.3. Koszul complexes. — Let us consider a $\Bbbk$-vector space V of dimension d and a surjective morphism $V \twoheadrightarrow V'$ with $V' \neq 0$. We then have an associated "Koszul complex," defined as the graded symmetric algebra of the complex of vector spaces

$$\cdots \to 0 \to V \to V' \to 0 \to \cdots$$

where V is in degree -1 and V' in degree 0. We "extract" from this complex the summand

$$\begin{aligned}(6.2.13) \quad \cdots \to 0 \to \wedge^d(V) \otimes \mathrm{S}^0(V') \to \wedge^{d-1}(V) \otimes \mathrm{S}^1(V') \to \cdots \\ \to \wedge^1(V) \otimes \mathrm{S}^{d-1}(V') \to \wedge^0(V) \otimes \mathrm{S}^d(V') \to 0 \to \cdots\end{aligned}$$

(Here, the nonzero terms are in degrees between $-d$ and 0.)

Lemma 6.2.6. — *The complex* (6.2.13) *is acyclic.*

Proof. — The "usual" Koszul complex, corresponding to the case $V' = V$, is well known to be a free resolution of the trivial $\mathrm{S}(V)$-module $\Bbbk$. The Koszul complex considered above is obtained by tensoring this complex with $\mathrm{S}(V')$ over $\mathrm{S}(V)$; its image in the derived category of vector spaces is therefore

$$\Bbbk \overset{L}{\otimes}_{\mathrm{S}(V)} \mathrm{S}(V') \cong \wedge^\bullet(V''),$$

where $V'' = \ker(V \to V')$. If we consider our Koszul complex as a complex of graded vector spaces, where V and V' are in internal degree 1, then (6.2.13) is the part of this complex of internal degree d. Since the cohomology of this complex is in internal degrees $< d$ (because $\dim(V'') < d$), the latter complex must then be acyclic. $\square$

Now let $\lambda \in \mathbf{X}_+^\vee$, and consider the $G^\vee$-module $\mathsf{N}(\lambda)$ as a $(G^\vee \times T^\vee)$-module with trivial action of $T^\vee$. We then have an associated $G^\vee \times T^\vee$-equivariant $\mathscr{O}(\mathcal{X})$-module $\mathsf{N}(\lambda) \otimes_{\Bbbk} \mathscr{O}(\mathcal{X})$. In view of (6.2.2), multiplication in $\mathscr{O}(\mathcal{X})$ induces a canonical morphism

$$\mathsf{N}(\lambda) \otimes_{\Bbbk} \mathscr{O}(\mathcal{X}) \to \Bbbk_{T^\vee}(\lambda) \otimes_{\Bbbk} \mathscr{O}(\mathcal{X}), \tag{6.2.14}$$

where in the right-hand side $\Bbbk_{T^\vee}(\lambda)$ is viewed as a $G^\vee \times T^\vee$-module with trivial action of $G^\vee$. More precisely, this morphism is obtained by taking global sections of the morphism of $G^\vee \times T^\vee$-equivariant coherent sheaves $\mathsf{N}(\lambda) \otimes \mathscr{O}_{G^\vee/U^\vee} \to \Bbbk_{T^\vee}(\lambda) \otimes \mathscr{O}_{G^\vee/U^\vee}$ which is the image of the morphism $\mathbf{f}_\lambda$ from §1.5.1 under the equivalence (6.2.4).

Applying the construction considered above for the morphism $\mathbf{f}_\lambda$, then taking the image of the complex so obtained under the equivalence (6.2.4), and finally taking global sections, we obtain a "Koszul complex"

$$\begin{aligned} \cdots \to 0 \to \wedge^{d_\lambda}(\mathsf{N}(\lambda)) \otimes \mathscr{O}(\mathcal{X}) \to \wedge^{d_\lambda - 1}(\mathsf{N}(\lambda)) \otimes \Bbbk_{T^\vee}(\lambda) \otimes \mathscr{O}(\mathcal{X}) \to \cdots \\ \to \wedge^1(\mathsf{N}(\lambda)) \otimes \Bbbk_{T^\vee}((d_\lambda - 1)\cdot\lambda) \otimes \mathscr{O}(\mathcal{X}) \to \Bbbk_{T^\vee}(d_\lambda \cdot \lambda) \otimes \mathscr{O}(\mathcal{X}) \to 0 \to \cdots \end{aligned} \tag{6.2.15}$$

where $d_\lambda = \dim(\mathsf{N}(\lambda))$. (This complex can also be described more directly using an extension of the construction above for morphisms of free modules over a ring, applied to (6.2.14).) This complex can be regarded as a complex of $G^\vee \times T^\vee$-equivariant coherent sheaves on $\mathcal{X}$; the pullback under the projection morphism $\widehat{\mathcal{N}}_{\mathcal{X}} \to \mathcal{X}$ of its tensor product with $\Bbbk_{T^\vee}(-d_\lambda \cdot \lambda)$ will be denoted $\mathscr{K}_\lambda$.

Lemma 6.2.7. — *For any $\lambda \in \mathbf{X}_+^\vee$, the restriction of the complex $\mathscr{K}_\lambda$ to $\widehat{\mathcal{N}}$ is acyclic.*

Proof. — By construction, the restriction of $\mathscr{K}_\lambda$ to $\widehat{\mathcal{N}}$ is the pullback of the restriction of (6.2.15) to $G^\vee/U^\vee$. The latter restriction is acyclic by construction and Lemma 6.2.6, which implies the lemma since the morphism $\widehat{\mathcal{N}} \to G^\vee/U^\vee$ is smooth, and hence flat. $\square$

6.2.4. Equivariant coherent sheaves on $\widetilde{\mathcal{N}}$. — The following claims are well known.

Lemma 6.2.8. — *The category $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ is generated (as a triangulated category) by the following classes of objects:*

1. *the line bundles $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$, for $\lambda \in \mathbf{X}^\vee$;*
2. *the objects of the form $V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ where $V \in \mathsf{Rep}(G^\vee)$ and $\lambda \in \mathbf{X}_+^\vee$.*

Proof. — (1) The proof of this property is adapted from that of its "graded" analogue in [**Ac1**, Lemma 5.7]. Namely, by [**Bri1**, Lemma 2] again, we have an equivalence

$$\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} \mathsf{Coh}^{B^\vee}(\mathfrak{n}^\vee) \tag{6.2.16}$$

induced by restriction to $\{1\} \times \mathfrak{n}^\vee \subset \widetilde{\mathcal{N}}$, and $\mathsf{Coh}^{B^\vee}(\mathfrak{n}^\vee)$ is equivalent to the category of $B^\vee$-equivariant $\mathscr{O}(\mathfrak{n}^\vee)$-modules which are of finite type over $\mathscr{O}(\mathfrak{n}^\vee)$. Therefore, to prove the claim it suffices to show that the bounded derived category of the latter category is generated, as a triangulated category, by the modules of the form $\Bbbk_{B^\vee}(\lambda) \otimes \mathscr{O}(\mathfrak{n}^\vee)$, or equivalently (as any $B^\vee$-module has a filtration with subquotients of the form $\Bbbk_{B^\vee}(\lambda)$) by the modules of the form $V \otimes \mathscr{O}(\mathfrak{n}^\vee)$ with V a finite-dimensional $B^\vee$-module.

Let 2ρ be the sum of the positive roots of G, or of the positive coroots of $G^\vee$. This is a cocharacter $2\rho : \mathbb{G}_\mathrm{m} \to T^\vee$. Via this cocharacter, we can regard any $T^\vee$- or $B^\vee$-module as a graded vector space. In particular, we use this cocharacter to regard any $B^\vee$-equivariant $\mathscr{O}(\mathfrak{n}^\vee)$-module as a graded $\mathscr{O}(\mathfrak{n}^\vee)$-module. (Note that although the subspace $(\mathfrak{n}^\vee)^* \subset \mathscr{O}(\mathfrak{n}^\vee)$ is not homogeneous, it is spanned by homogeneous elements of strictly positive even degrees.)

Clearly, any $B^\vee$-equivariant $\mathscr{O}(\mathfrak{n}^\vee)$-module of finite type M admits a resolution

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to M \to 0$$

where each P^j is of the form $V \otimes \mathscr{O}(\mathfrak{n}^\vee)$ with V a finite-dimensional $B^\vee$-module. Regard this as a sequence of graded $\mathscr{O}(\mathfrak{n}^\vee)$-modules as in the preceding paragraph. Setting $m := \dim(\mathfrak{n}^\vee)$, the Hilbert Syzygy Theorem (in the form stated e.g. in [**Ei**, Corollary 19.7]) implies that $\ker(P^{-m} \to P^{-m+1})$ is finitely generated and free as a graded $\mathscr{O}(\mathfrak{n}^\vee)$-module.

To finish the proof it suffices to show that any finitely generated $B^\vee$-equivariant $\mathscr{O}(\mathfrak{n}^\vee)$-module that is free over $\mathscr{O}(\mathfrak{n}^\vee)$ admits a filtration (as a $B^\vee$-equivariant $\mathscr{O}(\mathfrak{n}^\vee)$-module) with subquotients of the form $V \otimes \mathscr{O}(\mathfrak{n}^\vee)$ with V a finite-dimensional $B^\vee$-module. This is proved by induction on $\mathrm{rk}(M)$. Indeed, by assumption M admits a homogeneous basis. If k is the smallest degree of a vector in this basis, and if V is the subspace of M spanned by the basis vectors of degree k, then V is also the k-th graded component of M, and is stable under the $B^\vee$-action. It is easily seen that the morphism $V \otimes \mathscr{O}(\mathfrak{n}^\vee) \to M$ induced by the $\mathscr{O}(\mathfrak{n}^\vee)$-module structure is an embedding of $B^\vee$-equivariant $\mathscr{O}(\mathfrak{n}^\vee)$-modules, whose cokernel is again free over $\mathscr{O}(\mathfrak{n}^\vee)$, of rank $\mathrm{rk}(M) - \dim(V)$. We conclude using the induction hypothesis.

(2) In view of (1), it suffices to prove that for any $\mu \in \mathbf{X}^\vee$ the line bundle $\mathscr{O}_{\widetilde{\mathcal{N}}}(\mu)$ belongs to the triangulated subcategory generated by the objects $V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ with $V \in \mathsf{Rep}(G^\vee)$ and $\lambda \in \mathbf{X}^\vee_+$. If $\lambda \in \mathbf{X}^\vee_+$, then taking the Koszul complex associated with the surjection $\mathbf{f}_\lambda$ (see (6.2.13)) we obtain a complex of $B^\vee$-modules

$$\begin{aligned}(6.2.17) \quad \cdots \to 0 \to \wedge^{d_\lambda}(\mathsf{N}(\lambda)) \to \wedge^{d_\lambda - 1}(\mathsf{N}(\lambda)) \otimes \Bbbk_{B^\vee}(\lambda) \to \cdots \\ \to \wedge^1(\mathsf{N}(\lambda)) \otimes \Bbbk_{B^\vee}((d_\lambda - 1) \cdot \lambda) \to \Bbbk_{B^\vee}(d_\lambda \cdot \lambda) \to 0 \to \cdots,\end{aligned}$$

which is acyclic by Lemma 6.2.6. (Here, as in (6.2.15), we set $d_\lambda = \dim \mathsf{N}(\lambda)$.) Then, tensoring with $\mathscr{O}_{\mathfrak{n}^\vee}$ and using the equivalence (6.2.16) we obtain an acyclic complex

$$\begin{aligned}\cdots \to 0 \to \wedge^{d_\lambda}(\mathsf{N}(\lambda)) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}} \to \wedge^{d_\lambda - 1}(\mathsf{N}(\lambda)) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda) \to \cdots \\ \to \wedge^1(\mathsf{N}(\lambda)) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}((d_\lambda - 1) \cdot \lambda) \to \mathscr{O}_{\widetilde{\mathcal{N}}}(d_\lambda \cdot \lambda) \to 0 \to \cdots\end{aligned}$$

in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$. Tensoring with $\mathscr{O}_{\widetilde{\mathcal{N}}}(\mu) \otimes (\wedge^{d_\lambda}(\mathsf{N}(\lambda)))^*$ and choosing (as we may) λ such that $\mu+\lambda$ is dominant, then this complex shows that $\mathscr{O}_{\widetilde{\mathcal{N}}}(\mu)$ does indeed belong to the desired triangulated subcategory. □

The following lemma will be needed in Section 6.6 below.

Lemma 6.2.9. — *For any $\lambda \in \mathbf{X}^\vee$ there exists $V \in \mathsf{Rep}(G^\vee)$ and an embedding of $G^\vee$-equivariant coherent sheaves $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda) \hookrightarrow V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$.*

Proof. — For α a simple root of $(G^\vee, T^\vee)$, we denote by $\mathfrak{n}^\vee_{(\alpha)}$ the Lie algebra of the unipotent radical of the parabolic subgroup of $G^\vee$ containing $B^\vee$ associated with the subset $\{\alpha\}$ of the set of simple roots. Then we have an embedding $\widetilde{\mathcal{N}}_{(\alpha)} := G^\vee \times^{B^\vee} \mathfrak{n}^\vee_{(\alpha)} \subset \widetilde{\mathcal{N}}$, and an associated exact sequence of coherent sheaves

$$\mathscr{O}_{\widetilde{\mathcal{N}}}(\alpha) \hookrightarrow \mathscr{O}_{\widetilde{\mathcal{N}}} \twoheadrightarrow \mathscr{O}_{\widetilde{\mathcal{N}}_{(\alpha)}}.$$

In particular, using repeatedly the first map in this sequence, and tensoring, we obtain an embedding $\mathscr{O}_{\widetilde{\mathcal{N}}}(\nu) \hookrightarrow \mathscr{O}_{\widetilde{\mathcal{N}}}$ for any ν in $\mathbb{Z}_{\geq 0}\mathfrak{R}^\vee_+$.

Going back to our problem, we choose ν in $\mathbb{Z}_{\geq 0}\mathfrak{R}^\vee_+$ such that $\lambda - \nu \in -\mathbf{X}^\vee_+$. Then, tensoring the above embedding by $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda-\nu)$ we obtain an embedding $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda) \hookrightarrow \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda-\nu)$. Now since $\lambda-\nu$ is antidominant, we have an embedding of $B^\vee$-modules $\Bbbk_{B^\vee}(\lambda-\nu) \hookrightarrow \mathsf{N}(w_\circ(\lambda-\nu))$. Tensoring with $\mathscr{O}_{\mathfrak{n}^\vee}$ and inducing to $G^\vee$ (see (6.2.16)) we obtain an embedding $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda-\nu) \hookrightarrow \mathsf{N}(w_\circ(\lambda-\nu)) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$. Composing the two maps we have constructed we obtain the desired embedding. □

6.2.5. Equivariant coherent sheaves on $\widetilde{\mathcal{N}}$ as a quotient. — Let us denote by

$$\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})$$

the full additive subcategory of $\mathsf{Coh}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})$ whose objects are the "free" coherent sheaves, i.e. those of the form $V \otimes \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}$ with V a finite-dimensional $G^\vee \times T^\vee$-module. We will consider the composition

$$\begin{aligned}(6.2.18)\quad K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}}) &\to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}}) \\ &\to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}) \xrightarrow[\sim]{(6.2.7)} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}),\end{aligned}$$

where the first arrow is the obvious functor, and the second one is pullback under the open embedding $\widehat{\mathcal{N}} \subset \widehat{\mathcal{N}}_{\mathcal{X}}$ (see (6.2.10)). We will denote by $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$ the kernel of this functor, i.e. the full subcategory of $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})$ whose objects are those killed by this functor (in other words, those whose cohomology is supported set-theoretically[3] on the preimage of $\partial\mathcal{X}$).

In the next statement we will consider a Verdier quotient of a triangulated category; for that notion we refer to [**SP**, Tag 05RA]. By the universal property of this

[3] Recall that a quasi-coherent sheaf $\mathscr{F}$ on a scheme X is said to be set-theoretically supported on a closed subscheme $Y \subset X$ if its restriction to the open complement vanishes. If X is noetherian and $\mathscr{F}$ is coherent, then this condition is equivalent to requiring that the ideal $\mathscr{I} \subset \mathscr{O}_X$ defining Y acts nilpotently on $\mathscr{F}$.

construction (see [**SP**, Tag 05RJ]), our functor (6.2.18) factors uniquely through a triangulated functor

$$K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})/K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}} \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}). \tag{6.2.19}$$

Proposition 6.2.10. — *The functor* (6.2.19) *is an equivalence of triangulated categories.*

Proof. — By construction, the category on the left-hand side is generated (as a triangulated category) by the objects of the form $V\otimes\Bbbk_{T^\vee}(\mu)\otimes\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}$ where V is in $\mathsf{Rep}(G^\vee)$ and $\mu\in\mathbf{X}^\vee$. Since the images of these objects generate the category $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ (see Lemma 6.2.8), by Beĭlinson's lemma (as formulated e.g. in [**ABG**, Lemma 3.9.3]), to prove the proposition it will suffice to prove that for any V_1, V_2 in $\mathsf{Rep}(G^\vee)$, any $\mu_1,\mu_2\in\mathbf{X}^\vee$, and any $n\in\mathbb{Z}$, our functor induces an isomorphism from

$$\mathrm{Hom}_{K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})/K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}}(V_1\otimes\Bbbk_{T^\vee}(\mu_1)\otimes\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}, V_2\otimes\Bbbk_{T^\vee}(\mu_2)\otimes\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}[n])$$

to

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(V_1\otimes\mathscr{O}_{\widetilde{\mathcal{N}}}(\mu_1), V_2\otimes\mathscr{O}_{\widetilde{\mathcal{N}}}(\mu_2)[n]).$$

Easy arguments show that we can in fact assume that $V_2=\Bbbk$ and $\mu_2=0$. We will then write V for V_1 and μ for μ_1.

We will first prove that our map is injective. A morphism from $V\otimes\Bbbk_{T^\vee}(\mu)\otimes\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}$ to $\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}[n]$ in $K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})/K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$ can be represented by a diagram

$$V\otimes\Bbbk_{T^\vee}(\mu)\otimes\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}\xleftarrow{f}\mathscr{F}\xrightarrow{g}\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}[n] \tag{6.2.20}$$

where $\mathscr{F}$ is an object of $K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})$, and the cone of f belongs to the subcategory $K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$. Saying that the image of this morphism vanishes is equivalent to saying that the image of g under (6.2.18) vanishes.

Now choose $\lambda\in\mathbf{X}^\vee_+$, and recall the complex $\mathscr{K}_\lambda$ from §6.2.3. This complex is the cone of a morphism of complexes

$$\mathscr{G}_\lambda\to\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}} \tag{6.2.21}$$

where each component in $\mathscr{G}_\lambda$ is of the form $M\otimes\Bbbk_{T^\vee}(-i\cdot\lambda)\otimes\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}$ for some M in $\mathsf{Rep}(G^\vee)$ and some $i\in\mathbb{Z}_{>0}$. Lemma 6.2.7 says that the cone of (6.2.21) is supported on $\widehat{\mathcal{N}}_{\mathcal{X}}\smallsetminus\widehat{\mathcal{N}}$; hence the same will be true for the induced morphism $\mathscr{F}\otimes_{\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}}\mathscr{G}_\lambda\to\mathscr{F}$. In other words, this morphism is an isomorphism in the quotient category $K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})/K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$. If N is as in Lemma 6.2.4, this argument shows that in the diagram (6.2.20) we can assume that the terms of $\mathscr{F}$ are all direct sums of objects of the form $M\otimes\Bbbk_{T^\vee}(-\eta)\otimes\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}$ with M in $\mathsf{Rep}(G^\vee)$ and $\eta\in\mathbf{X}^\vee_+$ which satisfies $\langle\eta,\alpha\rangle\geq N$ for any simple root α.

For such M and η we have

$$\mathrm{Hom}_{K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})}(M\otimes \Bbbk_{T^\vee}(-\eta)\otimes \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}, \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}[n]) = \begin{cases} \big(M^*\otimes \mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})_{-\eta}\big)^{G^\vee} & \text{if } n=0;\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(M\otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(-\eta), \mathscr{O}_{\widetilde{\mathcal{N}}}[n]) = \begin{cases} \big(M^*\otimes \Gamma(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\eta))\big)^{G^\vee} & \text{if } n=0;\\ 0 & \text{otherwise}\end{cases}$$

(see (6.2.1) and Remark 6.2.5). In view of these isomorphisms and (6.2.8), Lemma 6.2.4 says that the functor (6.2.18) induces an isomorphism between these Hom-spaces. By the 5-lemma, it then follows that this functor induces an isomorphism

$$\mathrm{Hom}_{K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})}(\mathscr{G}, \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}[n]) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{G}', \mathscr{O}_{\widetilde{\mathcal{N}}}[n])$$

for any complex $\mathscr{G}$ whose components are direct sums of objects of this form (where $\mathscr{G}'$ is the image of $\mathscr{G}$). In particular, this property holds for the complex $\mathscr{F}$ considered above, which finishes the proof of injectivity.

The proof of surjectivity will use similar tools. Namely, consider a morphism $f: V\otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\mu)\to \mathscr{O}_{\widetilde{\mathcal{N}}}[n]$ in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$. Choose $\lambda\in \mathbf{X}_+^\vee$ such that $\lambda-\mu$ satisfies $\langle\eta,\alpha\rangle \geq N$ for any simple root α. Then if $\mathscr{G}_\lambda$ is as above and $\mathscr{G}'_\lambda$ is its image under (6.2.18), the arguments above show that the composition

$$V\otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\mu)\otimes \mathscr{G}'_\lambda \to V\otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\mu)\otimes \mathscr{O}_{\widetilde{\mathcal{N}}} \xrightarrow{f} \mathscr{O}_{\widetilde{\mathcal{N}}}[n]$$

(where the first map is induced by (6.2.21)) is the image of a morphism

$$V\otimes \Bbbk_{T^\vee}(\mu)\otimes \mathscr{G}_\lambda \to \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}[n]$$

in $K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})$. Hence f is the image of the morphism represented by the diagram

$$V\otimes \Bbbk_{T^\vee}(\mu)\otimes \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}} \leftarrow V\otimes \Bbbk_{T^\vee}(\mu)\otimes \mathscr{G}_\lambda \to \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}[n],$$

which finishes the proof. □

Remark 6.2.11. — In [**AB**, Lemma 20], the authors prove that the category $K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$ is the smallest full subcategory of $K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr}}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})$ which contains the objects $\mathscr{K}_\lambda$ (for $\lambda\in\mathbf{X}_+^\vee$) and is stable under tensoring with objects of $\mathsf{Rep}(G^\vee\times T^\vee)$, taking cones, applying cohomological shifts, and taking direct summands. It turns out that this statement is not strictly needed for the proof of the main result of this chapter. We will therefore not review its proof.

6.3. Construction of the functor

For the remainder of this chapter, we work with $\Bbbk = \overline{\mathbb{Q}}_\ell$.

6.3.1. Equivariant perverse sheaves on Fl_G**.** — As in §5.3.1 we will consider $\overline{\mathbb{Q}}_\ell$-sheaves on Fl_G. More specifically we will denote by $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ the I-equivariant derived category of such sheaves, and will consider its monoidal product $\star^I$. We will choose once and for all a trivialization of the sheaf $\mathbb{Z}_\ell(1)$ on $\mathrm{Spec}(\mathbb{F})$, and use it to omit Tate twists in all our constructions below. For brevity, we introduce the notation

$$\mathsf{P}_I := \mathsf{Perv}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \subset D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

and

$$\mathscr{Z} := \mathsf{Z} \circ \mathcal{S}^{-1} : \mathsf{Rep}(G^\vee) \to \mathsf{P}_I.$$

Our construction will use the Wakimoto sheaves from Section 4.2. We will only use these perverse sheaves in the case where the closed Weyl chamber is the dominant one, so that $\Lambda = \mathbf{X}^\vee_+$. For simplicity we will sometimes omit "$\mathbf{X}^\vee_+$" in some notation. Recall the exact category $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ of I-equivariant perverse sheaves admitting a Wakimoto filtration, see §4.3.2. Here, since we are working with field coefficients, this category is stable under the convolution product $\star^I$, and hence admits a natural structure of a monoidal category. Moreover, the convolution product is exact in each variable.

We are now ready to explain the construction of a functor

$$F : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}}) \to D^{\mathrm{b}}\mathsf{P}_I. \tag{6.3.1}$$

6.3.2. "Extension" of a restriction functor. — Before proceeding, we explain a general construction on which we will rely. Suppose we are given an algebraically closed field $\Bbbk$, a $\Bbbk$-algebraic group H, a subgroup $K \subset H$, and commutative $\Bbbk$-algebras A, A' endowed with algebraic actions of K and H respectively. The question we consider is as follows. We denote by $A\text{-}\mathsf{mod}^K$ the category of K-equivariant A-modules which are finitely generated over A, and consider the exact symmetric monoidal functor

$$\mathsf{Rep}(H) \to A\text{-}\mathsf{mod}^K \tag{6.3.2}$$

sending V to $V_{|K} \otimes A$ (endowed with the diagonal action of K, and the natural action of A). (The symmetric monoidal structure on $A\text{-}\mathsf{mod}^K$ we consider here is that provided by tensor product over A.) We also denote by $A'\text{-}\mathsf{mod}^H_{\mathrm{fr}}$ the full subcategory of the category of H-equivariant A'-modules whose objects are the "free" modules, i.e. those of the form $V \otimes A'$ with V in $\mathsf{Rep}(H)$. (As above, here the H-action is the diagonal action, and the A'-module structure is the natural one.) We want to consider the question of "extending" the functor (6.3.2) to $A'\text{-}\mathsf{mod}^H_{\mathrm{fr}}$, i.e. to define a $\Bbbk$-linear symmetric monoidal functor

$$A'\text{-}\mathsf{mod}^H_{\mathrm{fr}} \to A\text{-}\mathsf{mod}^K$$

whose composition with the natural functor $\mathsf{Rep}(H) \to A'\text{-}\mathsf{mod}^H_{\mathrm{fr}}$ (defined by $V \mapsto V \otimes A'$) is (6.3.2). We have no choice for the definition of this functor on objects: it must send $V \otimes A'$ to $V_{|K} \otimes A$. The monoidal structure is also determined. But we have to understand what this functor does on morphisms.

We claim that the datum of such an extension is equivalent to the datum of a K-equivariant algebra morphism $A' \to A$. Indeed, one direction is easy: given a morphism $A' \to A$, the tensor product $A \otimes_{A'} -$ defines the wished-for functor $A'\text{-}\mathsf{mod}^H_{\mathrm{fr}} \to A\text{-}\mathsf{mod}^K$. On the other hand, suppose we are given an extension of (6.3.2) to a symmetric monoidal functor

$$A'\text{-}\mathsf{mod}^H_{\mathrm{fr}} \to A\text{-}\mathsf{mod}^K. \tag{6.3.3}$$

Then the H-module $\mathscr{O}(H)$ (with respect to multiplication of H on itself on the left) defines in a natural way an ind-object[4] in $\mathsf{Rep}(H)$. Tensoring with A' we obtain an ind-object $\mathscr{O}(H) \otimes A'$ in $A'\text{-}\mathsf{mod}^H_{\mathrm{fr}}$, which satisfies

$$\mathrm{Hom}_{\mathrm{Ind}(A'\text{-}\mathsf{mod}^H_{\mathrm{fr}})}(A', \mathscr{O}(H) \otimes A') = A'$$

by Frobenius reciprocity. In fact this identification is even an algebra isomorphism, if the left-hand side is endowed with the product defined as follows: given $\varphi : A' \to \mathscr{O}(H) \otimes A'$ and $\psi : A' \to \mathscr{O}(H) \otimes A'$, the product $\varphi \cdot \psi$ is the composition

$$A' \xrightarrow{\varphi} \mathscr{O}(H) \otimes A' \xrightarrow{\mathrm{id} \otimes \psi} \mathscr{O}(H) \otimes \mathscr{O}(H) \otimes A' \to \mathscr{O}(H) \otimes A',$$

where the rightmost map is induced by the product morphism in the algebra $\mathscr{O}(H)$. This identification is also H-equivariant, for the H-action on the left-hand side induced by the *right* multiplication action of H on $\mathscr{O}(H)$.

The extension of our functor (6.3.3) to ind-objects provides an H-equivariant algebra morphism

$$\mathrm{Hom}_{\mathrm{Ind}(A'\text{-}\mathsf{mod}^H_{\mathrm{fr}})}(A', \mathscr{O}(H) \otimes A') \to \mathrm{Hom}_{\mathrm{Ind}(A\text{-}\mathsf{mod}^K)}(A, \mathscr{O}(H) \otimes A),$$

where the product and the H-action on the right-hand side are defined as above. Now $\mathrm{Hom}_{\mathrm{Ind}(A\text{-}\mathsf{mod}^K)}(A, \mathscr{O}(H) \otimes A)$ identifies with

$$(\mathscr{O}(H) \otimes A)^K,$$

and using the restriction morphism $\mathscr{O}(H) \to \mathscr{O}(K)$ we obtain a K-equivariant algebra morphism

$$(\mathscr{O}(H) \otimes A)^K \to (\mathscr{O}(K) \otimes A)^K = A.$$

Composing these two constructions we obtain the wished-for K-equivariant algebra morphism $A' \to A$.

It is an easy exercise to check that these two constructions are inverse to each other.

Example 6.3.1. — One case we will encounter in particular is when $A' = \mathscr{O}(\mathfrak{h})$ for $\mathfrak{h}$ the Lie algebra of H. In this case the category $A'\text{-}\mathsf{mod}^H_{\mathrm{fr}}$ identifies with the full subcategory $\mathsf{Coh}^H_{\mathrm{fr}}(\mathfrak{h})$ of the category of H-equivariant coherent sheaves on $\mathfrak{h}$ whose objects are of the form $V \otimes \mathscr{O}_{\mathfrak{h}}$ for some V in $\mathsf{Rep}(H)$. Note that the datum of a K-equivariant algebra morphism $\mathscr{O}(\mathfrak{h}) \to A$ is equivalent to the datum of a morphism of K-modules $\mathfrak{h}^* \to A$, i.e. of a K-invariant element in $\mathfrak{h} \otimes A$.

[4] For the general theory of ind-objects in categories, we refer to [**KS2**]. The category of ind-objects in a category A will be denoted $\mathrm{Ind}(\mathsf{A})$.

In this case, the correspondence between extensions of the functor (6.3.2) to $\mathsf{Coh}^H_{\mathrm{fr}}(\mathfrak{h})$ and K-invariant elements in $\mathfrak{h} \otimes A$ admits another equivalent formulation, as follows. Note that for any V in $\mathsf{Rep}(H)$, the object $V \otimes \mathscr{O}_{\mathfrak{h}}$ admits a canonical endomorphism $\mathsf{f}^{\mathrm{can}}_V$ (as an object of $\mathsf{Coh}^H_{\mathrm{fr}}(\mathfrak{h})$) defined at the level of global sections as the composition

$$V \otimes \mathscr{O}(\mathfrak{h}) \to V \otimes \mathfrak{h} \otimes \mathfrak{h}^* \otimes \mathscr{O}(\mathfrak{h}) \to V \otimes \mathscr{O}(\mathfrak{h})$$

where the first map sends $v \otimes f$ to $v \otimes \mathrm{id}_{\mathfrak{h}} \otimes f$ (here $\mathrm{id}_{\mathfrak{h}}$ is regarded as an element in $\mathrm{End}_{\Bbbk}(\mathfrak{h}) \cong \mathfrak{h} \otimes \mathfrak{h}^*$) and the second map sends $v \otimes x \otimes \xi \otimes f$ to $(x \cdot v) \otimes (\xi f)$. (Here in the first factor we consider the action of $\mathfrak{h}$ on V obtained from the H-action by differentiation, and in the second factor we consider the multiplication in $\mathscr{O}(\mathfrak{h}) = \mathrm{S}(\mathfrak{h}^*)$. This construction is very similar to that encountered in (6.2.9).) It is easily seen that for V_1, V_2 in $\mathsf{Rep}(H)$ we have

$$\mathsf{f}^{\mathrm{can}}_{V_1 \otimes V_2} = \mathsf{f}^{\mathrm{can}}_{V_1} \otimes \mathrm{id}_{V_2 \otimes \mathscr{O}_{\mathfrak{h}}} + \mathrm{id}_{V_1 \otimes \mathscr{O}_{\mathfrak{h}}} \otimes \mathsf{f}^{\mathrm{can}}_{V_2}.$$

Therefore, given an extension of the functor (6.3.2) to $\mathsf{Coh}^H_{\mathrm{fr}}(\mathfrak{h})$, by taking the images of these morphisms we obtain, for any V in $\mathsf{Rep}(H)$, a K-equivariant endomorphism f_V of $V \otimes A$, this collection satisfying

$$f_{V_1 \otimes V_2} = f_{V_1} \otimes_A \mathrm{id}_{V_2 \otimes A} + \mathrm{id}_{V_1 \otimes A} \otimes_A f_{V_2}$$

for V_1, V_2 in $\mathsf{Rep}(H)$. Forgetting the K-equivariance, Tannakian formalism ensures that such a datum determines an element in $\mathfrak{h} \otimes A$, see e.g. [**YZ**, p. 361]. The fact that each f_V is K-equivariant is equivalent to the property that this element is K-invariant.

We leave it to the reader to check that the element in $(\mathfrak{h} \otimes A)^K$ described by this procedure is the same as the one produced by the more general considerations of the present subsection. In this way we obtain that the data of the following structures are equivalent:

1. an extension of the functor $V \mapsto V \otimes A$ to a $\Bbbk$-linear (symmetric) monoidal functor $\mathsf{Coh}^H_{\mathrm{fr}}(\mathfrak{h}) \to A\text{-mod}^K$;
2. a K-equivariant algebra morphism $\mathscr{O}(\mathfrak{h}) \to A$;
3. a K-invariant element in $\mathfrak{h} \otimes A$;
4. for any V in $\mathsf{Rep}(H)$, a K-equivariant endomorphism f_V of $V \otimes A$, this collection satisfying
$$f_{V_1 \otimes V_2} = f_{V_1} \otimes_A \mathrm{id}_{V_2 \otimes A} + \mathrm{id}_{V_1 \otimes A} \otimes_A f_{V_2}$$
for V_1, V_2 in $\mathsf{Rep}(H)$.

Under this correspondence, the collection $(f_V : V \in \mathsf{Rep}(H))$ attached to an extension $\varphi : \mathsf{Coh}^H_{\mathrm{fr}}(\mathfrak{h}) \to A\text{-mod}^K$ is given by $f_V = \varphi(\mathsf{f}^{\mathrm{can}}_V)$.

6.3.3. Starting point: central and Wakimoto sheaves. — We start with the functor

$$\overline{F} : \mathsf{Rep}(G^\vee \times T^\vee) \to \mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

defined as follows. Any object V in $\mathsf{Rep}(G^\vee \times T^\vee)$ can be written canonically as a direct sum of tensor products

$$V = \bigoplus_{\lambda \in \mathbf{X}^\vee} V^\lambda \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda) \tag{6.3.4}$$

for some $V^\lambda \in \mathsf{Rep}(G^\vee)$ (with only finitely many nonzero terms). Our functor then sends V to

$$\bigoplus_\lambda \mathscr{Z}(V^\lambda) \star^I \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell).$$

One defines a monoidal structure on this functor as follows: given V_1 and V_2 in $\mathsf{Rep}(G^\vee \times T^\vee)$, with canonical decompositions (6.3.4)

$$V_1 = \bigoplus_{\lambda \in \mathbf{X}^\vee} V_1^\lambda \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda) \quad \text{and} \quad V_2 = \bigoplus_{\lambda \in \mathbf{X}^\vee} V_2^\lambda \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda),$$

we have

$$V_1 \otimes V_2 = \bigoplus_\lambda \left(\bigoplus_{\substack{\mu,\nu \\ \mu+\nu=\lambda}} V_1^\mu \otimes V_2^\nu \right) \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda).$$

We define the isomorphism

$$\overline{F}(V_1 \otimes V_2) \xrightarrow{\sim} \overline{F}(V_1) \star^I \overline{F}(V_2) \tag{6.3.5}$$

as the isomorphism

$$\bigoplus_\lambda \left(\bigoplus_{\substack{\mu,\nu \\ \mu+\nu=\lambda}} \mathscr{Z}(V_1^\mu \otimes V_2^\nu) \right) \star^I \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)$$
$$\xrightarrow{\sim} \left(\bigoplus_{\lambda'} \mathscr{Z}(V_1^{\lambda'}) \star^I \mathbf{J}_{\lambda'}(\overline{\mathbb{Q}}_\ell) \right) \star^I \left(\bigoplus_{\lambda''} \mathscr{Z}(V_2^{\lambda''}) \star^I \mathbf{J}_{\lambda''}(\overline{\mathbb{Q}}_\ell) \right)$$

induced by the "centrality" isomorphism $(\sigma_{\mathcal{S}^{-1}(V_2^{\lambda''}), \mathbf{J}_{\lambda'}(\overline{\mathbb{Q}}_\ell)})^{-1}$, the "monoidality" isomorphism $\phi_{\mathcal{S}^{-1}(V_1^{\lambda'}), \mathcal{S}^{-1}(V_2^{\lambda''})}$, and the canonical isomorphism $\mathbf{J}_{\lambda'}(\overline{\mathbb{Q}}_\ell) \star^I \mathbf{J}_{\lambda''}(\overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathbf{J}_{\lambda'+\lambda''}(\overline{\mathbb{Q}}_\ell)$ (see Lemma 4.2.6).

Given V_1, V_2 in $\mathsf{Rep}(G^\vee \times T^\vee)$, there exists a canonical isomorphism

$$\overline{F}(V_1) \star^I \overline{F}(V_2) \xrightarrow{\sim} \overline{F}(V_2) \star^I \overline{F}(V_1) \tag{6.3.6}$$

induced by the appropriate centrality isomorphisms and the composition of canonical isomorphisms $\mathbf{J}_{\lambda'}(\overline{\mathbb{Q}}_\ell) \star^I \mathbf{J}_{\lambda''}(\overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathbf{J}_{\lambda'+\lambda''}(\overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathbf{J}_{\lambda''}(\overline{\mathbb{Q}}_\ell) \star^I \mathbf{J}_{\lambda'}(\overline{\mathbb{Q}}_\ell)$ for $\lambda', \lambda'' \in \mathbf{X}^\vee$. Using Theorem 3.5.1, one can check that this morphism identifies, via the appropriate monoidality isomorphisms, with the image under $\overline{F}$ of the obvious commutativity isomorphism $V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$. In particular, its inverse is the similar morphism for the pair (V_2, V_1).

From the results of Section 4.8 we see that, identifying the category $\mathsf{Vect}_{\overline{\mathbb{Q}}_\ell}^{\mathbf{X}^\vee}$ of finite-dimensional $\mathbf{X}^\vee$-graded $\overline{\mathbb{Q}}_\ell$-vector spaces with the category of finite-dimensional algebraic $T^\vee$-modules, the composition $\mathrm{Grad}_{\mathbf{X}^\vee} \circ \overline{F}$ (see §4.7.2) identifies with the restriction functor associated with the diagonal embedding $T^\vee \hookrightarrow G^\vee \times T^\vee$.

6.3.4. Extending the functor to free coherent sheaves on $\widehat{\mathcal{N}}_{\mathcal{X}}$, I. — Our next goal is to "extend" the functor $\overline{F}$ to a monoidal functor

$$\widetilde{F} : \mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to \mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell),$$

where the category $\mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})$ is as in §6.2.5 (in the special case $\Bbbk = \overline{\mathbb{Q}}_\ell$). This will be done in two steps, carried out in this and the next subsection.

The first step is motivated by the following observation: the source category for our desired functor $\widetilde{F}$ (and also for the functor $\overline{F}$) is a *symmetric* monoidal category, while the target category is not symmetric. We will remedy this by replacing $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ by a (non-full!) subcategory which is "forced" to be symmetric. Namely, we denote by $\mathscr{C}$ the category with

– objects: those of $\mathsf{Rep}(G^\vee \times T^\vee)$;
– morphisms: for V, V' in $\mathsf{Rep}(G^\vee \times T^\vee)$, the space $\mathrm{Hom}_{\mathscr{C}}(V, V')$ consists of the morphisms $\varphi : \overline{F}(V) \to \overline{F}(V')$ in $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ such that for any V'' in $\mathsf{Rep}(G^\vee \times T^\vee)$ the diagram

$$\begin{tikzcd}
\overline{F}(V'') \star^I \overline{F}(V) \arrow[r, "\mathrm{id}_{\overline{F}(V'')} \star^I \varphi"] \arrow[d, "(6.3.6)"', "\wr"] & \overline{F}(V'') \star^I \overline{F}(V') \arrow[d, "(6.3.6)", "\wr"'] \\
\overline{F}(V) \star^I \overline{F}(V'') \arrow[r, "\varphi \star^I \mathrm{id}_{\overline{F}(V'')}"] & \overline{F}(V') \star^I \overline{F}(V'')
\end{tikzcd}$$

commutes.

Remark 6.3.2. — To check that a given morphism $\overline{F}(V) \to \overline{F}(V')$ belongs to $\mathscr{C}$, we a priori need to check the commutativity of the diagram above for all V'' in $\mathsf{Rep}(G^\vee \times T^\vee)$. However, using the decomposition (6.3.4) we see that it suffices to do so in the following two special cases:

1. when V'' is a $G^\vee$-module (with trivial $T^\vee$-action);
2. when $V'' = \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda)$ for some $\lambda \in \mathbf{X}^\vee$.

Moreover, in the former case the commutativity is automatic by the functoriality of the isomorphism $\sigma_{-,-}$ in the second variable. Hence only the objects $V'' = \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda)$ (with $\lambda \in \mathbf{X}^\vee$) need to be considered. In fact, since any such module is isomorphic to a tensor product of a module with λ dominant and the inverse of such a module, it even suffices to consider the case $\lambda \in \mathbf{X}^\vee_+$.

By definition, $\mathscr{C}$ is a monoidal category (with monoidal product denoted $\star$, which coincides with the tensor product $\otimes_{\overline{\mathbb{Q}}_\ell}$ on objects), and there exists a canonical faithful monoidal functor

$$\mathscr{C} \to \mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell). \tag{6.3.7}$$

Moreover, by construction $\overline{F}$ factors through a monoidal functor

$$\underline{F} : \mathsf{Rep}(G^\vee \times T^\vee) \to \mathscr{C}.$$

We claimed above that $\mathscr{C}$ is symmetric; we are now ready to prove this claim.

Lemma 6.3.3. — *The obvious commutativity constraint on* $\mathsf{Rep}(G^\vee \times T^\vee)$ *provides a commutativity constraint on the monoidal category* $\mathscr{C}$.

Proof. — The only property which requires a proof is the fact that our isomorphism $V_1 \star V_2 \xrightarrow{\sim} V_2 \star V_1$ is bifunctorial. For this we consider morphisms $f : V_1 \to V_1'$ and $g : V_2 \to V_2'$ in $\mathscr{C}$; what we need to prove is that the diagram

$$\begin{tikzcd}
\overline{F}(V_1) \star^I \overline{F}(V_2) \arrow[r, "(6.3.6)", "\sim"'] \arrow[d, "f\star g"'] & \overline{F}(V_2) \star^I \overline{F}(V_1) \arrow[d, "g\star f"] \\
\overline{F}(V_1') \star^I \overline{F}(V_2') \arrow[r, "(6.3.6)", "\sim"'] & \overline{F}(V_2') \star^I \overline{F}(V_1')
\end{tikzcd}$$

commutes. However this square can be decomposed into two smaller squares as follows:

$$\begin{tikzcd}
\overline{F}(V_1) \star^I \overline{F}(V_2) \arrow[r, "(6.3.6)", "\sim"'] \arrow[d, "f\star \mathrm{id}"'] & \overline{F}(V_2) \star^I \overline{F}(V_1) \arrow[d, "\mathrm{id}\star f"] \\
\overline{F}(V_1) \star^I \overline{F}(V_2') \arrow[r, "(6.3.6)", "\sim"'] \arrow[d, "\mathrm{id}\star g"'] & \overline{F}(V_2') \star^I \overline{F}(V_1) \arrow[d, "g\star \mathrm{id}"] \\
\overline{F}(V_1') \star^I \overline{F}(V_2') \arrow[r, "(6.3.6)", "\sim"'] & \overline{F}(V_2') \star^I \overline{F}(V_1').
\end{tikzcd}$$

Here the upper (resp. lower) square commutes because f (resp. g) is a morphism in $\mathscr{C}$; hence the outer square commutes as well. □

Below we will also require the following easy claim.

Lemma 6.3.4. — *For any* V *in* $\mathsf{Rep}(G^\vee \times T^\vee)$, *the functor* $V^* \star (-) : \mathscr{C} \to \mathscr{C}$ *is right adjoint to the functor* $V \star (-) : \mathscr{C} \to \mathscr{C}$.

Proof. — To prove that these functors are adjoint, we need to construct morphisms of functors

$$V \star V^* \star (-) \to \mathrm{id} \quad \text{and} \quad \mathrm{id} \to V^* \star V \star (-),$$

and check that these morphisms satisfy the zigzag condition. In fact these morphisms can be chosen as those induced by the natural morphisms of $G^\vee \times T^\vee$-modules $V \otimes V^* \to \overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}}_\ell \to V^* \otimes V$. □

We now consider the category $\mathrm{Ind}(\mathscr{C})$ of ind-objects in $\mathscr{C}$. Since the regular representation $\mathscr{O}(G^\vee \times T^\vee)$ (with the action induced by left multiplication in $G^\vee \times T^\vee$) is in a natural way an ind-object in the category $\mathsf{Rep}(G^\vee \times T^\vee)$, applying $\underline{F}$ we obtain an object $\underline{F}(\mathscr{O}(G^\vee \times T^\vee))$ in $\mathrm{Ind}(\mathscr{C})$. We set

$$A := \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(\mathscr{O}(G^\vee \times T^\vee))).$$

More concretely we have

$$A = \varinjlim_{V} \mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, \underline{F}(V))$$

where V runs over the set of finite-dimensional $(G^\vee \times T^\vee)$-submodules of $\mathscr{O}(G^\vee \times T^\vee)$ (ordered by inclusion), and if $V \subset V'$ the induced map

$$\mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, \underline{F}(V)) \to \mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, \underline{F}(V'))$$

is injective, since the same is true for the map

$$\mathrm{Hom}_{\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(1_{\mathscr{C}}, \overline{F}(V)) \to \mathrm{Hom}_{\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(1_{\mathscr{C}}, \overline{F}(V')).$$

We endow A with an associative product as follows. Given morphisms $\varphi : 1_{\mathscr{C}} \to \underline{F}(V)$ and $\psi : 1_{\mathscr{C}} \to \underline{F}(V')$ (where V and V' are finite-dimensional submodules of $\mathscr{O}(G^\vee \times T^\vee)$), we define the product $\varphi \cdot \psi$ as the composition

$$1_{\mathscr{C}} = 1_{\mathscr{C}} \star 1_{\mathscr{C}} \xrightarrow{\varphi \star \psi} \underline{F}(V) \star \underline{F}(V') \to \underline{F}(\mathscr{O}(G^\vee \times T^\vee)),$$

where the rightmost morphism is induced by the multiplication morphism $V \otimes V' \to \mathscr{O}(G^\vee \times T^\vee)$. Using the symmetry of $\mathscr{C}$, it is not difficult to check that this product endows A with the structure of a commutative $\overline{\mathbb{Q}}_\ell$-algebra. A very similar construction provides, for any V in $\mathscr{C}$, a structure of an A-module on

$$\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V) \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee))).$$

Since $\mathscr{O}(G^\vee \times T^\vee)$ admits a second $G^\vee \times T^\vee$-module structure (induced by *right* multiplication in $G^\vee \times T^\vee$), A also admits a $G^\vee \times T^\vee$-action by algebra automorphisms, which is easily seen (e.g. by reasoning in terms of comodules) to be algebraic. The same construction provides, for any V in $\mathscr{C}$, a $G^\vee \times T^\vee$-action on the A-module $\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V) \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)))$, which makes it a $G^\vee \times T^\vee$-equivariant A-module.

As in §6.3.2, we will denote by

$$A\text{-}\mathsf{mod}_{\mathrm{fr}}^{G^\vee \times T^\vee}$$

the full monoidal subcategory of the category of $G^\vee \times T^\vee$-equivariant A-modules whose objects are those of the form $V \otimes A$ with V in $\mathsf{Rep}(G^\vee \times T^\vee)$.

Proposition 6.3.5. — *There is an equivalence of symmetric monoidal categories* $H : \mathscr{C} \xrightarrow{\sim} A\text{-}\mathsf{mod}_{\mathrm{fr}}^{G^\vee \times T^\vee}$ *given by the formula*

$$H(V) = \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V) \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee))).$$

Proof. — Let $V \in \mathsf{Rep}(G^\vee \times T^\vee)$. In this proof, we consider four different actions of $G^\vee \times T^\vee$ on the vector space $V \otimes \mathscr{O}(G^\vee \times T^\vee)$

1. the “default” action, in which $G^\vee \times T^\vee$ acts by the given action on V, and by left multiplication on $\mathscr{O}(G^\vee \times T^\vee)$;
2. the “left-only” action, in which $G^\vee \times T^\vee$ acts trivially on V, and by left multiplication on $\mathscr{O}(G^\vee \times T^\vee)$;
3. the “right-only” action, in which $G^\vee \times T^\vee$ acts trivially on V, and by right multiplication on $\mathscr{O}(G^\vee \times T^\vee)$;
4. the “mixed” action, in which $G^\vee \times T^\vee$ acts by the given action on V, and by right multiplication on $\mathscr{O}(G^\vee \times T^\vee)$.

Recall that $V \otimes \mathscr{O}(G^\vee \times T^\vee)$ identifies canonically with the space of algebraic maps $G^\vee \times T^\vee \to V$. From this perspective, the four actions are given by the following formulas (here $g \in G^\vee \times T^\vee$, and $f : G^\vee \times T^\vee \to V$ is an algebraic map):

$$\begin{aligned}
&\text{default:} && (g \cdot_{\text{def}} f)(x) = gf(g^{-1}x),\\
&\text{left-only:} && (g \cdot_{\text{l-o}} f)(x) = f(g^{-1}x),\\
&\text{right-only:} && (g \cdot_{\text{r-o}} f)(x) = f(xg),\\
&\text{mixed:} && (g \cdot_{\text{mix}} f)(x) = gf(xg).
\end{aligned}$$

There is a vector space automorphism

$$\varphi : V \otimes \mathscr{O}(G^\vee \times T^\vee) \to V \otimes \mathscr{O}(G^\vee \times T^\vee) \qquad \text{given by} \qquad \varphi(f)(x) = x^{-1}f(x)$$

for $f \in V \otimes \mathscr{O}(G^\vee \times T^\vee)$ and $x \in G^\vee \times T^\vee$. This map has the following intertwining properties:

$$\varphi(g \cdot_{\text{def}} f) = g \cdot_{\text{l-o}} \varphi(f), \qquad \varphi(g \cdot_{\text{r-o}} f) = g \cdot_{\text{mix}} \varphi(f).$$

In the formula for $H(V)$, we regard $\mathscr{O}(G^\vee \times T^\vee)$ as a $G^\vee \times T^\vee$-module under left multiplication. By monoidality of $\underline{F}$, we have

$$\begin{aligned}
(6.3.8) \quad H(V) = \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V) \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)))&\\
\cong \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V \otimes \mathscr{O}(G^\vee \times T^\vee))),&
\end{aligned}$$

where on the right-hand side we use the default action on $V \otimes \mathscr{O}(G^\vee \times T^\vee)$. Now apply φ: we have

$$\begin{aligned}
(6.3.9) \quad \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V \otimes \mathscr{O}(G^\vee \times T^\vee))) \xrightarrow[\sim]{\varphi}&\\
\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V \otimes \mathscr{O}(G^\vee \times T^\vee)))&\\
\cong V \otimes \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(\mathscr{O}(G^\vee \times T^\vee))) \cong V \otimes A.&
\end{aligned}$$

In the second term, $V \otimes \mathscr{O}(G^\vee \times T^\vee)$ carries the left-only action. In particular, V is a trivial $G^\vee \times T^\vee$-representation, so it can be brought outside the Hom-group as shown in the third term. Combining (6.3.8) and (6.3.9), we obtain an isomorphism of A-modules

$$(6.3.10) \qquad H(V) \cong V \otimes A.$$

Recall that the $G^\vee \times T^\vee$-action on A, resp. on $\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V) \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)))$, is induced by the right multiplication action on $\mathscr{O}(G^\vee \times T^\vee)$, resp. the right-only action on $V \otimes \mathscr{O}(G^\vee \times T^\vee)$. Since the first isomorphism in (6.3.9) intertwines the right-only action with the mixed action, we see that (6.3.10) is in fact an isomorphism of $G^\vee \times T^\vee$-equivariant A-modules. We then conclude that the functor H is well defined (i.e. that it indeed takes values in $A\text{-}\mathsf{mod}_{\mathrm{fr}}^{G^\vee \times T^\vee}$), and essentially surjective.

Given V, V' in $\mathscr{C}$ we have a natural morphism $H(V) \otimes H(V') \to H(V \star V')$, or

$$\begin{aligned}
(6.3.11) \quad \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V) \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)))\otimes&\\
\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V') \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)))&\\
\to \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}(V \otimes V') \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee))),&
\end{aligned}$$

defined as follows: the image of $\varphi \otimes \psi \in H(V) \otimes H(V')$ is the composition

$$\begin{aligned} 1_{\mathscr{C}} \xrightarrow{\varphi \star \psi} \underline{F}(V) \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)) \star \underline{F}(V') \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)) \\ \cong \underline{F}(V) \star \underline{F}(V') \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)) \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)) \\ \to \underline{F}(V \otimes V') \star \underline{F}(\mathscr{O}(G^\vee \times T^\vee)), \end{aligned}$$

where the last arrow is induced by the monoidal structure on $\underline{F}$ and the product morphism $\mathscr{O}(G^\vee \times T^\vee) \otimes \mathscr{O}(G^\vee \times T^\vee) \to \mathscr{O}(G^\vee \times T^\vee)$. Combining (6.3.11) with (two instances of) the isomorphism (6.3.10), we obtain a monoidal structure on H.

To conclude, what we have to check is that H is also fully faithful, or in other words that for any V, V' in $\mathscr{C}$ the induced morphism

$$\mathrm{Hom}_{\mathscr{C}}(V', V) \to \mathrm{Hom}_{A\text{-mod}_{\mathrm{fr}}^{G^\vee \times T^\vee}}(V' \otimes A, V \otimes A) \tag{6.3.12}$$

is an isomorphism. In view of Lemma 6.3.4, we can assume that $V' = \overline{\mathbb{Q}}_\ell = 1_{\mathscr{C}}$; then the right-hand side identifies with $(V \otimes A)^{G^\vee \times T^\vee}$, where $G^\vee \times T^\vee$ acts diagonally. Now we consider the extension of H to ind-objects, and take V to be the ind-object $\mathscr{O}(G^\vee \times T^\vee)$. In this setting, the left-hand side in (6.3.12) identifies (by definition) with A, and in the right-hand side we find

$$(\mathscr{O}(G^\vee \times T^\vee) \otimes A)^{G^\vee \times T^\vee} = A.$$

It is easily seen that under these identifications the map (6.3.12) identifies with id_A, and hence is an isomorphism. Since any object in $\mathsf{Rep}(G^\vee \times T^\vee)$ is a direct summand in a direct sum of copies of $\mathscr{O}(G^\vee \times T^\vee)$, it follows that (6.3.12) is an isomorphism for any V in $\mathsf{Rep}(G^\vee \times T^\vee)$, which finishes the proof. □

6.3.5. Extending the functor to free coherent sheaves on $\widehat{\mathcal{N}}_{\mathcal{X}}$, II. — From the proof of Proposition 6.3.5 we see that, identifying $\mathscr{C}$ with $A\text{-mod}_{\mathrm{fr}}^{G^\vee \times T^\vee}$ through the equivalence of this statement, the functor $\underline{F}$ identifies with $V \mapsto V \otimes A$. We are thus in the setting of §6.3.2 (with $H = K = G^\vee \times T^\vee$), and now want to extend our functor to a symmetric monoidal functor

$$\mathsf{Coh}_{\mathrm{fr}}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to A\text{-mod}_{\mathrm{fr}}^{G^\vee \times T^\vee}.$$

For this we need to construct a morphism of $G^\vee \times T^\vee$-equivariant algebras

$$\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to A.$$

Here the left-hand side is by definition a quotient of $\mathscr{O}(\mathfrak{g}^\vee \times \mathcal{X}) = \mathscr{O}(\mathfrak{g}^\vee) \otimes \mathscr{O}(\mathcal{X})$; therefore we start by defining equivariant algebra morphisms $\mathscr{O}(\mathfrak{g}^\vee) \to A$ and $\mathscr{O}(\mathcal{X}) \to A$.

We recall that for any V in $\mathsf{Rep}(G^\vee)$ we have the logarithm of monodromy endomorphism $\mathsf{n}_{\mathscr{Z}(V)}$ of $\mathscr{Z}(V) = \overline{F}(V \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(0))$ (see §9.5.4), which in this chapter we will denote for simplicity by n_V. By an analogue of Proposition 3.4.2 we have

$$\mathsf{n}_{V \otimes V'} = \mathsf{n}_V \star^I \mathrm{id}_{\mathscr{Z}(V')} + \mathrm{id}_{\mathscr{Z}(V)} \star^I \mathsf{n}_{V'}. \tag{6.3.13}$$

As explained in Example 6.3.1, the datum of such a collection of endomorphisms determines a $G^\vee$-equivariant algebra morphism $\mathscr{O}(\mathfrak{g}^\vee) \to A$. In fact, since for any V

our automorphism of $V \otimes A$ is $G^\vee \times T^\vee$-equivariant, this algebra morphism is even $G^\vee \times T^\vee$-equivariant (where $T^\vee$ acts trivially on $\mathscr{O}(\mathfrak{g}^\vee)$).

Next, we turn to the construction of the $G^\vee \times T^\vee$-equivariant algebra morphism $\mathscr{O}(\mathcal{X}) \to A$. In view of (6.2.3), to define a morphism of $G^\vee \times T^\vee$-modules $\mathscr{O}(\mathcal{X}) \to A$ we need to define, for any $\lambda \in \mathbf{X}^\vee_+$, a morphism of $G^\vee \times T^\vee$-modules

$$\mathsf{N}(\lambda) \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(-\lambda) \to A. \tag{6.3.14}$$

Recall that in §4.6.3 (see also Lemma 1.5.2) we have considered, for any $\lambda \in \mathbf{X}^\vee_+$, a canonical surjective morphism

$$\mathfrak{f}_\lambda : \mathscr{Z}(\mathsf{N}(\lambda)) \to \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)$$

in $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. Here the left-hand side identifies with $\underline{F}(\mathsf{N}(\lambda) \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(0))$, and the right-hand side with $\underline{F}(\overline{\mathbb{Q}}_\ell \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda))$, where here $\overline{\mathbb{Q}}_\ell$ denotes the trivial $G^\vee$-module.

Lemma 6.3.6. — *The morphism $\mathfrak{f}_\lambda$ is a morphism in $\mathscr{C}$.*

Proof. — In view of Remark 6.3.2, what we have to prove is that for any $\mu \in \mathbf{X}^\vee_+$ the following diagram commutes, where the right-hand vertical maps are the isomorphisms provided by Lemma 4.2.6:

$$\begin{tikzcd}
\mathbf{J}_\mu(\overline{\mathbb{Q}}_\ell) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda, \overline{\mathbb{Q}}_\ell)) \arrow[r, "\mathrm{id} \star^I \mathfrak{f}_\lambda"] \arrow[dd, "(\sigma_{\mathcal{J}_*(\lambda,\overline{\mathbb{Q}}_\ell), \mathbf{J}_\mu(\overline{\mathbb{Q}}_\ell)})^{-1}"'] & \mathbf{J}_\mu(\overline{\mathbb{Q}}_\ell) \star^I \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell) \arrow[d, "\wr"] \\
& \mathbf{J}_{\lambda+\mu}(\overline{\mathbb{Q}}_\ell) \arrow[d, "\wr"] \\
\mathsf{Z}(\mathcal{J}_*(\lambda, \overline{\mathbb{Q}}_\ell)) \star^I \mathbf{J}_\mu(\overline{\mathbb{Q}}_\ell) \arrow[r, "\mathfrak{f}_\lambda \star^I \mathrm{id}"] & \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell) \star^I \mathbf{J}_\mu(\overline{\mathbb{Q}}_\ell).
\end{tikzcd}$$

To do this it suffices to check that the compositions of the two maps considered here with the surjective morphism

$$\mathfrak{f}_\mu \star^I \mathrm{id} : \mathsf{Z}(\mathcal{J}_*(\mu, \overline{\mathbb{Q}}_\ell)) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda, \overline{\mathbb{Q}}_\ell)) \to \mathbf{J}_\mu(\overline{\mathbb{Q}}_\ell) \star^I \mathsf{Z}(\mathcal{J}_*(\lambda, \overline{\mathbb{Q}}_\ell))$$

coincide. This is exactly the content of Proposition 4.6.12. □

Applying the equivalence of Proposition 6.3.5 to $\mathfrak{f}_\lambda$ we obtain a canonical morphism

$$\mathsf{N}(\lambda) \otimes A \to \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda) \otimes A.$$

Restricting to $\mathsf{N}(\lambda) \otimes 1$ and then tensoring with $\overline{\mathbb{Q}}_{\ell T^\vee}(-\lambda)$, we deduce the sought-after morphism (6.3.14). Then Lemma 1.5.5, Remark 4.6.13 and Lemma 6.2.1 imply that the morphism $\mathscr{O}(\mathcal{X}) \to A$ so constructed is an algebra morphism.

Combining the two constructions above we obtain a $G^\vee \times T^\vee$-equivariant algebra morphism

$$\mathscr{O}(\mathfrak{g}^\vee \times \mathcal{X}) \to A. \tag{6.3.15}$$

Lemma 6.3.7. — *The morphism* (6.3.15) *factors (uniquely) through a $G^\vee \times T^\vee$-equivariant algebra morphism*

$$\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to A.$$

Proof. — By definition (see §6.2.2), the algebra $\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}})$ is the quotient of $\mathscr{O}(\mathfrak{g}^\vee \times \mathcal{X})$ by the ideal generated by the images of the maps (6.2.11) for all $\lambda \in \mathbf{X}^\vee_+$. Now, recall from Lemma 4.6.9 (see also the comments at the end of §4.6.3) that we have

$$\mathfrak{f}_\lambda \circ \mathsf{m}_{\mathsf{N}(\lambda)} = \mathfrak{f}_\lambda,$$

so that

$$\mathfrak{f}_\lambda \circ \mathsf{n}_{\mathsf{N}(\lambda)} = 0.$$

Applying the equivalence of Proposition 6.3.5 (see also Example 6.3.1), this shows that the composition of (6.2.11) with (6.3.15) vanishes, which implies our lemma. □

Finally, we can define the monoidal functor

$$\widetilde{F} : \mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to \mathsf{Perv}^{\mathbf{X}^\vee_+}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

as the composition

$$\mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to A\text{-mod}^{G^\vee \times T^\vee}_{\mathrm{fr}} \xrightarrow{\sim} \mathscr{C} \xrightarrow{(6.3.7)} \mathsf{Perv}^{\mathbf{X}^\vee_+}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

where the first arrow is given by tensor product with A (with respect to the morphism of Lemma 6.3.7) and the second one is the inverse of the equivalence of Lemma 6.3.5. By construction, for any V in $\mathsf{Rep}(G^\vee)$ and $\lambda \in \mathbf{X}^\vee$ we have

$$\widetilde{F}(V \otimes \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}) = \mathscr{Z}(V), \qquad \widetilde{F}(\overline{\mathbb{Q}}_{\ell T^\vee}(\lambda) \otimes \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}) = \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell). \tag{6.3.16}$$

Moreover, for any $\lambda \in \mathbf{X}^\vee_+$, the image of the canonical morphism $\mathsf{N}(\lambda) \otimes \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}} \to \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda) \otimes \mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}$ obtained from the multiplication map $\mathsf{N}(\lambda) \otimes \mathscr{O}(\mathcal{X}) \to \overline{\mathbb{Q}}_{\ell T^\vee}(\lambda) \otimes \mathscr{O}(\mathcal{X})$ (see (6.2.3)) is $\mathfrak{f}_\lambda$.

For the next statement, recall the "base point" $e_{\widehat{\mathcal{N}}} \in \widehat{\mathcal{N}}$ from §6.2.2. This point is stabilized by the diagonal copy of $T^\vee$ in $G^\vee \times T^\vee$. As a consequence, restriction to this point defines a functor

$$\mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to \mathsf{Rep}(T^\vee). \tag{6.3.17}$$

Lemma 6.3.8. — *The composition*

$$\mathrm{Grad}_{\mathbf{X}^\vee} \circ \widetilde{F} : \mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to \mathsf{Rep}(T^\vee)$$

is isomorphic to (6.3.17).

Proof. — As explained in §6.3.3, the composition of $\widetilde{F}$ with the natural functor $\mathsf{Rep}(G^\vee \times T^\vee) \to \mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})$ is the restriction functor associated with the diagonal embedding $T^\vee \hookrightarrow G^\vee \times T^\vee$. We are thus again in the setting considered in §6.3.2, now with $H = G^\vee \times T^\vee$ and $K = T^\vee$ (embedded diagonally). The functor $\mathrm{Grad}_{\mathbf{X}^\vee} \circ \widetilde{F}$ must be induced by a $T^\vee$-equivariant algebra morphism $\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to \overline{\mathbb{Q}}_\ell$, which itself is determined by the datum of a suitable endomorphism of the composition of the restriction to $\mathsf{Rep}(G^\vee)$ with the forgetful functor $\mathsf{Rep}(T^\vee) \to \mathsf{Vect}_{\overline{\mathbb{Q}}_\ell}$ (which will define an algebra morphism $\mathscr{O}(\mathfrak{g}^\vee) \to \overline{\mathbb{Q}}_\ell$), and of suitable morphisms of $T^\vee$-modules $\mathsf{N}(\lambda) \otimes \overline{\mathbb{Q}}_{\ell T^\vee}(-\lambda) \to \overline{\mathbb{Q}}_\ell$ (which will define an algebra morphism $\mathscr{O}(\mathcal{X}) \to \overline{\mathbb{Q}}_\ell$). In the present setting the first datum is the trivial endomorphism (by Lemma 4.6.9) and the second datum is induced by the projection $\mathsf{N}(\lambda) \to \mathsf{N}(\lambda)_\lambda$ on the highest weight line (see §4.6.3). Therefore the associated morphism $\mathscr{O}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to \overline{\mathbb{Q}}_\ell$ is evaluation at $e_{\widehat{\mathcal{N}}}$, and the desired claim follows. ☐

6.3.6. Factorization through coherent sheaves on $\widetilde{\mathcal{N}}$. — Now that we have the functor $\widetilde{F}$, to deduce the desired functor (6.3.1) we will use the description of the category $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ provided by Proposition 6.2.10.

Proposition 6.3.9. — *There exists a unique triangulated functor*

$$F : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{P}_I$$

such that the following diagram (where the left vertical arrow is induced by restriction to the open subset $\widehat{\mathcal{N}}$ followed by the equivalence (6.2.7)*, and the right vertical arrow is the obvious functor) commutes up to isomorphism:*

$$\begin{tikzcd}
K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}}) \arrow[r, "K^{\mathrm{b}}(\widetilde{F})"] \arrow[d] & K^{\mathrm{b}}\mathsf{P}_I \arrow[d] \\
D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \arrow[r, "F"] & D^{\mathrm{b}}\mathsf{P}_I.
\end{tikzcd}$$

Proof. — By Proposition 6.2.10 and the universal property of Verdier quotients (see [**SP**, Tag 05RJ]), what we have to prove is that the composition of $K^{\mathrm{b}}(\widetilde{F})$ with the canonical functor $K^{\mathrm{b}}\mathsf{P}_I \to D^{\mathrm{b}}\mathsf{P}_I$ vanishes on the triangulated subcategory $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$, or in other words that for any $\mathscr{F}$ in $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$ the complex $K^{\mathrm{b}}(\widetilde{F})(\mathscr{F})$ is acyclic. Now, recall that $\widetilde{F}$ takes values in the subcategory $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. By Proposition 4.6.1, to finish the proof, it suffices to show that for any $\mathscr{F}$ in $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$ the complex $K^{\mathrm{b}}(\mathrm{Grad}_{\mathbf{X}^\vee} \circ \widetilde{F})(\mathscr{F})$ is acyclic. However, in view of Lemma 6.3.8, the complex $K^{\mathrm{b}}(\mathrm{Grad}_{\mathbf{X}^\vee} \circ \widetilde{F})(\mathscr{F})$ is the pullback of $\mathscr{F}$ under the embedding $\{e_{\widehat{\mathcal{N}}}\} \hookrightarrow \widehat{\mathcal{N}}_{\mathcal{X}}$. Denoting (temporarily) this map by f, since any object in $\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})$ is flat over $\mathscr{O}_{\widehat{\mathcal{N}}_{\mathcal{X}}}$ we have a commutative diagram

$$\begin{tikzcd}
K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}}) \arrow[r, "K^{\mathrm{b}}(f^*)"] \arrow[d] & K^{\mathrm{b}}\mathsf{Rep}(T^\vee) \arrow[r, "\mathsf{For}^{T^\vee}"] & K^{\mathrm{b}}\mathsf{Vect}_{\overline{\mathbb{Q}}_\ell} \arrow[d] \\
D^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}}) \arrow[rr, "Lf^*"] & & D^{-}\mathsf{Vect}_{\overline{\mathbb{Q}}_\ell},
\end{tikzcd}$$

where the vertical arrows are the obvious functors. Now the bottom arrow factors through the restriction functor $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}(\widehat{\mathcal{N}})$, which kills all objects in $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$. We deduce, as desired, that for any $\mathscr{F}$ in $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee\times T^\vee}_{\mathrm{fr}}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$ the complex $K^{\mathrm{b}}(\mathrm{Grad}_{\mathbf{X}^\vee} \circ \widetilde{F})(\mathscr{F})$ is acyclic. ☐

From (6.3.16) we obtain that for any V in $\mathsf{Rep}(G^\vee)$ and $\lambda \in \mathbf{X}^\vee$ we have

$$F(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) = \mathscr{Z}(V), \qquad F(\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) = \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell). \tag{6.3.18}$$

Remark 6.3.10. — 1. Below we will not really work with the functor F, but rather with its composition with the realization functor

$$\mathrm{real} : D^{\mathrm{b}}\mathsf{P}_I \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell). \tag{6.3.19}$$

Both $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}})$ and $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ are monoidal categories (where the product on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}})$ is the derived tensor product over $\mathscr{O}_{\widetilde{\mathscr{N}}}$). We claim that $\mathrm{real} \circ F$ is canonically equipped with the structure of a monoidal functor. To see this, we must recall some details of the construction of the realization functor (see [**Beĭ1**]; see also [**Rd**] and [**AMRW**, §2.5]). One first considers a filtered version $\widetilde{\mathcal{D}}$ of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$, constructed e.g. using appropriate filtered complexes on acyclic resolutions of Fl_G in the sense of [**BL**]. Then the convolution bifunctor $\star^I$ induces a similar bifunctor on $\widetilde{\mathcal{D}}$ such that the forgetful functor $\widetilde{\mathcal{D}} \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ is monoidal. One considers the full subcategory $\widetilde{\mathcal{A}} \subset \widetilde{\mathcal{D}}$ whose objects are the complexes $\mathscr{F}$ such that $\mathrm{gr}_i(\mathscr{F}) \in \mathsf{P}_I[-i]$ for all $i \in \mathbb{Z}$. Then one observes that $\widetilde{\mathcal{A}}$ is canonically equivalent to $C^{\mathrm{b}}\mathsf{P}_I$, and that the composition

$$C^{\mathrm{b}}\mathsf{P}_I \xleftarrow{\sim} \widetilde{\mathcal{A}} \hookrightarrow \widetilde{\mathcal{D}} \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

factors through $D^{\mathrm{b}}\mathsf{P}_I$, giving rise to (6.3.19). One can similarly consider the subcategory $\widetilde{\mathcal{A}}' \subset \widetilde{\mathcal{D}}$ whose objects are the complexes $\mathscr{F}$ such that $\mathrm{gr}_i(\mathscr{F}) \in \mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)[-i]$ for all $i \in \mathbb{Z}$. Then $\widetilde{\mathcal{A}}'$ is canonically isomorphic to the category $C^{\mathrm{b}}\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$, and the composition

$$C^{\mathrm{b}}\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \xleftarrow{\sim} \widetilde{\mathcal{A}}' \hookrightarrow \widetilde{\mathcal{D}} \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

factors through a functor $K^{\mathrm{b}}\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$, which coincides with the composition of (6.3.19) with the natural functor $K^{\mathrm{b}}\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}\mathsf{P}_I$. In this variant, since $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ is stable under convolution, the subcategory $\widetilde{\mathcal{A}}'$ is stable under the convolution product on $\widetilde{\mathcal{D}}$, and it is not difficult to check that the resulting functor $K^{\mathrm{b}}\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ is monoidal. Hence its composition with the monoidal functor $K^{\mathrm{b}}(\widetilde{F})$ is also monoidal. Then, by construction of F, we deduce that the composition of (6.3.19) with F is monoidal.

2. Let us sketch a slightly different construction of the functor F, which does not rely on Proposition 6.2.10. Since the functor of taking $(G^\vee \times T^\vee)$-fixed points is exact, any object of $\mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathscr{N}}_{\mathcal{X}})$ is projective in the category $\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathscr{N}}_{\mathcal{X}})$. Moreover, any object $\mathscr{F}$ of $\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathscr{N}}_{\mathcal{X}})$ is a quotient of an object of $\mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathscr{N}}_{\mathcal{X}})$: in fact it is a quotient of $V \otimes \mathscr{O}_{\widehat{\mathscr{N}}_{\mathcal{X}}}$ where V is any finite-dimensional $(G^\vee \times T^\vee)$-stable generating subspace of $\Gamma(\widehat{\mathscr{N}}_{\mathcal{X}}, \mathscr{F})$ (as a module over $\mathscr{O}(\widehat{\mathscr{N}}_{\mathcal{X}})$). As a consequence, the natural functor

$$K^-\mathsf{Coh}^{G^\vee \times T^\vee}_{\mathrm{fr}}(\widehat{\mathscr{N}}_{\mathcal{X}}) \to D^-\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathscr{N}}_{\mathcal{X}})$$

is an equivalence of categories. Using this equivalence, from $\widetilde{F}$ we obtain a functor

$$D^{-}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}}) \to K^{-}\mathsf{Perv}_{I}^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell),$$

which we can then restrict to $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})$.

Consider the full subcategory $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}} \subset D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})$ of objects supported set-theoretically on the preimage of $\partial\mathcal{X}$. The arguments in the proof of Proposition 6.3.9 show that the functor so obtained sends any object of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$ to a complex whose image under the associated graded functor is acyclic, hence which is acyclic by Remark 4.6.3. Hence it factors through the Verdier quotient $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})/D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$. Now by [**AriB**, Remark after Lemma 2.12], restriction induces an equivalence

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})/D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}} \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee \times T^\vee}(\widehat{\mathcal{N}}) \cong D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}).$$

Using this equivalence we obtain a functor

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{-}\mathsf{P}_I.$$

Since this functor sends the line bundles $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ to bounded complexes, in view of Lemma 6.2.8 it takes values in $D^{\mathrm{b}}\mathsf{P}_I$.

6.4. Antispherical and Iwahori–Whittaker categories

6.4.1. The antispherical category. — Since the I-orbits on Fl_G are parametrized by W, so are the simple objects in P_I. More precisely, recall that for any $w \in W$ we have the standard and costandard perverse sheaves $\Delta^I_w(\overline{\mathbb{Q}}_\ell)$ and $\nabla^I_w(\overline{\mathbb{Q}}_\ell)$. (In this chapter, for simplicity we will rather denote these objects by Δ^I_w and ∇^I_w respectively.) Then there exists, up to scalar, a unique nonzero morphism $\Delta^I_w \to \nabla^I_w$; its image is simple, and denoted $\mathscr{IC}^I_w$ (see §4.1.2). Then the assignment $w \mapsto \mathscr{IC}^I_w$ induces a bijection between W and the set of isomorphism classes of simple objects in P_I.

We will denote by ${}^{\mathrm{f}}W \subset W$ the subset of elements w which are minimal in $W_{\mathrm{f}}w$. (Here, "minimal" means that the element w is minimal in the Bruhat order among the elements in $W_{\mathrm{f}}w$. The general theory of Coxeter systems guarantees that there exists one such element in each W_{f}-coset, and that this element is also characterized as the unique element of minimal length in $W_{\mathrm{f}}w$.)

In this section, what we call the "antispherical category" is the Serre quotient $\mathsf{P}_I^{\mathrm{asph}}$ of the abelian category P_I by the Serre subcategory generated by the simple objects $\mathscr{IC}^I_w$ with $w \notin {}^{\mathrm{f}}W$. The quotient functor $\mathsf{P}_I \to \mathsf{P}_I^{\mathrm{asph}}$ will be denoted Π_{asph}. (For the construction and main properties of Serre quotients of abelian categories, we refer to [**Gab**].)

6.4.2. The Iwahori–Whittaker category. — Recall the Borel subgroup $B^+ \subset G$ opposite to B, and denote by $I^+ \subset \mathrm{L}^+G$ the corresponding Iwahori subgroup. We also let I^+_{u} be the pro-unipotent radical of I^+, i.e. the preimage of the unipotent radical U^+ of B^+ under the natural map $I^+ \to B^+$. We choose, for each simple root α, an isomorphism between the root subgroup of U^+ associated with α and the additive group $\mathbb{G}_{\mathrm{a},\mathbb{F}}$. We then denote by χ' the group homomorphism obtained as the composition

$$U^+ \to U^+/(U^+,U^+) \cong \prod_{\alpha \text{ simple root}} U_\alpha \cong \prod_{\alpha \text{ simple root}} \mathbb{G}_{\mathrm{a},\mathbb{F}} \to \mathbb{G}_{\mathrm{a},\mathbb{F}}$$

where the last arrow is the sum map. We will also denote by χ the composition of χ' with the projection $I_{\mathrm{u}}^+ \to U^+$.

We fix an Artin–Schreier local system $\mathscr{L}_{\mathrm{AS}}$ on $\mathbb{G}_{\mathrm{a},\mathbb{F}}$. Recall that, concretely, this means that we choose a primitive p-th root of unity ζ in $\overline{\mathbb{Q}}_\ell$, and denote by $\mathscr{L}_{\mathrm{AS}}$ the summand of the pushforward of the constant sheaf under the Galois covering $\mathbb{G}_{\mathrm{a},\mathbb{F}} \to \mathbb{G}_{\mathrm{a},\mathbb{F}}$ defined by $x \mapsto x^p - x$ on which the Galois group $\mathbb{Z}/p\mathbb{Z}$ acts via the character $\overline{n} \mapsto \zeta^n$. The essential property of this local system is that it satisfies

$$\mathsf{H}^\bullet(\mathbb{G}_{\mathrm{a},\mathbb{F}}, \mathscr{L}_{\mathrm{AS}}) = \mathsf{H}^\bullet_c(\mathbb{G}_{\mathrm{a},\mathbb{F}}, \mathscr{L}_{\mathrm{AS}}) = 0. \tag{6.4.1}$$

We then define the derived category $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ of Iwahori–Whittaker sheaves on Fl_G as the $(I_{\mathrm{u}}^+, \chi^*(\mathscr{L}_{\mathrm{AS}}))$-equivariant derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on Fl_G. Concretely, this category will be an inductive limit of subcategories of sheaves on each closed finite union of I_{u}^+-orbits in Fl_G. Given such a union X of orbits, there exists a normal subgroup $K \subset I_{\mathrm{u}}^+$ contained in $\ker(\chi)$ (so that χ factors through a morphism $\chi_K : I_{\mathrm{u}}^+/K \to \mathbb{G}_{\mathrm{a},\mathbb{F}}$) such that I_{u}^+/K is of finite type, and such that the I_{u}^+-action on X factors through an action of I_{u}^+/K. (These considerations are similar to those encountered in [**BR**, §1.16.4].) Then $D^{\mathrm{b}}_{\mathcal{IW}}(X, \overline{\mathbb{Q}}_\ell)$ is defined as the full subcategory of the derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on X whose objects are the complexes $\mathscr{F}$ whose pullback under the action morphism $I_{\mathrm{u}}^+/K \times X \to X$ is isomorphic to $\chi_K^*(\mathscr{L}_{\mathrm{AS}}) \boxtimes \mathscr{F}$. (It is easily checked that this category does not depend on the choice of K up to canonical equivalence.)

Even though this is not clear from the definition recalled above, it turns out that the subcategory $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ of the derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on Fl_G is triangulated; see [**AR1**, Appendix A] for some details. It is then easily seen that the perverse t-structure on the derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on Fl_G restricts to a t-structure on $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ (still called the perverse t-structure). The heart of this t-structure will be denoted $\mathsf{P}_{\mathcal{IW}}$.

Note that the same constructions as for the convolution product $\star^I$ on $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ can be used to define a bifunctor

$$(-) \star^I (-) : D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \times D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

which makes $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ into a right module category over the monoidal category $(D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell), \star^I)$.

6.4.3. Statement. — The I_{u}^+-orbits on Fl_G are parametrized in the standard way by W. Those which support a nonzero $(I_{\mathrm{u}}^+, \chi^*(\mathscr{L}_{\mathrm{AS}}))$-equivariant local system are those corresponding to elements in the subset ${}^{\mathrm{f}}W \subset W$. For any $\lambda \in \mathbf{X}$ we will denote by w_λ the minimal length element in $W_{\mathrm{f}} \cdot \mathsf{t}(\lambda)$, by $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ the corresponding orbit, and by $\mathscr{L}_{\chi,\lambda}$ the unique rank-1 $(I_{\mathrm{u}}^+, \chi^*(\mathscr{L}_{\mathrm{AS}}))$-equivariant local system on $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$. We also denote by $j_\lambda^{\mathcal{IW}} : \mathrm{Fl}_{G,\lambda}^{\mathcal{IW}} \to \mathrm{Fl}_G$ the embedding, and set

$$\Delta_\lambda^{\mathcal{IW}} := (j_\lambda^{\mathcal{IW}})_! \mathscr{L}_{\chi,\lambda}[\dim(\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}})], \quad \nabla_\lambda^{\mathcal{IW}} := (j_\lambda^{\mathcal{IW}})_* \mathscr{L}_{\chi,\lambda}[\dim(\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}})].$$

(These objects are perverse sheaves by [**BBDG**, Corollaire 4.1.3], since $j_\lambda^{\mathcal{IW}}$ is an affine morphism.) As usual we have

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta_\lambda^{\mathcal{IW}}, \nabla_\mu^{\mathcal{IW}}[n]) \cong \begin{cases} \overline{\mathbb{Q}}_\ell & \text{if } \lambda = \mu \text{ and } n = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{6.4.2}$$

Moreover, the image of any nonzero morphism $\Delta_\lambda^{\mathcal{IW}} \to \nabla_\lambda^{\mathcal{IW}}$ is simple; this simple object will be denoted $\mathscr{IC}_\lambda^{\mathcal{IW}}$. It is well known (and follows from the techniques of [**BGS**, §§3.2–3.3]) that the category $\mathsf{P}_{\mathcal{IW}}$ has a natural structure of a highest weight category (in the sense considered in [**BR**, §1.12.3]), with weight poset $\mathbf{X}$ (for the order defined by $\lambda \preceq \mu$ iff $\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}} \subset \overline{\mathrm{Fl}_{G,\mu}^{\mathcal{IW}}}$) and standard, resp. costandard, objects $(\Delta_\lambda^{\mathcal{IW}} : \lambda \in \mathbf{X})$, resp. $(\nabla_\lambda^{\mathcal{IW}} : \lambda \in \mathbf{X})$.

Remark 6.4.1. — For any $\lambda \in \mathbf{X}^\vee$, the orbit $\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}}$ is dense in the L^+G-orbit $\mathrm{L}^+G \cdot z^\lambda I/I$. Therefore, the inclusion relations among these orbits is closely related to the inclusion relations among I-orbits in Gr_G, which is discussed e.g. in [**AR2**, §9.4]. In particular, according to [**AR2**, Lemma 9.12], if $\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}} \subset \overline{\mathrm{Fl}_{G,\mu}^{\mathcal{IW}}}$, then:

1. either $\mu - \lambda \in \mathbb{Z}\mathfrak{R}$, and λ belongs to the convex hull of $W_{\mathrm{f}}(\mu)$ but not to $W_{\mathrm{f}}(\mu)$
2. or $\lambda \in W_{\mathrm{f}}(\mu)$, and if ν is the unique dominant weight in $W_{\mathrm{f}}(\mu)$ and if $x, y \in W_{\mathrm{f}}$ are the unique elements of minimal length such that $\lambda = x(\nu)$ and $\mu = y(\nu)$ respectively, we have $y \leq_{\mathrm{Bru}} x$.

It also follows from [**AR2**, Lemma 9.12] that in the case where $\mu \in \mathbf{X}^\vee_+$, the coweights λ such that $\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}} \subset \overline{\mathrm{Fl}_{G,\mu}^{\mathcal{IW}}}$ are exactly those such that $\mathsf{N}(\mu)_\lambda \neq 0$.

Note that since the closure $\overline{\mathrm{Fl}_{G,0}^{\mathcal{IW}}} = \mathrm{L}^+G/I \cong G/B$ does not contain any orbit $\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}}$ with $\lambda \neq 0$, the natural morphism $\Delta_0^{\mathcal{IW}} \to \nabla_0^{\mathcal{IW}}$ is an isomorphism. We can then consider the functor

$$\mathsf{Av}_{\mathcal{IW}} : D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

defined by

$$\mathsf{Av}_{\mathcal{IW}}(\mathscr{F}) = \Delta_0^{\mathcal{IW}} \star^I \mathscr{F}.$$

The main result of the present section is the following statement.

Theorem 6.4.2. —
1. *The functor $\mathsf{Av}_{\mathcal{IW}}$ is t-exact with respect to the perverse t-structures on $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ and $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$.*
2. *The restriction of this functor to the hearts of these t-structures factors through a fully faithful functor $\mathsf{P}_I^{\mathrm{asph}} \to \mathsf{P}_{\mathcal{IW}}$.*

Remark 6.4.3. — We will see much later (see Corollary 6.6.2) that the functor $\mathsf{P}_I^{\mathrm{asph}} \to \mathsf{P}_{\mathcal{IW}}$ in Theorem 6.4.2(2) is in fact an equivalence of categories.

6.4.4. Some preliminaries. — In order to prove Theorem 6.4.2 we will need some preparation. We begin with the following lemma.

Lemma 6.4.4. — *For $w \in W$ we have $\mathsf{Av}_{\mathcal{IW}}(\mathscr{IC}^I_w) = 0$ unless $w \in {}^{\mathrm{f}}W$.*

Proof. — By definition we have $\mathsf{Av}_{\mathcal{IW}}(\mathscr{IC}^I_w) = \Delta^{\mathcal{IW}}_0 \star^I \mathscr{IC}^I_w$. Now if $w \notin {}^{\mathrm{f}}W$, there exists a simple reflection s in W_{f} such that $sw < w$. Let $J_s \subset \mathrm{L}^+G$ denote the corresponding "parahoric" subgroup; i.e., the preimage under $\mathrm{L}^+G \to G$ of the parabolic subgroup of G containing B and associated with the subset $\{s\}$ of the simple reflections. Then the perverse sheaf $\mathscr{IC}^I_w$ is J_s-equivariant. In particular, there exists a complex $\mathscr{F}$ in $D^{\mathrm{b}}_{J_s}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ such that $\mathscr{IC}^I_w = \mathsf{For}^{J_s}_I(\mathscr{F})$. (Here, $\mathsf{For}^{J_s}_I : D^{\mathrm{b}}_{J_s}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ is the natural forgetful functor.) Then by standard properties of the convolution construction we have

$$\Delta^{\mathcal{IW}}_0 \star^I \mathscr{IC}^I_w \cong ((\pi_s)_* \Delta^{\mathcal{IW}}_0) \star^{J_s} \mathscr{F}$$

where $\pi_s : \mathrm{Fl}_G \to \mathrm{L}G/J_s$ is the quotient morphism and $\star^{J_s}$ is the convolution product of complexes on $\mathrm{L}G/J_s$ with J_s-equivariant complexes on Fl_G. Now using (6.4.1) one can check that $(\pi_s)_* \Delta^{\mathcal{IW}}_0 = 0$, which finishes the proof. □

Lemma 6.4.5. — *For any $w \in W$ we have*

$$\mathsf{Av}_{\mathcal{IW}}(\Delta^I_w) \cong \Delta^{\mathcal{IW}}_\lambda \quad \text{and} \quad \mathsf{Av}_{\mathcal{IW}}(\nabla^I_w) \cong \nabla^{\mathcal{IW}}_\lambda,$$

where $\lambda \in \mathbf{X}$ is the unique element such that $W_{\mathrm{f}} \cdot w = W_{\mathrm{f}} \cdot w_\lambda$ (or equivalently $W_{\mathrm{f}} \cdot w = W_{\mathrm{f}} \cdot \mathsf{t}(\lambda)$).

Proof. — When $w \in {}^{\mathrm{f}}W$, the claim follows (using arguments similar to those encountered in the proof of Lemma 4.1.4) from the observation that the multiplication map induces an isomorphism $\mathrm{Fl}^{\mathcal{IW}}_{G,0} \mathbin{\widetilde{\times}} \mathrm{Fl}_{G,w} \xrightarrow{\sim} \mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ (where $w = w_\lambda$), together with the fact that $\Delta^{\mathcal{IW}}_0 \cong \nabla^{\mathcal{IW}}_0$. (Here, following the same conventions as in §1.3.1, we write $\mathrm{Fl}^{\mathcal{IW}}_{G,0} \mathbin{\widetilde{\times}} \mathrm{Fl}_{G,w}$ for $(p_{\mathrm{Gr}})^{-1}(\mathrm{Fl}^{\mathcal{IW}}_{G,0}) \times^{\mathrm{L}^+G} \mathrm{Fl}_{G,w}$.)

To deduce the first isomorphism in the general case, one proves that there exists a morphism $\Delta^I_{w_\lambda} \to \Delta^I_w$ whose cone is killed by $\mathsf{Av}_{\mathcal{IW}}$. Indeed, write $w = xw_\lambda$ with $x \in W_{\mathrm{f}}$. By Lemma 4.1.3, there exists an embedding $\mathscr{IC}^I_e \hookrightarrow \Delta^I_x$ whose cokernel has only composition factors of the form $\mathscr{IC}^I_y$ with $y \in W_{\mathrm{f}} \smallsetminus \{e\}$. Convolving on the right with $\Delta^I_{w_\lambda}$ and using Lemma 4.1.4(1) we obtain a morphism $\Delta^I_{w_\lambda} \to \Delta^I_w$ whose cone belongs to the triangulated subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ generated by the objects of the form $\mathscr{IC}^I_y \star^I \Delta^I_{w_\lambda}$ with $y \in W_{\mathrm{f}}$. Hence, to prove that this cone is indeed killed by $\mathsf{Av}_{\mathcal{IW}}$, we only have to note that in view of Lemma 6.4.4 we have $\mathsf{Av}_{\mathcal{IW}}(\mathscr{IC}^I_y) = 0$ for such y; it follows that

$$\mathsf{Av}_{\mathcal{IW}}(\mathscr{IC}^I_y \star^I \Delta^I_{w_\lambda}) = \mathsf{Av}_{\mathcal{IW}}(\mathscr{IC}^I_y) \star^I \Delta^I_{w_\lambda} = 0,$$

which implies our claim.

The proof of the second isomorphism is similar. □

Remark 6.4.6. — It is not difficult to show that the morphism $\Delta^I_{w_\lambda} \to \Delta^I_w$ considered in the proof of Lemma 6.4.5 is in fact an embedding of perverse sheaves (but this fact will not be needed here).

Corollary 6.4.7. — *The functor* $\mathsf{Av}_{\mathcal{IW}}$ *is t-exact. Moreover, for any* $\lambda \in \mathbf{X}$ *we have*

$$\mathsf{Av}_{\mathcal{IW}}(\mathscr{IC}^I_{w_\lambda}) \cong \mathscr{IC}^{\mathcal{IW}}_\lambda.$$

Proof. — The first claim follows from Lemma 6.4.5 and the fact that the nonpositive, resp. nonnegative, part of the perverse t-structure on $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ is the subcategory generated under extensions by objects of the form $\Delta^I_w[n]$ with $w \in W$ and $n \in \mathbb{Z}_{\geq 0}$, resp. by the objects of the form $\nabla^I_w[n]$ with $w \in W$ and $n \in \mathbb{Z}_{\leq 0}$.

Once this fact is known, we observe that $\mathscr{IC}^I_{w_\lambda}$ is the image of any nonzero morphism $f : \Delta^I_{w_\lambda} \to \nabla^I_{w_\lambda}$; hence its image under $\mathsf{Av}_{\mathcal{IW}}$ is the image of $\mathsf{Av}_{\mathcal{IW}}(f)$. By Lemma 6.4.5 this morphism identifies with a morphism $\Delta^{\mathcal{IW}}_\lambda \to \nabla^{\mathcal{IW}}_\lambda$, and by considering its restriction to $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ we see that $\mathsf{Av}_{\mathcal{IW}}(f) \neq 0$. It follows that $\mathsf{Av}_{\mathcal{IW}}(\mathscr{IC}^I_{w_\lambda}) \cong \mathscr{IC}^{\mathcal{IW}}_\lambda$, as desired. □

6.4.5. Proof of Theorem 6.4.2. — By Lemma 6.4.4 and the universal property of Serre quotients, the restriction of $\mathsf{Av}_{\mathcal{IW}}$ to the hearts of the perverse t-structures factors through an exact functor

$$\mathsf{Av}^{\mathrm{asph}}_{\mathcal{IW}} : \mathsf{P}^{\mathrm{asph}}_I \to \mathsf{P}_{\mathcal{IW}}.$$

The only thing that remains to be seen is that $\mathsf{Av}^{\mathrm{asph}}_{\mathcal{IW}}$ is fully faithful.

For this, denote by I_0 the intersection of I and I^+_{u}, or in other words the kernel of the natural map $\mathrm{L}^+G \to G$. We consider the associated equivariant derived category $D^{\mathrm{b}}_{I_0}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$, and the induction functor

$$^*\mathrm{Ind}^I_{I_0} : D^{\mathrm{b}}_{I_0}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

sending a complex $\mathscr{F}$ to $a_*(\underline{\overline{\mathbb{Q}}_\ell} \widetilde{\boxtimes} \mathscr{F})$, where $a : I \times^{I_0} \mathrm{Fl}_G \to \mathrm{Fl}_G$ is the action morphism and $\underline{\overline{\mathbb{Q}}_\ell} \widetilde{\boxtimes} \mathscr{F}$ is the unique complex whose pullback to $I \times \mathrm{Fl}_G$ is $\underline{\overline{\mathbb{Q}}_\ell} \boxtimes \mathscr{F}$.

Lemma 6.4.8. — *There exists a morphism*

$$\mathscr{IC}^I_e \to {}^*\mathrm{Ind}^I_{I_0}(\Delta^{\mathcal{IW}}_0)[-\dim(T)]$$

whose cone belongs to the subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ *generated under extensions by the objects of the form* $\mathscr{IC}^I_w[n]$ *with* $(w, n) \in W_{\mathrm{f}} \times \mathbb{Z}_{\leq 0}$ *and either* $n < 0$ *or* $n = 0$ *and* $w \neq e$.

Proof. — The L^+G-action on the base point of Fl_G induces an isomorphism between $\overline{\mathrm{Fl}_{G,w_\circ}}$ and the flag variety G/B, which induces equivalences of categories

$$D^{\mathrm{b}}_I(\overline{\mathrm{Fl}_{G,w_\circ}}, \overline{\mathbb{Q}}_\ell) \cong D^{\mathrm{b}}_B(G/B, \overline{\mathbb{Q}}_\ell),$$
$$D^{\mathrm{b}}_{(I^+_{\mathrm{u}}, \chi^*(\mathscr{L}_{\mathrm{AS}}))}(\overline{\mathrm{Fl}_{G,w_\circ}}, \overline{\mathbb{Q}}_\ell) \cong D^{\mathrm{b}}_{(U^+, (\chi')^*(\mathscr{L}_{\mathrm{AS}}))}(G/B, \overline{\mathbb{Q}}_\ell).$$

Using these equivalences, we can apply [**BeR2**, Lemma 12.1 and Remark 12.2(1)], which implies that the complex $^*\mathrm{Ind}^I_{I_0}(\Delta^{\mathcal{IW}}_0)$ is concentrated in perverse degrees $\geq -\dim(T)$, and that moreover we have

$${}^{\mathrm{p}}\mathscr{H}^{-\dim(T)}\left({}^*\mathrm{Ind}^I_{I_0}(\Delta^{\mathcal{IW}}_0)\right) \cong \Delta^I_{w_\circ}.$$

The claim follows, since by Lemma 4.1.3 there exists an embedding $\mathscr{IC}^I_e \hookrightarrow \Delta^I_{w_\circ}$ whose cokernel has all of its composition factors of the form $\mathscr{IC}^I_w$ with $\ell(w) > 0$. □

Remark 6.4.9. — The proof of this lemma in [**AB**] is incomplete. Filling this gap was one of the motivations for the work performed in [**BeR2**].

Now we set

$$\Xi := \Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^0\big({}^*\mathrm{Ind}^I_{I_0}(-)[-\dim(T)]\big) : \mathsf{P}_{\mathcal{IW}} \to \mathsf{P}^{\mathrm{asph}}_I,$$

where we omit the forgetful functor $\mathsf{P}_{\mathcal{IW}} \to D^{\mathrm{b}}_{I_0}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$.

Lemma 6.4.10. — *There exists an isomorphism of functors* $\Xi \circ \mathsf{Av}^{\mathrm{asph}}_{\mathcal{IW}} \cong \mathrm{id}$.

Proof. — The proof will exploit the following observation. Let $\mathscr{A}$ be an abelian category, let $\mathscr{B} \subset \mathscr{A}$ be a Serre subcategory, and let $\mathscr{C}$ be another category. Denote by $\Pi : \mathscr{A} \to \mathscr{A}/\mathscr{B}$ the quotient functor. Given functors $X_1, X_2 : \mathscr{A}/\mathscr{B} \to \mathscr{C}$, then defining a morphism of functors $X_1 \to X_2$ is equivalent to defining a morphism of functors $X_1 \circ \Pi \to X_2 \circ \Pi$. In fact, given a morphism $X_1 \to X_2$ we obtain a morphism $X_1 \circ \Pi \to X_2 \circ \Pi$ by composing with Π. And conversely, given a morphism $\varphi : X_1 \circ \Pi \to X_2 \circ \Pi$, since the objects in $\mathscr{A}/\mathscr{B}$ are the same as those of $\mathscr{A}$ we can obtain a morphism of functors $X_1 \to X_2$ by declaring that the morphism $X_1(A) \to X_2(A)$ is the morphism $\varphi(A) : X_1 \circ \Pi(A) \to X_2 \circ \Pi(A)$ for any A in $\mathscr{A}/\mathscr{B}$, where in the latter case A is regarded as an object in $\mathscr{A}$. (It is left to the reader to check that this indeed defines a morphism of functors $X_1 \to X_2$, and that these two constructions are inverse to each other.)

From this remark we deduce that to prove the lemma it suffices to construct an isomorphism of functors $\Pi_{\mathrm{asph}} \xrightarrow{\sim} \Xi \circ \mathsf{Av}^{\mathrm{asph}}_{\mathcal{IW}} \circ \Pi_{\mathrm{asph}}$.

By definition, and since the functor ${}^*\mathrm{Ind}^I_{I_0}$ commutes with convolution on the right, for $\mathscr{F}$ in P_I we have

$$\begin{aligned}\Xi \circ \mathsf{Av}^{\mathrm{asph}}_{\mathcal{IW}}(\Pi_{\mathrm{asph}}(\mathscr{F})) &= \Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^0\big({}^*\mathrm{Ind}^I_{I_0}(\Delta^{\mathcal{IW}}_0 \star^I \mathscr{F})[-\dim(T)]\big) \\ &\cong \Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^0\big({}^*\mathrm{Ind}^I_{I_0}(\Delta^{\mathcal{IW}}_0) \star^I \mathscr{F}[-\dim(T)]\big).\end{aligned}$$

Now we claim that if $\mathscr{G}$ belongs to the subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ generated under extensions by objects of the form $\mathscr{IC}^I_w[n]$ with $(w,n) \in W_{\mathrm{f}} \times \mathbb{Z}_{\leq 0}$ and either $n < 0$ or $n = 0$ and $w \neq e$, then we have

$$\Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^{-1}(\mathscr{G} \star^I \mathscr{F}) = \Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{G} \star^I \mathscr{F}) = 0.$$

Indeed, to prove this we can assume that $\mathscr{G} = \mathscr{IC}^I_w[n]$ with (w,n) as above. If $w \neq e$, and if $s \in S_{\mathrm{f}}$ is a simple reflection such that $sw < w$, then $\mathscr{G}$ belongs to the essential image of the forgetful functor $D^{\mathrm{b}}_{J_s}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$, where J_s is as in the proof of Lemma 6.4.4. Hence the same property holds for $\mathscr{G} \star^I \mathscr{F}$, and then for its perverse cohomology objects, which implies that $\Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^m(\mathscr{G} \star^I \mathscr{F}) = 0$ for any $m \in \mathbb{Z}$. Now if $w = e$ then $n < 0$, so that $\Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^m(\mathscr{G} \star^I \mathscr{F}) = \Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^{n+m}(\mathscr{F}) = 0$ for $m \in \{0, -1\}$ (and even for any $m \in \mathbb{Z}_{\leq 0}$).

Now we consider a morphism $\mathscr{I}\mathscr{C}_e^I \to {}^*\mathrm{Ind}_{I_0}^I(\Delta_0^{\mathcal{IW}})[-\dim(T)]$ as in Lemma 6.4.8, and denote its cone by $\mathscr{G}$. Convolving with $\mathscr{F}$ on the right we obtain a distinguished triangle

$$\mathscr{F} \to {}^*\mathrm{Ind}_{I_0}^I(\Delta_0^{\mathcal{IW}}) \star^I \mathscr{F}[-\dim(T)] \to \mathscr{G} \star^I \mathscr{F} \xrightarrow{[1]},$$

and then applying the cohomological functor $\Pi_{\mathrm{asph}} \circ {}^{\mathrm{p}}\mathscr{H}^0$ and using the claim of the previous paragraph we deduce a functorial isomorphism

$$\Pi_{\mathrm{asph}}(\mathscr{F}) \xrightarrow{\sim} \Xi \circ \mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(\Pi_{\mathrm{asph}}(\mathscr{F})),$$

which finishes the proof. □

Corollary 6.4.11. — *For any X, Y in $\mathsf{P}_I^{\mathrm{asph}}$ the morphism*

$$\mathrm{Ext}^1_{\mathsf{P}_I^{\mathrm{asph}}}(X,Y) \to \mathrm{Ext}^1_{\mathsf{P}_{\mathcal{IW}}}(\mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(X), \mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(Y))$$

induced by $\mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}$ is injective.

Proof. — A nonzero element in $\mathrm{Ext}^1_{\mathsf{P}_I^{\mathrm{asph}}}(X,Y)$ corresponds to a nonsplit exact sequence $Y \hookrightarrow Z \twoheadrightarrow X$, and its image under $\mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}$ is the class of the exact sequence $\mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(Y) \hookrightarrow \mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(Z) \twoheadrightarrow \mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(X)$. The latter exact sequence cannot split, since any choice of splitting would provide, by applying Ξ and using Lemma 6.4.10, a splitting of the original exact squence. □

We can finally finish the proof of Theorem 6.4.2.

Proof of Theorem 6.4.2. — Recall that what we have to prove is that the functor $\mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}$ is fully faithful, or in other words that for any X, Y in $\mathsf{P}_I^{\mathrm{asph}}$ the induced morphism

$$\mathrm{Hom}_{\mathsf{P}_I^{\mathrm{asph}}}(X,Y) \to \mathrm{Hom}_{\mathsf{P}_{\mathcal{IW}}}(\mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(X), \mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(Y))$$

is an isomorphism. We prove this by induction on the sum of the lengths of X and Y. If X and Y are simple then these objects are images of simple objects in P_I, sowor $\mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(X)$ and $\mathsf{Av}_{\mathcal{IW}}^{\mathrm{asph}}(Y)$ are also simple by Corollary 6.4.7, and moreover they are isomorphic iff X and Y are; so the claim is clear in this case. The general case follows using Corollary 6.4.11 and the 5-lemma. □

6.5. Central sheaves and tilting Iwahori–Whittaker perverse sheaves

6.5.1. Statement. — We now introduce the notation

$$\mathscr{Z}^{\mathcal{IW}} := \mathsf{Av}_{\mathcal{IW}} \circ \mathscr{Z} : \mathsf{Rep}(G^\vee) \to \mathsf{P}_{\mathcal{IW}}.$$

Recall (see §6.4.3) that the category $\mathsf{P}_{\mathcal{IW}}$ has a natural structure of a highest weight category. In such a category one can consider the *tilting* objects, i.e. those which possess both a filtration whose subquotients are standard objects, and a filtration whose subquotients are costandard objects. By the general theory of highest weight categories, if $\mathscr{F}$ is a tilting object, then the number of occurrences of a given standard object $\Delta_\lambda^{\mathcal{IW}}$, resp. of a given costandard object $\nabla_\lambda^{\mathcal{IW}}$, in a filtration with standard subquotients, resp. with costandard subquotients, does not depend on the choice of filtration; it will be denoted

$$(\mathscr{F} : \Delta_\lambda^{\mathcal{IW}}), \quad \text{resp.} \quad (\mathscr{F} : \nabla_\lambda^{\mathcal{IW}}).$$

In fact we have

$$\dim \operatorname{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)}(\Delta_\lambda^{\mathcal{IW}}, \mathscr{F}) = (\mathscr{F} : \nabla_\lambda^{\mathcal{IW}}); \tag{6.5.1}$$

$$\dim \operatorname{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)}(\mathscr{F}, \nabla_\lambda^{\mathcal{IW}}) = (\mathscr{F} : \Delta_\lambda^{\mathcal{IW}}). \tag{6.5.2}$$

Another general property that will be needed below is that a direct summand of a tilting object is again tilting.

The following lemma is easy to check (see [**BBM**] for the origin of these ideas).

Lemma 6.5.1. — *Let $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. Then $\mathscr{F}$ is a tilting object in $\mathsf{P}_{\mathcal{IW}}$ iff for any $\lambda \in \mathbf{X}^\vee$ the complexes $(j_\lambda^{\mathcal{IW}})^* \mathscr{F}$ and $(j_\lambda^{\mathcal{IW}})^! \mathscr{F}$ are concentrated in degree $-\dim(\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}})$.*

The main result of the present section plays a key role in proving the main equivalence, but it is also interesting in its own right: it stipulates that the perverse sheaves $\mathscr{Z}^{\mathcal{IW}}(V)$ are tilting for all V in $\mathsf{Rep}(G^\vee)$.

Theorem 6.5.2. — *For any V in $\mathsf{Rep}(G^\vee)$, the perverse sheaf $\mathscr{Z}^{\mathcal{IW}}(V)$ is tilting. Moreover, for any $\lambda \in \mathbf{X}^\vee$ we have*

$$(\mathscr{Z}^{\mathcal{IW}}(V) : \Delta_\lambda^{\mathcal{IW}}) = (\mathscr{Z}^{\mathcal{IW}}(V) : \nabla_\lambda^{\mathcal{IW}}) = \dim(V_\lambda). \tag{6.5.3}$$

The strategy of proof of this theorem will be to first prove it "by hand" for some "explicit enough" representations, and then propagate this result by taking tensor products.

Remark 6.5.3. —

1. It is natural to ask whether the perverse sheaf $\mathscr{Z}^{\mathcal{IW}}(V)$ is indecomposable if V is simple. We will see that this is indeed the case (see §6.6.4), but only after the Arkhipov–Bezrukavnikov equivalence is proved.
2. Theorem 6.5.2 should be seen as a statement which is "Koszul dual" to Lemma 2.5.1. Namely, one can define a "Koszul duality" autoequivalence of a certain category closely related with the category of I-constructible $\overline{\mathbb{Q}}_\ell$-sheaves on Fl_G, see [**BeY**]. This construction has a "parabolic–singular" analogue relating I-constructible sheaves on Gr_G and Iwahori–Whittaker sheaves on Fl_G, and these equivalences intertwine the functor π_* and a certain "averaging" functor similar to $\mathsf{Av}_{\mathcal{IW}}$. It was suggested by M. Finkelberg (see [**AB**, Footnote on p. 174]) that the appropriate lifts of the central sheaves $\mathscr{Z}(V)$ should be fixed by Koszul duality. Combining all these considerations, one sees Theorem 6.5.2 as the mirror of Lemma 2.5.1 under Koszul duality.

6.5.2. Computing multiplicities. — We start by explaining how to deduce the equalities (6.5.3) from the fact that $\mathscr{Z}^{\mathcal{IW}}(V)$ is tilting. The argument will be based on the following proposition.

Proposition 6.5.4. — *For any V in $\mathsf{Rep}(G^\vee)$ and any $\mu \in \mathbf{X}^\vee$ we have*

$$\sum_{i\geq 0}(-1)^i \cdot \dim\bigl(\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta^{\mathcal{IW}}_\mu, \mathscr{Z}^{\mathcal{IW}}(V)[i])\bigr) = \dim(V_\mu),$$
$$\sum_{i\geq 0}(-1)^i \cdot \dim\bigl(\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\mathscr{Z}^{\mathcal{IW}}(V), \nabla^{\mathcal{IW}}_\mu[i])\bigr) = \dim(V_\mu).$$

Proof. — We prove the first equality; the second one can be justified similarly. It is clear that the assignment $\mathscr{F} \mapsto \sum_{i\geq 0}(-1)^i \cdot \dim\bigl(\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta^{\mathcal{IW}}_\mu, \mathscr{F}[i])\bigr)$ factors through the Grothendieck group of $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. Now by Lemma 4.8.1, in the Grothendieck group of P_I we have

$$[\mathscr{Z}(V)] = \sum_{\lambda\in\mathbf{X}^\vee} \dim(V_{w_\circ(\lambda)}) \cdot [\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)] = \sum_{\lambda\in\mathbf{X}^\vee} \dim(V_\lambda) \cdot [\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)].$$

In view of Lemma 4.1.9, this implies that

$$[\mathscr{Z}(V)] = \sum_{\lambda\in\mathbf{X}^\vee} \dim(V_\lambda) \cdot [\nabla^I_{\mathsf{t}(\lambda)}],$$

and then (using Lemma 6.4.5) that

$$[\mathscr{Z}^{\mathcal{IW}}(V)] = \sum_{\lambda\in\mathbf{X}^\vee} \dim(V_\lambda) \cdot [\nabla^{\mathcal{IW}}_\lambda].$$

The claim follows, since we have

$$\sum_{i\geq 0}(-1)^i \cdot \dim\bigl(\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta^{\mathcal{IW}}_\mu, \nabla^{\mathcal{IW}}_\lambda[i])\bigr) = \delta_{\lambda,\mu}$$

by (6.4.2). □

Remark 6.5.5. — By the same arguments as in the proof of Lemma 4.1.9, Proposition 6.5.4 implies that for any $\lambda \in \mathbf{X}^\vee_+$ the perverse sheaf $\mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\lambda))$ is supported on $\overline{\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}}$. (Moreover, for $\mu \in \mathbf{X}^\vee$, this closure contains the orbit $\mathrm{Fl}^{\mathcal{IW}}_{G,\mu}$ iff $\mathsf{N}(\lambda)_\mu \neq 0$: see Remark 6.4.1.)

Corollary 6.5.6. — *Let V in $\mathsf{Rep}(G^\vee)$, and assume that $\mathscr{Z}^{\mathcal{IW}}(V)$ is tilting. Then for any $\lambda \in \mathbf{X}^\vee$ we have*

$$(\mathscr{Z}^{\mathcal{IW}}(V) : \Delta^{\mathcal{IW}}_\lambda) = (\mathscr{Z}^{\mathcal{IW}}(V) : \nabla^{\mathcal{IW}}_\lambda) = \dim(V_\lambda).$$

Proof. — From (6.4.2) we deduce that under our assumption we have

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta^{\mathcal{IW}}_\lambda, \mathscr{Z}^{\mathcal{IW}}(V)[i]) = \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\mathscr{Z}^{\mathcal{IW}}(V), \nabla^{\mathcal{IW}}_\lambda[i]) = 0$$

for any $i > 0$. Moreover, by (6.5.1)–(6.5.2) we have

$$\dim \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta^{\mathcal{IW}}_\lambda, \mathscr{Z}^{\mathcal{IW}}(V)) = (\mathscr{Z}^{\mathcal{IW}}(V) : \nabla^{\mathcal{IW}}_\lambda);$$
$$\dim \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\mathscr{Z}^{\mathcal{IW}}(V), \nabla^{\mathcal{IW}}_\lambda) = (\mathscr{Z}^{\mathcal{IW}}(V) : \Delta^{\mathcal{IW}}_\lambda).$$

Then the claim follows from Proposition 6.5.4. □

Once this corollary is proved, to complete the proof of Theorem 6.5.2 it suffices to prove that $\mathscr{Z}^{\mathcal{IW}}(V)$ is tilting for all V in $\mathsf{Rep}(G^\vee)$.

6.5.3. Propagation through tensor products. — The following proposition is the key fact that will allow us to reduce the proof of the first claim in Theorem 6.5.2 to certain special cases.

Proposition 6.5.7. — *If V, V' are objects in $\mathsf{Rep}(G^\vee)$ such that $\mathscr{Z}^{\mathcal{IW}}(V)$ and $\mathscr{Z}^{\mathcal{IW}}(V')$ are tilting, then $\mathscr{Z}^{\mathcal{IW}}(V \otimes V')$ is tilting.*

Before proving this proposition, we start with the following well-known result.

Lemma 6.5.8. — *For any $x, y \in W$, the object $\Delta^I_x \star^I \Delta^I_y$ belongs to the full subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ generated under extensions by the objects of the form $\Delta^I_z[n]$ with $z \in W$ and $n \leq 0$.*

Dually, for any $x, y \in W$, the object $\nabla^I_x \star^I \nabla^I_y$ belongs to the full subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ generated under extensions by the objects of the form $\nabla^I_z[n]$ with $z \in W$ and $n \geq 0$.

Proof. — We prove the first claim; the second one can be obtained similarly. The proof proceeds by induction on $\ell(y)$. If $\ell(y) = 0$ then we have $\Delta^I_x \star^I \Delta^I_y \cong \Delta^I_{xy}$ by Lemma 4.1.4(1), so that the claim is clear. Otherwise, choose $s \in S$ such that $sy < y$; then we have $\Delta^I_y \cong \Delta^I_s \star^I \Delta^I_{sy}$ again by Lemma 4.1.4(1), so that

$$\Delta^I_x \star^I \Delta^I_y \cong (\Delta^I_x \star^I \Delta^I_s) \star^I \Delta^I_{sy}.$$

If $xs > x$ then we have $\Delta^I_x \star^I \Delta^I_s \cong \Delta^I_{xs}$ (once again by Lemma 4.1.4(1)), and the desired claim follows by induction. Otherwise, using the exact sequences of perverse sheaves

$$\mathscr{IC}^I_e \hookrightarrow \Delta^I_s \twoheadrightarrow \mathscr{IC}^I_s \quad \text{and} \quad \mathscr{IC}^I_s \hookrightarrow \nabla^I_s \twoheadrightarrow \mathscr{IC}^I_e$$

on $\overline{\mathrm{Fl}_{G,s}} \cong \mathbb{P}^1$ and the fact that $\Delta^I_x \star^I \nabla^I_s \cong \Delta^I_{xs}$ (as follows from Lemma 4.1.4(1)–(3)), we obtain distinguished triangles

$$\Delta^I_x \to \Delta^I_x \star^I \Delta^I_s \to \Delta^I_x \star^I \mathscr{IC}^I_s \xrightarrow{[1]} \quad \text{and} \quad \Delta^I_x \star^I \mathscr{IC}^I_s \to \Delta^I_{xs} \to \Delta^I_x \xrightarrow{[1]}.$$

These triangles show that $\Delta^I_x \star^I \Delta^I_s$ belongs to the full subcategory of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ generated under extensions by the objects of the form $\Delta^I_z[n]$ with $z \in W$ and $n \leq 0$. Once again, we deduce the same property for $\Delta^I_x \star^I \Delta^I_y$ using induction. □

Proof of Proposition 6.5.7. — In view of Lemma 6.5.1, what we have to prove is that for any $\lambda \in \mathbf{X}^\vee$ the restriction and the corestriction of $\mathsf{Av}_{\mathcal{IW}}(\mathscr{Z}(V \otimes V'))$ to $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ are both concentrated in perverse degree 0. First, since this object is perverse its restriction is concentrated in degrees ≤ 0, and its corestriction in degrees ≥ 0.

Now, by assumption $\mathsf{Av}_{\mathcal{IW}}(\mathscr{Z}(V))$ admits a filtration with subquotients of the form $\Delta^{\mathcal{IW}}_\mu$; therefore $\mathsf{Av}_{\mathcal{IW}}(\mathscr{Z}(V \otimes V')) \cong \mathsf{Av}_{\mathcal{IW}}(\mathscr{Z}(V)) \star^I \mathscr{Z}(V')$ admits a filtration (in the sense of triangulated categories[5]) with subquotients of the form $\Delta^{\mathcal{IW}}_\mu \star^I \mathscr{Z}(V')$.

[5] We say that an object admits "a filtration in the sense of triangulated categories with subquotients in a set C" if it lies in the smallest additive subcategory that contains C and is closed under extensions.

We observe that

$$\Delta^{\mathcal{IW}}_{\mu} \star^I \mathscr{Z}(V') \cong \Delta^{\mathcal{IW}}_0 \star^I \Delta^I_{w_\mu} \star^I \mathscr{Z}(V') \\ \cong \Delta^{\mathcal{IW}}_0 \star^I \mathscr{Z}(V') \star^I \Delta^I_{w_\mu} \cong \mathsf{Av}_{\mathcal{IW}}(\mathscr{Z}(V')) \star^I \Delta^I_{w_\mu},$$

where the first isomorphism comes from Lemma 6.4.5, and the second one from the fact that $\mathscr{Z}(V')$ is central (see Theorem 3.2.3). Now, again by assumption $\mathsf{Av}_{\mathcal{IW}}(\mathscr{Z}(V'))$ admits a filtration with subquotients of the form $\Delta^{\mathcal{IW}}_{\nu}$; therefore $\Delta^{\mathcal{IW}}_{\mu} \star^I \mathscr{Z}(V')$ admits a filtration (again in the sense of triangulated categories) with subquotients of the form $\Delta^{\mathcal{IW}}_{\nu} \star^I \Delta^I_{w_\mu} \cong \mathsf{Av}_{\mathcal{IW}}(\Delta^I_{w_\nu} \star^I \Delta^I_{w_\mu})$. Finally, Lemma 6.4.5 and Lemma 6.5.8 imply that the restriction of each $\mathsf{Av}_{\mathcal{IW}}(\Delta^I_{w_\nu} \star^I \Delta^I_{w_\mu})$ to $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ is concentrated in degrees ≥ 0. Hence the same is true for $\mathsf{Av}_{\mathcal{IW}}(\mathscr{Z}(V \otimes V'))$.

Dual arguments show that the corestriction of this object to $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ is concentrated in degrees ≤ 0, and the proof is complete. □

6.5.4. Minuscule and quasi-minuscule coweights. — Now we want to show that $\mathscr{Z}^{\mathcal{IW}}(V)$ is tilting for any V in $\mathsf{Rep}(G^\vee)$. If $H = Z(G)^\circ$, then the natural morphism $\mathrm{Gr}_G \to \mathrm{Gr}_{G/H}$ is surjective, and restricts to a universal homeomorphism on each connected component of Gr_G. (To justify this property, one may invoke [**PR**, §6.a and Proposition 6.6]; see also [**BR**, Footnote on p. 3].) Moreover, the pushforward under this map corresponds, under the Satake equivalences for G and G/H, to restriction along the natural embedding $(G/H)^\vee \hookrightarrow G^\vee$ (see [**BR**, p. 118]). From these remarks we conclude that it suffices to prove the desired claim in the case when G is semisimple.

Assume that G is semisimple, and recall (see §1.2.1.6 or [**NP**, Lemme 1.1]) that a dominant coweight $\lambda \in \mathbf{X}^\vee_+$ is called *minuscule* if for all $\alpha \in \mathfrak{R}$ we have $|\langle \lambda, \alpha \rangle| \leq 1$, and that in this case the orbit Gr^λ_G is closed. On the other hand, $\lambda \in \mathbf{X}^\vee_+ \smallsetminus \{0\}$ is called *quasi-minuscule* if it is minimal in $\mathbf{X}^\vee_+ \smallsetminus \{0\}$ (for the order such that μ is smaller than ν iff $\nu - \mu$ is a sum of positive roots) and not minuscule; in this case there exists a unique root γ such that $\langle \lambda, \gamma \rangle \geq 2$, and $\lambda = \gamma^\vee$; moreover we have $\overline{\mathrm{Gr}^\lambda_G} = \mathrm{Gr}^\lambda_G \sqcup \mathrm{Gr}^0_G$.

In the following subsections we will prove the following claim. (More precisely, the case of minuscule coweights will be treated in §6.5.5, and the case of quasi-minuscule coweights will be treated in §6.5.10.)

Proposition 6.5.9. — *Assume that G is semisimple. If V is a simple $G^\vee$-module whose highest weight is either minuscule or quasi-minuscule, then $\mathscr{Z}^{\mathcal{IW}}(V)$ is tilting.*

Before proving the proposition, we explain why this claim is sufficient to complete the proof of Theorem 6.5.2.

Proof of Theorem 6.5.2. — As explained in §6.5.2, all that remains to be proved is that $\mathscr{Z}^{\mathcal{IW}}(V)$ is tilting for any V in $\mathsf{Rep}(G^\vee)$. Moreover, as explained above, for this we can assume that G is semisimple. In addition, since the category $\mathsf{Rep}(G^\vee)$ is semisimple we can assume that V is simple. Recall (see e.g. [**NP**, Lemme 10.3]) that any simple $G^\vee$-module is isomorphic to a direct summand of a tensor product of simple modules whose highest weights are either minuscule or quasi-minuscule. In view of Proposition 6.5.7 and Proposition 6.5.9, $\mathscr{Z}^{\mathcal{IW}}(V)$ is then a direct summand of a tilting perverse sheaf, and hence is itself tilting. □

Remark 6.5.10. — In case $\mathrm{G} = \mathrm{GL}_n(\mathbb{F})$, with the notation above we have $G/H = \mathrm{PGL}_n(\mathbb{F})$, and then $(G/H)^\vee = \mathrm{SL}_n(\overline{\mathbb{Q}}_\ell)$. In this case, all the fundamental coweights are minuscule, and any simple $(G/H)^\vee$-module is isomorphic to a direct summand of a tensor product of simple modules with *minuscule* highest weights. (See e.g. Lemma 8.4.3 below for a similar statement in a modular context.) In particular, in this special case we do not need the case of quasi-minuscule coweights in Proposition 6.5.9 to prove Theorem 6.5.2.

6.5.5. Extremal coweights. — Our goal in this subsection is to prove that when G is semisimple and λ is minuscule, the perverse sheaf $\mathscr{Z}(\mathsf{N}(\lambda))$ is tilting. For this we start with the following lemma, which does not require any assumption.

Lemma 6.5.11. — *For any V in $\mathsf{Rep}(G^\vee)$, any $\lambda \in \mathbf{X}^\vee$, any $x \in W_{\mathrm{f}}$ and any $n \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(\Delta^{\mathcal{IW}}_\lambda, \mathscr{Z}^{\mathcal{IW}}(V)[n]\big) \cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(\Delta^{\mathcal{IW}}_{x(\lambda)}, \mathscr{Z}^{\mathcal{IW}}(V)[n]\big)$$

and

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(\mathscr{Z}^{\mathcal{IW}}(V), \nabla^{\mathcal{IW}}_\lambda[n]\big) \cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(\mathscr{Z}^{\mathcal{IW}}(V), \nabla^{\mathcal{IW}}_{x(\lambda)}[n]\big).$$

Proof. — We prove the first isomorphism; the second one can be justified similarly. We can assume that $\lambda \in \mathbf{X}^\vee_+$; then we have $w_\lambda = \mathsf{t}(\lambda)$ and $w_{x(\lambda)} = \mathsf{t}(\lambda) \cdot y$ with $\ell(w_{x(\lambda)}) = \ell(w_\lambda) - \ell(y)$, where $y \in W_{\mathrm{f}}$ is of minimal length with $yx(\lambda) = \lambda$ (see e.g. [**MR2**, Lemma 2.4]). Hence we have

$$\Delta^{\mathcal{IW}}_\lambda \cong \Delta^{\mathcal{IW}}_0 \star^I \Delta^I_{w_\lambda} \cong \Delta^{\mathcal{IW}}_0 \star^I \Delta^I_{w_{x(\lambda)}} \star^I \Delta^I_{y^{-1}} \cong \Delta^{\mathcal{IW}}_{x(\lambda)} \star^I \Delta^I_{y^{-1}},$$

where we have used Lemma 6.4.5 and Lemma 4.1.4(1). Now since the object $\Delta^I_{y^{-1}}$ is invertible in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ (see Lemma 4.1.4(3)), we have

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}&(\Delta^{\mathcal{IW}}_{x(\lambda)}, \mathscr{Z}^{\mathcal{IW}}(V)[n]) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta^{\mathcal{IW}}_{x(\lambda)} \star^I \Delta^I_{y^{-1}}, \mathscr{Z}^{\mathcal{IW}}(V) \star^I \Delta^I_{y^{-1}}[n]) \\ &\qquad \cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta^{\mathcal{IW}}_\lambda, \mathscr{Z}^{\mathcal{IW}}(V) \star^I \Delta^I_{y^{-1}}[n]).\end{aligned}$$

Finally we observe that

$$\begin{aligned}\mathscr{Z}^{\mathcal{IW}}(V) \star^I \Delta^I_{y^{-1}} = \Delta^{\mathcal{IW}}_0 \star^I \mathscr{Z}(V) \star^I \Delta^I_{y^{-1}} &\cong \Delta^{\mathcal{IW}}_0 \star^I \Delta^I_{y^{-1}} \star^I \mathscr{Z}(V) \\ &\cong \Delta^{\mathcal{IW}}_0 \star^I \mathscr{Z}(V) = \mathscr{Z}^{\mathcal{IW}}(V),\end{aligned}$$

where the isomorphism on the upper line uses the centrality of $\mathscr{Z}(V)$ (see Theorem 3.2.3), and the one on the lower line uses Lemma 6.4.5. This completes the proof. □

Corollary 6.5.12. — *For any $\lambda \in \mathbf{X}^\vee_+$ and any $\mu \in W_{\mathrm{f}}(\lambda)$, the complexes $(j^{\mathcal{IW}}_\mu)^* \mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\lambda))$ and $(j^{\mathcal{IW}}_\mu)^! \mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\lambda))$ are concentrated in degree $-\dim(\mathrm{Fl}^{\mathcal{IW}}_{G,\mu})$.*

Proof. — Proving the lemma amounts to proving that

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(\Delta^{\mathcal{IW}}_\mu, \mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\lambda))[n]\big) = \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(\mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\lambda)), \nabla^{\mathcal{IW}}_\mu[n]\big) = 0$$

for any $n > 0$. In view of Lemma 6.5.11 it suffices to consider the case $\mu = \lambda$. Now, by Remark 6.5.5 the orbit $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ is open in the support of $\mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\lambda))$; thus, the claim is obvious in this case. □

In the special case when G is semisimple and λ is minuscule, the only elements $\mu \in \mathbf{X}^\vee$ such that $\mathrm{Fl}^{\mathcal{IW}}_{G,\mu} \subset \overline{\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}}$ are those in $W_{\mathrm{f}}(\lambda)$. In view of Lemma 6.5.1, Remark 6.5.5 and Corollary 6.5.12, this shows Proposition 6.5.9 in this case.

6.5.6. The regular quotient. — The rest of this section is devoted to the proof of the case of quasi-minuscule coweights in Proposition 6.5.9. This case will require much more work than that of minuscule coweights. Part of the constructions involved in this proof do not require G to be semisimple, so we do not impose this assumption at this point.

We consider the Serre subcategory $\langle \mathscr{IC}^I_w : \ell(w) > 0\rangle_{\mathrm{Serre}} \subset \mathsf{P}_I$ generated by the objects $\mathscr{IC}^I_w$ with $\ell(w) > 0$, and the Serre quotient

$$\mathsf{P}^0_I := \mathsf{P}_I / \langle \mathscr{IC}^I_w : \ell(w) > 0\rangle_{\mathrm{Serre}}.$$

The quotient functor $\mathsf{P}_I \to \mathsf{P}^0_I$ will be denoted Π^0. Each object in P^0_I has finite length, and the simple objects in this category are those of the form $\Pi^0(\mathscr{IC}^I_w)$ with $w \in W$ and $\ell(w) = 0$. (In particular, this category has only finitely many simple objects in case G is semisimple.)

Lemma 6.5.13. — 1. *If $\mathscr{F}$ belongs to $\langle \mathscr{IC}^I_w : \ell(w) > 0\rangle_{\mathrm{Serre}}$ and $\mathscr{G}$ is any object of P_I, then for any $n \in \mathbb{Z}$ the perverse sheaves ${}^{\mathrm{p}}\mathscr{H}^n(\mathscr{F} \star^I \mathscr{G})$ and ${}^{\mathrm{p}}\mathscr{H}^n(\mathscr{G} \star^I \mathscr{F})$ belong to $\langle \mathscr{IC}^I_w : \ell(w) > 0\rangle_{\mathrm{Serre}}$.*
2. *For $\mathscr{F}, \mathscr{G}$ in P_I and any $n \in \mathbb{Z} \smallsetminus \{0\}$ we have $\Pi^0({}^{\mathrm{p}}\mathscr{H}^n(\mathscr{F} \star^I \mathscr{G})) = 0$.*

Proof. — (1) Of course, using long exact sequences of cohomology one can assume that $\mathscr{F} = \mathscr{IC}^I_w$ for some $w \in W$ with $\ell(w) > 0$. Choose a simple reflection s such that $\ell(sw) < \ell(w)$. As in the proof of Lemma 6.4.4, $\mathscr{F}$ is J_s-equivariant, where J_s is the corresponding minimal parahoric subgroup of $\mathrm{L}G$. That is, $\mathscr{F}$ belongs to the image of the forgetful functor $D^{\mathrm{b}}_{J_s}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. Then the same will be true for $\mathscr{F} \star^I \mathscr{G}$, so that all of its perverse cohomology objects will be J_s-equivariant. The composition factors of J_s-equivariant perverse sheaves are all of the form $\mathscr{IC}^I_x$ with $sx < x$; in particular such objects belong to $\langle \mathscr{IC}^I_w : \ell(w) > 0\rangle_{\mathrm{Serre}}$, which finishes the proof of the first claim.

The proof of the second claim is similar, using the fact that any $\mathscr{IC}^I_w$ with $\ell(w) > 0$ is the pullback of a perverse sheaf on the partial affine flag variety $\mathrm{L}G/J_s$ for some simple reflection s.

(2) Here again we can assume that $\mathscr{F} = \mathscr{IC}^I_x$ and $\mathscr{G} = \mathscr{IC}^I_y$ for some $x, y \in W$. If $\ell(x) > 0$ or $\ell(y) > 0$ then the claim follows from (1). And if $\ell(x) = \ell(y) = 0$ then we have $\mathscr{IC}^I_x \star^I \mathscr{IC}^I_y \cong \mathscr{IC}^I_{xy}$, so that the claim is obvious. □

Lemma 6.5.13 has the following consequence.

Proposition 6.5.14. — 1. *The bifunctor $\mathsf{P}_I \times \mathsf{P}_I \to \mathsf{P}_I^0$ sending a pair $(\mathscr{F}, \mathscr{G})$ to $\Pi^0({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \star^I \mathscr{G}))$ factors through a bifunctor*

$$\circledast : \mathsf{P}_I^0 \times \mathsf{P}_I^0 \to \mathsf{P}_I^0.$$

2. *The bifunctor $\circledast$ is exact on both sides, and admits natural associativity and unit constraints.*

Proof. — Lemma 6.5.13 implies that the bifunctor $(\mathscr{F}, \mathscr{G}) \mapsto \Pi^0({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \star^I \mathscr{G}))$ is exact on both sides, and vanishes on $\langle \mathscr{IC}_w^I : \ell(w) > 0 \rangle_{\mathrm{Serre}} \times \mathsf{P}_I$ and on $\mathsf{P}_I \times \langle \mathscr{IC}_w^I : \ell(w) > 0 \rangle_{\mathrm{Serre}}$. Therefore it factors through a bifunctor $\circledast$, which is exact on both sides. This observation also shows that for $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ in P_I^0 we have canonical isomorphisms

$$(\mathscr{F}_1 \circledast \mathscr{F}_2) \circledast \mathscr{F}_3 \cong \Pi^0({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F}_1 \star^I \mathscr{F}_2 \star^I \mathscr{F}_3)) \cong \mathscr{F}_1 \circledast (\mathscr{F}_2 \circledast \mathscr{F}_3),$$

where in the middle term we use the canonical identification between objects in P_I^0 and P_I to consider $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ as objects in P_I. This provides the desired associativity constraint. Similar considerations show that the object

$$\delta^0 := \Pi^0(\mathscr{IC}_e^I)$$

admits a canonical structure of a unit object. □

Thanks to Proposition 6.5.14, we can now consider the abelian monoidal category $(\mathsf{P}_I^0, \circledast)$. We set

$$\mathscr{Z}^0 := \Pi^0 \circ \mathscr{Z} : \mathsf{Rep}(G^\vee) \to \mathsf{P}_I^0.$$

Lemma 6.5.15. — *The functor $\mathscr{Z}^0$ admits a natural structure of a central functor.*

Proof. — To prove the lemma we have to endow $\mathscr{Z}^0$ with "monoidality" and "centrality" isomorphisms. These will be induced by the corresponding isomorphisms for Z. Namely, for V, V' in $\mathsf{Rep}(G^\vee)$ we have a canonical isomorphism

$$\mathsf{Z}(\mathcal{S}^{-1}(V)) \star^I \mathsf{Z}(\mathcal{S}^{-1}(V')) \xrightarrow{\sim} \mathsf{Z}(\mathcal{S}^{-1}(V \otimes V')),$$

see Theorem 3.4.1. Since both sides are perverse sheaves, appying Π^0 we deduce a canonical isomorphism

$$\mathscr{Z}^0(V) \circledast \mathscr{Z}^0(V') \xrightarrow{\sim} \mathscr{Z}^0(V \otimes V').$$

We leave it to the reader to check that these isomorphisms define a monoidal structure on $\mathscr{Z}^0$.

Next, for V in $\mathsf{Rep}(G^\vee)$ and $\mathscr{F}$ in P_I^0 we have a canonical isomorphism

$$\mathsf{Z}(\mathcal{S}^{-1}(V)) \star^I \mathscr{F} \xrightarrow{\sim} \mathscr{F} \star^I \mathsf{Z}(\mathcal{S}^{-1}(V)),$$

see Theorem 3.4.1. (Here, as above we use the identification between objects in P_I^0 and P_I to regard $\mathscr{F}$ as an object in P_I.) Since both sides are perverse (see Corollary 3.2.5), once again applying Π^0 we deduce a canonical isomorphism

$$\mathscr{Z}^0(V) \circledast \mathscr{F} \xrightarrow{\sim} \mathscr{F} \circledast \mathscr{Z}^0(V).$$

We leave it to the reader to check (using the analogous claim for the functor Z) that these isomorphisms define a central structure on $\mathscr{Z}^0$. □

For V in $\mathsf{Rep}(G^\vee)$ we set

$$\mathsf{n}_V^0 := \Pi^0(\mathsf{n}_V) \in \mathrm{End}_{\mathsf{P}_I^0}(\mathscr{Z}^0(V)).$$

The following claim is an immediate consequence of Proposition 2.4.6(2).

Lemma 6.5.16. — *For any V, V' in $\mathsf{Rep}(G^\vee)$ and any $f \in \mathrm{Hom}_{\mathsf{P}_I^0}(\mathscr{Z}^0(V), \mathscr{Z}^0(V'))$ we have $f \circ \mathsf{n}_V^0 = \mathsf{n}_{V'}^0 \circ f$.*

6.5.7. Description of the regular quotient. — We denote by $\widetilde{\mathsf{P}}_I^0$ the full abelian subcategory of P_I^0 whose objects are the subquotients of objects of the form $\mathscr{Z}^0(V)$ with V in $\mathsf{Rep}(G^\vee)$. Obviously, the functor $\mathscr{Z}^0$ factors through a functor

$$\widetilde{\mathscr{Z}^0} : \mathsf{Rep}(G^\vee) \to \widetilde{\mathsf{P}}_I^0.$$

Lemma 6.5.17. — *If $\mathscr{F}$ and $\mathscr{G}$ belong to $\widetilde{\mathsf{P}}_I^0$, then $\mathscr{F} \circledast \mathscr{G}$ also belongs to $\widetilde{\mathsf{P}}_I^0$.*

Proof. — If $\mathscr{F}$ is a subquotient of $\mathscr{Z}^0(V)$ and $\mathscr{G}$ a subquotient of $\mathscr{Z}^0(V')$, then by exactness of the product $\circledast$ (see Proposition 6.5.14) the object $\mathscr{F} \circledast \mathscr{G}$ is a subquotient of $\mathscr{Z}^0(V) \circledast \mathscr{Z}^0(V') \cong \mathscr{Z}^0(V \otimes V')$. □

As a consequence of this lemma, $\widetilde{\mathsf{P}}_I^0$ admits a natural structure of an abelian monoidal category. Clearly, the structure of a central functor on $\mathscr{Z}^0$ restricts to a structure of a central functor on $\widetilde{\mathscr{Z}^0}$.

The following proposition will be the key to the proof of the quasi-minuscule case in Proposition 6.5.9. It will also play a technical role in Section 6.6.

Proposition 6.5.18. — *There exist*

1. *a closed subgroup $H \subset G^\vee$;*
2. *a nilpotent element $n_0 \in \mathfrak{g}^\vee$ such that $H \subset Z_{G^\vee}(n_0)$;*
3. *an equivalence of monoidal categories*
$$\Phi^0 : (\widetilde{\mathsf{P}}_I^0, \circledast) \xrightarrow{\sim} (\mathsf{Rep}(H), \otimes);$$
4. *and an isomorphism of functors $\eta : \Phi^0 \circ \widetilde{\mathscr{Z}^0} \xrightarrow{\sim} \mathsf{For}_H^{G^\vee}$*

such that for any V in $\mathsf{Rep}(G^\vee)$, the endomorphism $\eta(\Phi^0(\mathsf{n}_V^0))$ coincides with the action of n_0 on V.

Proposition 6.5.18 is a particular case of a more general result proved in [**Be2**], to which we refer for further details. Below we review how its proof works, for later comparison with an analogue over a field of positive characteristic in Chapter 8. (The details of this proof will however not be needed for the rest of the present chapter.)

The key idea of the proof is to check that the forgetful functor $\mathsf{Rep}(G^\vee) \to \mathsf{Vect}_{\overline{\mathbb{Q}}_\ell}$ factors through the functor $\widetilde{\mathscr{Z}^0}$. For this, we start from the observation that the (left) regular $G^\vee$-module $\mathscr{O}(G^\vee)$ defines a ring object in the monoidal category of ind-objects in $\mathsf{Rep}(G^\vee)$, which we will denote $\underline{\mathscr{O}}(G^\vee)$. Therefore we can consider its image $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G^\vee))$, a ring object in the category $\mathrm{Ind}(\widetilde{\mathsf{P}}_I^0)$ of ind-objects in $\widetilde{\mathsf{P}}_I^0$. By definition, a left ideal subobject in $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G^\vee))$ is a subobject $\mathscr{M}$ (in the abelian[6] category $\mathrm{Ind}(\widetilde{\mathsf{P}}^0_I)$) such that the multiplication map sends $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G^\vee)) \circledast \mathscr{M}$ into $\mathscr{M}$. Using the fact that the ring $\mathscr{O}(G^\vee)$ is commutative and Theorem 3.5.1, it is not difficult to check that any left ideal is automatically a right ideal.

Lemma 6.5.19. — *There exist maximal left ideal subobjects in $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G^\vee))$.*

Proof. — To prove this lemma we will use Zorn's lemma. Namely, the first observation is that by [**KS2**, Theorem 6.1.8], the category $\mathrm{Ind}(\widetilde{\mathsf{P}}^0_I)$ admits small filtrant inductive limits. We next consider the poset of left ideal subobjects in $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G^\vee))$. Given a chain $(\mathscr{M}_i : i \in I)$ in this poset, one can consider the associated inductive limit $\widetilde{\mathscr{M}}$. By the universal property of inductive limits, this object is endowed with a canonical morphism $\widetilde{\mathscr{M}} \to \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G^\vee))$, and the image of this morphism is an upper bound for our chain. By Zorn's lemma we conclude that our poset has a maximal element, i.e. that $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G^\vee))$ admits a maximal left ideal subobject. □

We next choose a maximal left ideal subobject $\mathscr{J}$ in $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G^\vee))$, and denote by $\underline{\mathscr{O}}(H)$ the associated quotient. Since our left ideal is also a right ideal, $\underline{\mathscr{O}}(H)$ has a natural structure of a ring object in $\mathrm{Ind}(\widetilde{\mathsf{P}}^0_I)$. Let us denote by $\mathsf{Mod}_{\underline{\mathscr{O}}(H)}$ the category of left $\underline{\mathscr{O}}(H)$-modules in $\mathrm{Ind}(\widetilde{\mathsf{P}}^0_I)$. Then $\mathsf{Mod}_{\underline{\mathscr{O}}(H)}$ is an abelian category, and $\underline{\mathscr{O}}(H)$ is a simple object in this category. Therefore, the ring

$$K := \mathrm{End}_{\mathsf{Mod}_{\underline{\mathscr{O}}(H)}}(\underline{\mathscr{O}}(H)) \tag{6.5.4}$$

is a division algebra, and the functor $V \mapsto V \otimes_K \underline{\mathscr{O}}(H)$ defines an equivalence of categories between the category of right finite K-modules and the full subcategory of $\mathsf{Mod}_{\underline{\mathscr{O}}(H)}$ whose objects are finite direct sums of copies of $\underline{\mathscr{O}}(H)$. (Note for later use that the latter subcategory is closed under subquotients.) Let us also remark that if $\mathscr{M}$ is a left $\underline{\mathscr{O}}(H)$-module and $\mathscr{N}$ is any object in P^0_I, then the product $\mathscr{M} \circledast \mathscr{N}$ admits a canonical structure of a left $\underline{\mathscr{O}}(H)$-module.

The next step is to remark that restriction to $\widetilde{\mathscr{Z}^0}(\overline{\mathbb{Q}}_\ell) = \delta^0 \subset \underline{\mathscr{O}}(H)$ induces an isomorphism

$$K \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ind}(\widetilde{\mathsf{P}}^0_I)}(\delta^0, \underline{\mathscr{O}}(H)).$$

In these terms, the product $f \cdot g$ of two maps $f, g : \delta^0 \to \underline{\mathscr{O}}(H)$ is the composition

$$\delta^0 = \delta^0 \circledast \delta^0 \xrightarrow{f \circledast g} \underline{\mathscr{O}}(H) \circledast \underline{\mathscr{O}}(H) \to \underline{\mathscr{O}}(H),$$

where the map on the right is multiplication in the ring-object $\underline{\mathscr{O}}(H)$. Since $\mathscr{O}(G^\vee)$ can be written as a formal direct limit of finite-dimensional modules parametrized by $\mathbb{Z}_{\geq 0}$, $\underline{\mathscr{O}}(H)$ is a formal direct limit of objects of $\widetilde{\mathsf{P}}^0_I$ parametrized by $\mathbb{Z}_{\geq 0}$, which implies that K has at most countable dimension over $\overline{\mathbb{Q}}_\ell$. Since $\overline{\mathbb{Q}}_\ell(X)$ has uncountable dimension (e.g. because the fractions $\frac{1}{X-\lambda}$ for $\lambda \in \overline{\mathbb{Q}}_\ell$ are linearly independent), the only division algebra over $\overline{\mathbb{Q}}_\ell$ which has at most countable dimension is $\overline{\mathbb{Q}}_\ell$. It follows that $K = \overline{\mathbb{Q}}_\ell$.

[6] Recall that the category of ind-objects in an abelian category is itself abelian: see [**KS2**, Theorem 8.6.5(i)].

The next important step is the following.

Lemma 6.5.20. — 1. *For any $\mathscr{M}$ in $\widetilde{\mathsf{P}}^0_I$, the product $\underline{\mathscr{O}}(H) \circledast \mathscr{M}$ is isomorphic (as an $\underline{\mathscr{O}}(H)$-module) to a (finite) direct sum of copies of $\underline{\mathscr{O}}(H)$.*

2. *The functor $\mathfrak{v} : \widetilde{\mathsf{P}}^0_I \to \mathsf{Vect}_{\overline{\mathbb{Q}}_\ell}$ defined by*

$$\mathfrak{v}(\mathscr{M}) := \mathrm{Hom}_{\mathsf{Mod}_{\underline{\mathscr{O}}(H)}}(\underline{\mathscr{O}}(H), \underline{\mathscr{O}}(H) \circledast \mathscr{M})$$

is exact and faithful, and admits a natural monoidal structure. Moreover, the composition $\mathfrak{v} \circ \widetilde{\mathscr{Z}}^0$ is canonically isomorphic (as a monoidal functor) to the forgetful functor $\mathsf{Rep}(G^\vee) \to \mathsf{Vect}_{\overline{\mathbb{Q}}_\ell}$.

Proof. — (1) First we consider the case $\mathscr{M} = \widetilde{\mathscr{Z}}^0(V)$ for some V in $\mathsf{Rep}(G^\vee)$. We have a canonical isomorphism of $G^\vee$-modules and of $\mathscr{O}(G^\vee)$-modules $\underline{\mathscr{O}}(G^\vee) \otimes V \cong \underline{\mathscr{O}}(G^\vee) \otimes \underline{V}$ where $\underline{V}$ is the underlying vector space of V (with trivial $G^\vee$-action). We deduce a canonical isomorphism of $\widetilde{\mathscr{Z}}^0(\underline{\mathscr{O}}(G^\vee))$-modules

$$\widetilde{\mathscr{Z}}^0(\underline{\mathscr{O}}(G^\vee)) \circledast \widetilde{\mathscr{Z}}^0(V) \cong \widetilde{\mathscr{Z}}^0(\underline{\mathscr{O}}(G^\vee)) \otimes_{\overline{\mathbb{Q}}_\ell} \underline{V}.$$

Being an isomorphism of $\widetilde{\mathscr{Z}}^0(\underline{\mathscr{O}}(G^\vee))$-modules, it must send $\mathscr{J} \cdot \big(\widetilde{\mathscr{Z}}^0(\underline{\mathscr{O}}(G^\vee)) \circledast \widetilde{\mathscr{Z}}^0(V)\big) = \mathscr{J} \circledast \widetilde{\mathscr{Z}}^0(V)$ to $\mathscr{J} \cdot \big(\widetilde{\mathscr{Z}}^0(\underline{\mathscr{O}}(G^\vee)) \otimes_{\overline{\mathbb{Q}}_\ell} \underline{V}\big) = \mathscr{J} \otimes_{\overline{\mathbb{Q}}_\ell} \underline{V}$, and therefore induce a canonical isomorphism

$$\underline{\mathscr{O}}(H) \circledast \widetilde{\mathscr{Z}}^0(V) \cong \underline{\mathscr{O}}(H) \otimes_{\overline{\mathbb{Q}}_\ell} \underline{V}. \tag{6.5.5}$$

This proves the desired claim for the object $\mathscr{M} = \widetilde{\mathscr{Z}}^0(V)$. Since any object of $\widetilde{\mathsf{P}}^0_I$ is by definition of subquotient of an object of this form, and since any subquotient (in $\mathsf{Mod}_{\underline{\mathscr{O}}(H)}$) of a direct sum of copies of $\underline{\mathscr{O}}(H)$ is itself isomorphic to a direct sum of copies of $\underline{\mathscr{O}}(H)$, this finishes the proof.

(2) From the fact that $K = \overline{\mathbb{Q}}_\ell$ and (1) we deduce that for any $\mathscr{M}$ in $\widetilde{\mathsf{P}}^0_I$ we have

$$\underline{\mathscr{O}}(H) \circledast \mathscr{M} \cong \underline{\mathscr{O}}(H) \otimes_{\overline{\mathbb{Q}}_\ell} \mathfrak{v}(\mathscr{M}).$$

From this one obtains a monoidal structure by observing that

$$\begin{aligned}\underline{\mathscr{O}}(H) \otimes_{\overline{\mathbb{Q}}_\ell} \mathfrak{v}(\mathscr{M} \circledast \mathscr{N}) \cong \underline{\mathscr{O}}(H) \circledast \mathscr{M} \circledast \mathscr{N} &\cong (\underline{\mathscr{O}}(H) \otimes_{\overline{\mathbb{Q}}_\ell} \mathfrak{v}(\mathscr{M})) \circledast \mathscr{N} \\ &\cong (\underline{\mathscr{O}}(H) \circledast \mathscr{N}) \otimes_{\overline{\mathbb{Q}}_\ell} \mathfrak{v}(\mathscr{M}) \cong \underline{\mathscr{O}}(H) \otimes_{\overline{\mathbb{Q}}_\ell} \mathfrak{v}(\mathscr{M}) \otimes_{\overline{\mathbb{Q}}_\ell} \mathfrak{v}(\mathscr{N})\end{aligned}$$

and then applying the functor $\mathrm{Hom}_{\mathsf{Mod}_{\underline{\mathscr{O}}(H)}}(\underline{\mathscr{O}}(H), -)$. The claim about $\mathfrak{v} \circ \widetilde{\mathscr{Z}}^0$ is clear from (6.5.5).

Now, let us prove that $\mathfrak{v}$ is exact. Consider an exact sequence $\mathscr{M}_1 \hookrightarrow \mathscr{M}_2 \twoheadrightarrow \mathscr{M}_3$ in $\widetilde{\mathsf{P}}^0_I$. By exactness of $\circledast$, applying the functor $\underline{\mathscr{O}}(H) \circledast (-)$ we obtain an exact sequence

$$\underline{\mathscr{O}}(H) \circledast \mathscr{M}_1 \hookrightarrow \underline{\mathscr{O}}(H) \circledast \mathscr{M}_2 \twoheadrightarrow \underline{\mathscr{O}}(H) \circledast \mathscr{M}_3$$

in $\mathsf{Mod}_{\underline{\mathscr{O}}(H)}$. By (1), each object in this sequence is a direct sum of copies of the simple module $\underline{\mathscr{O}}(H)$; the sequence must therefore split. Applying the functor $\mathrm{Hom}_{\mathsf{Mod}_{\underline{\mathscr{O}}(H)}}(\underline{\mathscr{O}}(H), -)$, we deduce an exact sequence

$$\mathfrak{v}(\mathscr{M}_1) \hookrightarrow \mathfrak{v}(\mathscr{M}_2) \twoheadrightarrow \mathfrak{v}(\mathscr{M}_3),$$

proving the desired exactness.

Finally we prove that $\mathfrak{v}$ is faithful. For this, since this functor is now known to be exact, it suffices to prove that it does not kill any object, or even that it does not kill any *simple* object in $\widetilde{\mathsf{P}}^0_I$. However, any simple object in $\widetilde{\mathsf{P}}^0_I$ is invertible for the product $\circledast$. (In fact, such an object is of the form $\Pi^0(\mathscr{IC}^I_\omega)$ for some $\omega \in \Omega$; its inverse is then $\Pi^0(\mathscr{IC}^I_{\omega^{-1}})$.) Since $\mathfrak{v}$ is a monoidal functor, it must send each of these objects to a nonzero (in fact, 1-dimensional) $\overline{\mathbb{Q}}_\ell$-vector space, which finishes the proof. □

Remark 6.5.21. — As above in the description of K, one can check that for any $\mathscr{M}$ in $\widetilde{\mathsf{P}}^0_I$ we have a canonical isomorphism $\mathfrak{v}(\mathscr{M}) \cong \mathrm{Hom}_{\mathrm{Ind}(\widetilde{\mathsf{P}}^0_I)}(\delta^0, \underline{\mathscr{O}}(H) \circledast \mathscr{M})$.

Once our functor $\mathfrak{v}$ is constructed, since this functor is exact and faithful, Tannakian formalism provides a bialgebra $\mathscr{A}(H)$ over $\overline{\mathbb{Q}}_\ell$ and an equivalence of monoidal categories

$$\widetilde{\mathsf{P}}^0_I \xrightarrow{\sim} \mathsf{Comod}_{\mathscr{A}(H)} \tag{6.5.6}$$

(where the right-hand side is the category of right $\mathscr{A}(H)$-comodules which are finite-dimensional over $\overline{\mathbb{Q}}_\ell$) whose composition with the forgetful functor $\mathsf{Comod}_{\mathscr{A}(H)} \to \mathsf{Vect}_{\overline{\mathbb{Q}}_\ell}$ is $\mathfrak{v}$ (see [**DM**, p. 137]; see also [**BR**, §1.2]). In more detail, $\mathscr{A}(H)$ is defined as an as an inductive limit over objects X of $\widetilde{\mathsf{P}}^0_I$ of certain coalgebras $\mathscr{A}(H)_X$ defined in [**BR**, Proposition 1.2.2]. This set-up is enough to obtain the equivalence (6.5.6), although at this stage, $\mathscr{A}(H)$ only has the structure of a coalgebra. Then, the monoidal structure on $\widetilde{\mathsf{P}}^0_I$ lets us define a multiplication map on $\mathscr{A}(H)$, making it into a bialgebra.

From the monoidal functor $\widetilde{\mathscr{Z}^0} : \mathsf{Rep}(G^\vee) \to \widetilde{\mathsf{P}}^0_I$ we obtain a canonical bialgebra morphism $\mathscr{O}(G^\vee) \to \mathscr{A}(H)$, see [**BR**, Proposition 1.2.6(1)]. In fact, this morphism can be obtained as an inductive limit of morphisms $\mathscr{O}(G^\vee)_Y \to \mathscr{A}(H)_{\widetilde{\mathscr{Z}^0}(Y)}$ where Y runs over objects in $\mathsf{Rep}(G^\vee)$; here $\mathscr{O}(G^\vee)_Y$ is obtained as above for $\mathscr{A}(H)_X$. From the fact that (by definition) any object in $\widetilde{\mathsf{P}}^0_I$ is a subquotient of an object $\widetilde{\mathscr{Z}^0}(V)$, we obtain that this morphism is surjective. (The argument for this can be copied from that in [**DM**, Proposition 2.21].) This implies that $\mathscr{A}(H)$ is commutative, and that its spectrum H is a submonoid scheme of $G^\vee$. Finally one can check that any submonoid scheme of a group scheme of finite type over a noetherian ring is a subgroup scheme (see [**Be2**, Lemma 2]), which lets us conclude that H is the desired subgroup of $G^\vee$.

To finish the proof of Proposition 6.5.18, it now only remains to explain the construction of n_0 and the proof of its stated properties. For this, recall the endomorphisms $(\mathsf{n}^0_V : V \in \mathsf{Rep}(G^\vee))$ from §6.5.6. Composing with $\mathfrak{v}$, we obtain an endomorphism of the functor $\mathfrak{v} \circ \widetilde{\mathscr{Z}^0} \cong \mathsf{For}^{G^\vee}$, which satisfies a compatibility with the tensor product similar to that in (6.3.13). As in §6.3.5, by Tannakian formalism such a datum defines a element $n_0 \in \mathfrak{g}^\vee$. The fact that this endomorphism is induced by an endomorphism of the forgetful functor $\mathsf{For}^{G^\vee}_H : \mathsf{Rep}(G^\vee) \to \mathsf{Rep}(H)$ means that H stabilizes n_0, i.e. that $H \subset Z_{G^\vee}(n_0)$. Finally, the fact that n_0 is nilpotent is clear from the fact that the endomorphisms n^0_V are nilpotent.

The following commutative diagram summarizes the functors that have appeared in the proof of Proposition 6.5.18:

$$\begin{tikzcd}
& & \mathsf{Rep}(H) \arrow[dd, "\mathsf{For}^H"] \\
\mathsf{Rep}(G^\vee) \arrow[r, "\widetilde{\mathscr{Z}}^0"] \arrow[urr, "\mathsf{For}^{G^\vee}_H"] \arrow[drr, "\mathsf{For}^{G^\vee}"'] & \widetilde{\mathsf{P}}^0_I \arrow[ur, "\Phi^0"'] \arrow[dr, "\mathfrak{v}"] & \\
& & \mathsf{Vect}_{\overline{\mathbb{Q}}_\ell}.
\end{tikzcd}$$

Remark 6.5.22. — 1. The subgroup H and the element n_0 are defined only up to conjugation. In fact, their construction depends on the initial choice of a maximal proper left ideal subobject in $\mathscr{Z}^0(\underline{\mathscr{O}}(G^\vee))$, which cannot be made canonical.
2. We will see in Section 7.2 that in fact $\widetilde{\mathsf{P}}^0_I = \mathsf{P}^0_I$, and $H = Z_{G^\vee}(n_0)$. These properties will however not be needed for the proof of the Arkhipov–Bezrukavnikov equivalence.

6.5.8. Regularity of n_0. — The other crucial property of n_0 we will need is the following.

Proposition 6.5.23. — *The element $n_0 \in \mathfrak{g}^\vee$ from Proposition 6.5.18 is regular.*

The proof of this property is given (as a particular case of a more general statement) in [**Be2**, Theorem 2]. In the special case under consideration, this proof simplifies considerably, as we explain below (following a suggestion in [**AB**, Remark 9]).

Remark 6.5.24. — In §8.4.4 we will prove a modular analogue of Proposition 6.5.23 in the case where G is a general linear group. This proof also applies for characteristic-0 coefficients, and provides a more direct argument proving Proposition 6.5.23 for general linear groups, which does not rely on Theorem 9.5.6.

Proof of Proposition 6.5.23. — Recall the notion of Jacobson–Morozov–Deligne filtration associated with a nilpotent endomorphism e of an object X in an abelian category, see §9.5.4. When X is a finite-dimensional vector space over some field, this filtration is described explicitly in [**De3**, §1.6.7]; from this description we see that the number of Jordan blocks of e (i.e. the dimension of its kernel) is $\dim(\mathrm{gr}^F_0(X)) + \dim(\mathrm{gr}^F_1(X))$.

Now, recall the setting considered in §5.3.4. Consider a representation V in $\mathsf{Rep}(G^\vee)$, and let $\mathscr{F} \in \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell)$ be the perverse sheaf such that $V = \mathcal{S}(\mathscr{F})$. Here $\mathscr{F}$ is a direct sum of intersection cohomology complexes associated with L^+G-orbits in Gr_G; therefore there exists a semisimple mixed perverse sheaf $\mathscr{F}_\circ$ of weight 0 on $\mathrm{Gr}_{G,\circ}$ such that $\mathscr{F} = \varkappa(\mathscr{F}_\circ)$, where $\varkappa$ denotes (as in §5.3.3) the base-change functor to $\mathbb{F}$. Then we have

$$\mathscr{Z}(V) = \mathsf{Z}(\mathscr{F}) = \varkappa(\mathsf{Z}^{\mathrm{mix}}(\mathscr{F}_\circ)),$$

see Lemma 9.5.5.

By Theorem 9.5.6, the Jacobson–Morozov–Deligne filtration associated with the nilpotent endomorphism n_V of $\mathscr{X}(V)$ is the image under $\varkappa$ of the weight filtration on $\mathsf{Z}^{\mathrm{mix}}(\mathscr{F}_\circ)$. Note that by uniqueness the Jacobson–Morozov–Deligne filtration associated with the nilpotent endomorphism n_V^0 of $\mathscr{X}^0(V)$ is the image under Π^0 of the aforementioned filtration on $\mathscr{X}(V)$, and then that the Jacobson–Morozov–Deligne filtration associated with the action of n_0 of V is deduced by applying the functor $\mathfrak{v}$ of Lemma 6.5.20(2). In summary, the Jacobson–Morozov–Deligne filtration associated with the action of n_0 of V is obtained from the weight filtration on $\mathsf{Z}^{\mathrm{mix}}(\mathscr{F}_\circ)$ by applying the composition $\mathfrak{v} \circ \Pi^0 \circ \varkappa$. In particular, since Π^0 sends each $\mathscr{IC}_w^I$ with $\ell(w) > 0$ to 0, and since $\dim \mathfrak{v}(\Pi^0(\mathscr{IC}_\omega^I)) = 1$ for $\omega \in \Omega$ (because $\Pi^0(\mathscr{IC}_\omega^I)$ is an invertible object in P_I^0), we deduce that the dimension of the i-th part of the associated graded of the Jacobson–Morozov–Deligne filtration associated with the action of n_0 on V is given by

$$\sum_{\omega \in \Omega} [\varkappa(\mathrm{gr}_i^{\mathrm{W}}(\mathsf{Z}^{\mathrm{mix}}(\mathscr{F}_\circ))) : \mathscr{IC}_\omega^I]. \tag{6.5.7}$$

To compute the integer (6.5.7) we use the combinatorial recipe described in §5.3.4. Consider the unique $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism

$$\eta : \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$$

such that $\eta(H_w) = (-v)^{\ell(w)}$. Then it is easily seen that

$$\eta(\underline{H}_w) = \begin{cases} 1 & \text{if } \ell(w) = 0; \\ 0 & \text{otherwise} \end{cases}$$

and that

$$\eta(\theta_\lambda^{\mathbf{X}_+^\vee}) = v^{\langle \lambda, 2\rho \rangle}.$$

In view of Proposition 5.3.5, it follows that

$$\sum_{i \in \mathbb{Z}} \sum_{\omega \in \Omega} [\varkappa(\mathrm{gr}_i^{\mathrm{W}}(\mathsf{Z}^{\mathrm{mix}}(\mathscr{F}_\circ))) : \mathscr{IC}_\omega^I] \cdot v^i = \sum_{\mu \in \mathbf{X}^\vee} \dim(V_\mu) \cdot v^{\langle \mu, 2\rho \rangle}.$$

Combining these considerations, we finally obtain that the dimension of the kernel of the action of n_0 on V is given by

$$\dim(V^{n_0}) = \sum_{\substack{\mu \in \mathbf{X}^\vee \\ \langle \mu, 2\rho \rangle \in \{0,1\}}} \dim(V_\mu). \tag{6.5.8}$$

In particular, when V is the adjoint representation, we obtain that the dimension of $\mathfrak{z}_{\mathfrak{g}^\vee}(n_0)$ is equal to the rank of $T^\vee$, i.e. that n_0 is regular. □

6.5.9. Consequence for the stalks and costalks of central sheaves. — The consequence of Proposition 6.5.18 which will be relevant in this section is the following. (Here, we write V^{n_0} for the kernel of the action of n_0 on V.)

Lemma 6.5.25. — *For any V in $\mathsf{Rep}(G^\vee)$, we have*

$$\dim\Big(\mathrm{Hom}_{\mathsf{P}_{\mathcal{IW}}}\big(\Delta_0^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V)\big)\Big) \leq \dim(V^{n_0}),$$
$$\dim\Big(\mathrm{Hom}_{\mathsf{P}_{\mathcal{IW}}}\big(\mathscr{Z}^{\mathcal{IW}}(V), \nabla_0^{\mathcal{IW}}\big)\Big) \leq \dim(V^{n_0}).$$

Proof. — We prove the first claim only; the second one can be treated similarly (using the fact that for an endomorphism f of a finite-dimensional vector space we have $\dim(\ker(f)) = \dim(\mathrm{cok}(f))$).

By Lemma 6.4.5 we have $\Delta_0^{\mathcal{IW}} = \mathsf{Av}_{\mathcal{IW}}(\mathscr{IC}_e^I)$; hence Theorem 6.4.2 provides an isomorphism

$$\mathrm{Hom}_{\mathsf{P}_{\mathcal{IW}}}\big(\Delta_0^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V)\big) \cong \mathrm{Hom}_{\mathsf{P}_I^{\mathrm{asph}}}\big(\Pi_{\mathrm{asph}}(\mathscr{IC}_e^I), \Pi_{\mathrm{asph}}(\mathscr{Z}(V))\big).$$

Now, by definition the functor Π^0 factors through the functor Π_{asph}. Since $\Pi_{\mathrm{asph}}(\mathscr{IC}_e^I)$ is simple, any nonzero morphism $\Pi_{\mathrm{asph}}(\mathscr{IC}_e^I) \to \Pi_{\mathrm{asph}}(\mathscr{Z}(V))$ is injective, and its image is not killed by the quotient functor $\mathsf{P}_I^{\mathrm{asph}} \to \mathsf{P}_I^0$; we deduce that this functor induces an injective map

$$\mathrm{Hom}_{\mathsf{P}_I^{\mathrm{asph}}}\big(\Pi_{\mathrm{asph}}(\mathscr{IC}_e^I), \Pi_{\mathrm{asph}}(\mathscr{Z}(V))\big) \hookrightarrow \mathrm{Hom}_{\mathsf{P}_I^0}\big(\mathscr{Z}^0(\overline{\mathbb{Q}}_\ell), \mathscr{Z}^0(V)\big).$$

Next, since $\mathsf{n}^0_{\overline{\mathbb{Q}}_\ell} = 0$, using Lemma 6.5.16 we see that

$$\mathrm{Hom}_{\mathsf{P}_I^0}\big(\mathscr{Z}^0(\overline{\mathbb{Q}}_\ell), \mathscr{Z}^0(V)\big) = \mathrm{Hom}_{\mathsf{P}_I^0}\big(\mathscr{Z}^0(\overline{\mathbb{Q}}_\ell), \ker(\mathsf{n}_V^0)\big).$$

And then using Proposition 6.5.18 we obtain an injection

$$\mathrm{Hom}_{\mathsf{P}_I^0}\big(\mathscr{Z}^0(\overline{\mathbb{Q}}_\ell), \ker(\mathsf{n}_V^0)\big) \hookrightarrow \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(\overline{\mathbb{Q}}_\ell, V^{n_0}) = V^{n_0}.$$

The desired inequality follows. ☐

6.5.10. The case of quasi-minuscule coweights. — We are finally in a position to prove the case of quasi-minuscule coweights in Proposition 6.5.9. So, in this subsection we assume that G is semisimple and that V is a simple $G^\vee$-module with quasi-minuscule highest weight λ. Observe that in this case we have

$$\dim(V^{n_0}) = \dim(V_0). \tag{6.5.9}$$

To see this, apply (6.5.8) and note that the nonzero weights μ of V lie in the root lattice of $\mathfrak{g}^\vee$, and hence satisfy $\langle \mu, 2\rho\rangle \in 2\mathbb{Z}$.

Going back to geometry, we note that by Remark 6.5.5 $\mathscr{Z}^{\mathcal{IW}}(V)$ is supported on $\overline{\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}}}$, which contains the orbit $\mathrm{Fl}_{G,\mu}^{\mathcal{IW}}$ iff $\mu \in W_{\mathrm{f}}(\lambda) \cup \{0\}$. By Corollary 6.5.12, the restriction and corestriction of $\mathscr{Z}^{\mathcal{IW}}(V)$ to any orbit $\mathrm{Fl}_{G,\mu}^{\mathcal{IW}}$ with $\mu \in W_{\mathrm{f}}(\lambda)$ is perverse, so to conclude it suffices to prove the analogous claim for $\mathrm{Fl}_{G,0}^{\mathcal{IW}}$. Let $i : \overline{\mathrm{Fl}_{G,0}^{\mathcal{IW}}} \to \mathrm{Fl}_G$ be the embedding, and let j be the embedding of the open complement. Then we have distinguished triangles

$$j_!j^*\mathscr{Z}^{\mathcal{IW}}(V) \to \mathscr{Z}^{\mathcal{IW}}(V) \to i_*i^*\mathscr{Z}^{\mathcal{IW}}(V) \xrightarrow{[1]},$$
$$i_*i^!\mathscr{Z}^{\mathcal{IW}}(V) \to \mathscr{Z}^{\mathcal{IW}}(V) \to j_*j^*\mathscr{Z}^{\mathcal{IW}}(V) \xrightarrow{[1]}.$$

What we have proved so far implies that $j^*\mathscr{Z}^{\mathcal{IW}}(V)$ (an Iwahori–Whittaker perverse sheaf on $\mathrm{Fl}_G \smallsetminus \overline{\mathrm{Fl}^{\mathcal{IW}}_{G,0}}$) admits a filtration with standard subquotients, and a filtration with costandard subquotients; hence both $j_!j^*\mathscr{Z}^{\mathcal{IW}}(V)$ and $j_*j^*\mathscr{Z}^{\mathcal{IW}}(V)$ are perverse sheaves. Using the triangles above we deduce that $i^*\mathscr{Z}^{\mathcal{IW}}(V)$ is concentrated in perverse degrees 0 and -1, while $i^!\mathscr{Z}^{\mathcal{IW}}(V)$ is concentrated in perverse degrees 0 and 1. However, in view of (6.5.9), Lemma 6.5.25 and Proposition 6.5.4 imply that

$$\dim \mathrm{Hom}_{\mathsf{P}_{\mathcal{IW}}}\big(\Delta_0^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V)\big) \leq \sum_{n\geq 0} (-1)^n \cdot \dim \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)}(\Delta_0^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V)[n]).$$

Our remarks above imply that the right-hand side equals

$$\dim \mathrm{Hom}_{\mathsf{P}_{\mathcal{IW}}}\big(\Delta_0^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V)\big) - \dim \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)}\big(\Delta_0^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V)[1]\big),$$

so that we necessarily have $\dim \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)}\big(\Delta_0^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V)[1]\big) = 0$; in other words, the corestriction of $\mathscr{Z}^{\mathcal{IW}}(V)$ to $\mathrm{Fl}^{\mathcal{IW}}_{G,0}$ is perverse. Similar arguments show that the restriction of this perverse sheaf to $\mathrm{Fl}^{\mathcal{IW}}_{G,0}$ is perverse, and the proof is complete.

6.5.11. Restriction to the regular orbit. — We finish this section by proving a lemma that will play a technical role in a step of the proof of the Arkhipov–Bezrukavnikov equivalence in Section 6.6.

Recall that the Springer map $\widetilde{\mathcal{N}} \to \mathfrak{g}^\vee$ (defined by $[g,x] \mapsto g\cdot x$) restricts to an isomorphism $\widetilde{\mathcal{O}}_{\mathrm{r}} \to \mathcal{O}_{\mathrm{r}}$, where $\mathcal{O}_{\mathrm{r}} \subset \mathfrak{g}^\vee$ is the regular nilpotent orbit and $\widetilde{\mathcal{O}}_{\mathrm{r}} \subset \widetilde{\mathcal{N}}$ is its preimage (see e.g. [**J2**, §6.10]). In particular, the element $n_0 \in \mathfrak{g}^\vee$ admits a unique preimage $\tilde{n}_0$ in $\widetilde{\mathcal{N}}$, and moreover since the canonical map $G^\vee/Z_{G^\vee}(n_0) \to \mathcal{O}_{\mathrm{r}}$ is an isomorphism of varieties we have a canonical equivalence of categories

$$\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}}) \xrightarrow{\sim} \mathsf{Rep}(Z_{G^\vee}(n_0)), \tag{6.5.10}$$

see [**Bri1**, Lemma 2]. If we denote by $\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}})$ the full subcategory of $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ whose objects are those of the form $V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$ for some V in $\mathsf{Rep}(G^\vee)$, we can therefore consider the composition

$$\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}}) \to \mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}}) \xrightarrow[\sim]{(6.5.10)} \mathsf{Rep}(Z_{G^\vee}(n_0)) \to \mathsf{Rep}(H) \tag{6.5.11}$$

where the first arrow is given by restriction to the open subset $\widetilde{\mathcal{O}}_{\mathrm{r}} \subset \widetilde{\mathcal{N}}$, and the third one is the restriction functor associated with the embedding $H \hookrightarrow Z_{G^\vee}(n_0)$. By construction this functor is monoidal, for the monoidal structure on $\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}})$ induced by the tensor product of coherent sheaves.

On the other hand, we can also consider the composition

$$\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}}) \xrightarrow{\Pi^0\circ F} \widetilde{\mathsf{P}}^0_I \xrightarrow[\sim]{\Phi^0} \mathsf{Rep}(H), \tag{6.5.12}$$

where the first arrow is defined as follows: the functor F restricts to a functor $\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}}) \to \mathsf{P}_I$, and then the composition $\Pi^0 \circ F : \mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}}) \to \mathsf{P}^0_I$ factors through $\widetilde{\mathsf{P}}^0_I$ by (6.3.18).

Using Remark 6.3.10 we see that (6.5.12) admits a canonical monoidal structure.

Lemma 6.5.26. — *The functors* (6.5.11) *and* (6.5.12) *are isomorphic as monoidal functors.*

Proof. — The objects in the category $\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}})$ are in canonical bijection with those in $\mathsf{Rep}(G^\vee)$ (through $V \mapsto V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$), and it is clear that both of our functors send $V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$ to $V_{|H}$. Hence what we have to prove is that for any V_1, V_2 in $\mathsf{Rep}(G^\vee)$ the maps

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(V_1 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}, V_2 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \to \mathrm{Hom}_{\mathsf{Rep}(H)}(V_1, V_2) \tag{6.5.13}$$

induced by these two functors coincide.

Using (6.2.1) we see that

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(V_1 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}, V_2 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \cong \big(V_1^* \otimes V_2 \otimes \mathscr{O}(\widetilde{\mathcal{N}})\big)^{G^\vee},$$

and that similarly we have

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\mathfrak{g}^\vee)}(V_1 \otimes \mathscr{O}_{\mathfrak{g}^\vee}, V_2 \otimes \mathscr{O}_{\mathfrak{g}^\vee}) \cong \big(V_1^* \otimes V_2 \otimes \mathscr{O}(\mathfrak{g}^\vee)\big)^{G^\vee}.$$

Let $\mathcal{N} \subset \mathfrak{g}^\vee$ be the nilpotent cone (i.e. the closed subvariety consisting of nilpotent elements); then the Springer map induces a projective birational morphism $p_{\mathrm{Spr}} : \widetilde{\mathcal{N}} \to \mathcal{N}$. Since $\mathcal{N}$ is known to be normal (see [**J2**, §8.5] for references), by Zariski's main theorem (see the proof of [**Har**, Chap. III, Corollary 11.4]) we deduce that

$$(p_{\mathrm{Spr}})_* \mathscr{O}_{\widetilde{\mathcal{N}}} \cong \mathscr{O}_{\mathcal{N}}, \tag{6.5.14}$$

hence that $\mathscr{O}(\widetilde{\mathcal{N}}) = \mathscr{O}(\mathcal{N})$. In particular, pullback under the Springer morphism $\widetilde{\mathcal{N}} \to \mathfrak{g}^\vee$ induces a surjection $\mathscr{O}(\mathfrak{g}^\vee) \twoheadrightarrow \mathscr{O}(\widetilde{\mathcal{N}})$, and hence a surjection

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\mathfrak{g}^\vee)}(V_1 \otimes \mathscr{O}_{\mathfrak{g}^\vee}, V_2 \otimes \mathscr{O}_{\mathfrak{g}^\vee}) \to \mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(V_1 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}, V_2 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}).$$

Therefore, to prove that the two morphisms (6.5.13) coincide, it suffices to prove that their compositions with this surjection coincide. In other words, the lemma will follow if we show that the two compositions

$$\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\mathfrak{g}^\vee) \xrightarrow{\text{pullback}} \mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}}) \underset{(6.5.12)}{\overset{(6.5.11)}{\rightrightarrows}} \mathsf{Rep}(H) \tag{6.5.15}$$

are isomorphic. Let us further compose with the natural functor $\mathsf{Rep}(G^\vee) \to \mathsf{Coh}^{G^\vee}(\mathfrak{g}^\vee)$. It is clear that both functors

$$\mathsf{Rep}(G^\vee) \longrightarrow \mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\mathfrak{g}^\vee) \xrightarrow{\text{pullback}} \mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}}) \underset{(6.5.12)}{\overset{(6.5.11)}{\rightrightarrows}} \mathsf{Rep}(H)$$

identify with the restriction functor associated with the embedding $H \hookrightarrow G^\vee$. As explained in §6.3.2 (see in particular Example 6.3.1), an extension of this functor to a $\Bbbk$-linear monoidal functor $\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\mathfrak{g}^\vee) \to \mathsf{Rep}(H)$ is determined by an H-equivariant algebra homomorphism $\mathscr{O}(\mathfrak{g}^\vee) \to \overline{\mathbb{Q}}_\ell$, which in turn is determined by an endomorphism of the restriction functor $\mathsf{Rep}(G^\vee) \to \mathsf{Rep}(H)$ (which should correspond to the canonical endomorphisms of the objects $V \otimes \mathscr{O}_{\mathfrak{g}^\vee}$). In our case, for both functors in (6.5.15), this endomorphism is given by the action of n_0; therefore these functors must be isomorphic. □

6.6. Proof of the equivalence

6.6.1. Statement. — We will consider the functor

$$F_{\mathcal{IW}} : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

defined as the composition

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \xrightarrow{F} D^{\mathrm{b}}\mathsf{P}_I \xrightarrow{(6.3.19)} D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathsf{Av}_{\mathcal{IW}}} D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell).$$

By construction (see (6.3.18)) we have

$$(6.6.1) \qquad F_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) = \mathscr{Z}^{\mathcal{IW}}(V), \qquad F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) = \mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell))$$

for V in $\mathsf{Rep}(G^\vee)$ and $\lambda \in \mathbf{X}^\vee$.

In view of the t-exactness of the functor $\mathsf{Av}_{\mathcal{IW}}$ (see Theorem 6.4.2(1)) and general results on realization functors (see [**Beĭ1**, Lemma A.7.1]), $F_{\mathcal{IW}}$ is canonically isomorphic to the composition

$$(6.6.2) \qquad D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \xrightarrow{F} D^{\mathrm{b}}\mathsf{P}_I \xrightarrow{D^{\mathrm{b}}(\mathsf{Av}^{\mathsf{P}}_{\mathcal{IW}})} D^{\mathrm{b}}\mathsf{P}_{\mathcal{IW}} \xrightarrow{\sim} D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$$

where $\mathsf{Av}^{\mathsf{P}}_{\mathcal{IW}}$ is the restriction of $\mathsf{Av}_{\mathcal{IW}}$ to the hearts of the perverse t-structures, and the last arrow is the realization functor. (Here this functor is an equivalence of categories, as follows from the techniques of [**BGS**, §§3.2–3.3].)

Our goal in this section is to prove the following.

Theorem 6.6.1. — *The functor $F_{\mathcal{IW}}$ is an equivalence of categories.*

Before explaining the proof of this result, let us note the following consequence.

Corollary 6.6.2. — *The functor*

$$\mathsf{Av}^{\mathrm{asph}}_{\mathcal{IW}} : \mathsf{P}^{\mathrm{asph}}_I \to \mathsf{P}_{\mathcal{IW}}$$

(see Theorem 6.4.2(2)) is an equivalence of categories.

Proof. — As noted in Theorem 6.4.2 our functor is fully faithful; what remains to be proved is that it is essentially surjective. However, Theorem 6.6.1 implies that for any $\mathscr{F}$ in $\mathsf{P}_{\mathcal{IW}}$, there exists $\mathscr{G}$ in $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ such that $\mathscr{F} \cong \mathsf{Av}_{\mathcal{IW}}(\mathscr{G})$. By exactness of $\mathsf{Av}_{\mathcal{IW}}$ (see Theorem 6.4.2(1)) we then have

$$\mathscr{F} \cong \mathsf{Av}_{\mathcal{IW}}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{G})) \cong \mathsf{Av}^{\mathrm{asph}}_{\mathcal{IW}}\big(\Pi_{\mathrm{asph}}({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{G}))\big),$$

which finishes the proof. □

In view of the description of the functor $F_{\mathcal{IW}}$ as the composition (6.6.2), Theorem 6.6.1 and Corollary 6.6.2 show that F also induces an equivalence of triangulated categories

$$(6.6.3) \qquad F^{\mathrm{asph}} := D^{\mathrm{b}}(\Pi_{\mathrm{asph}}) \circ F : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{P}^{\mathrm{asph}}_I.$$

6.6.2. Preliminaries. — The proof of Theorem 6.6.1 will be based on the techniques of Beĭlinson's lemma; so we will have to analyze the action of this functor on morphisms between certain objects, and show that these objects (resp. their images) generate the source category (resp. the target category).

Lemma 6.6.3. — *The objects* $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell))$ *with* $\lambda \in \mathbf{X}^\vee$ *generate* $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ *as a triangulated category.*

Proof. — By Lemma 4.1.9, the class of $\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)$ in the Grothendieck group of P_I (or of $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$) coincides with that of $\Delta^I_{\mathsf{t}(\lambda)}(\overline{\mathbb{Q}}_\ell)$. Using Lemma 6.4.5, we deduce that the class of $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell))$ in the Grothendieck group of $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ coincides with that of $\Delta^{\mathcal{IW}}_\lambda$. Using an Euler characteristic argument (as in the proof of Lemma 4.1.9), it follows that $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell))$ is supported on $\overline{\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}}$, and that its restriction to $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ has rank 1. Then standard arguments (based on induction on the support) show that these objects generate $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ as a triangulated category. □

Lemma 6.6.4. — *For any V in $\mathsf{Rep}(G^\vee)$, the morphism*

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)}(F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}), F_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}))$$

induced by $F_{\mathcal{IW}}$ is injective.

Proof. — Both $\mathscr{O}_{\widetilde{\mathcal{N}}}$ and $V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$ belong to $\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathcal{N}})$ (in the notation of §6.5.11), and both

$$F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}) = \Delta^{\mathcal{IW}}_0 \quad \text{and} \quad F_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) = \mathscr{Z}^{\mathcal{IW}}(V)$$

belong to the essential image of the fully faithful functor from Theorem 6.4.2(2). Hence what we have to show is that the composition of Π_{asph} with F induces an injective map

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \to \mathrm{Hom}_{\mathsf{P}^{\mathrm{asph}}_I}(\Pi_{\mathrm{asph}}(\Delta^I_e), \Pi_{\mathrm{asph}}(\mathscr{Z}(V))).$$

As observed in the course of the proof of Lemma 6.5.25, the functor Π^0 factors through Π_{asph}; therefore, to conclude it suffices to prove that the composition of Π^0 with F induces an injective morphism

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \to \mathrm{Hom}_{\mathsf{P}^0_I}(\delta^0, \mathscr{Z}^0(V)).$$

Finally, in view of Lemma 6.5.26, proving this claim amounts to proving that restriction to the preimage of n_0 induces an injective morphism

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \to \mathrm{Hom}_{\mathsf{Rep}(H)}(\overline{\mathbb{Q}}_\ell, V). \tag{6.6.4}$$

This is however well known. Namely, as in the proof of Lemma 6.5.26, the left-hand side identifies with

$$\big(V \otimes \mathscr{O}(\mathcal{N})\big)^{G^\vee},$$

where $\mathcal{N} \subset \mathfrak{g}^\vee$ is the nilpotent cone. The regular nilpotent orbit $\mathcal{O}_{\mathrm{r}} \subset \mathcal{N}$ is an open subset of the normal variety $\mathcal{N}$, whose complement has codimension 2. Therefore restriction induces an isomorphism $\mathscr{O}(\mathcal{N}) \xrightarrow{\sim} \mathscr{O}(\mathcal{O}_{\mathrm{r}})$, and hence an isomorphism

$$\big(V \otimes \mathscr{O}(\mathcal{N})\big)^{G^\vee} \xrightarrow{\sim} \big(V \otimes \mathscr{O}(\mathcal{O}_{\mathrm{r}})\big)^{G^\vee}.$$

Then we have $\mathscr{O}(\mathcal{O}_{\mathrm{r}}) = \mathrm{Ind}_{Z_{G^\vee}(n_0)}^{G^\vee}(\overline{\mathbb{Q}}_\ell)$, so that

$$\big(V \otimes \mathscr{O}(\mathcal{O}_{\mathrm{r}})\big)^{G^\vee} = \big(V \otimes \mathrm{Ind}_{Z_{G^\vee}(n_0)}^{G^\vee}(\overline{\mathbb{Q}}_\ell)\big)^{G^\vee} = V^{Z_{G^\vee}(n_0)}.$$

Under this identification, (6.6.4) can be identified with the inclusion map $V^{Z_{G^\vee}(n_0)} \hookrightarrow V^H$, which is indeed injective. □

Corollary 6.6.5. — *For any V in $\mathsf{Rep}(G^\vee)$, any $\lambda \in \mathbf{X}_+^\vee$ and any $n \in \mathbb{Z}$, the morphism*

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)[n]) \\ \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}), F_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda))[n])$$

induced by $F_{\mathcal{IW}}$ is injective.

Proof. — Using (6.2.1) we see that

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)[n]) = \big(V \otimes \mathsf{H}^n(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda))\big)^{G^\vee}.$$

By [**Bro**, Theorem 2.4] we have $\mathsf{H}^n(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) = 0$ unless $n=0$; it follows that

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)[n]) = 0 \tag{6.6.5}$$

unless $n = 0$. So, the only case we have to consider is when $n = 0$.

By Lemma 6.2.9, there exists V' in $\mathsf{Rep}(G^\vee)$ and an embedding $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda) \hookrightarrow V' \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$ in $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$. We deduce an embedding

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) \hookrightarrow \mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes V' \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}). \tag{6.6.6}$$

By Lemma 6.6.4, the morphism

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, V \otimes V' \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \\ \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}), F_{\mathcal{IW}}(V \otimes V' \otimes \mathscr{O}_{\widetilde{\mathcal{N}}})) \tag{6.6.7}$$

is injective. Now by functoriality the composition of (6.6.6) and (6.6.7) (which is injective by our arguments above) coincides with the composition of the morphism in the statement (for $n = 0$) with the morphism

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}), F_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda))) \\ \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}), F_{\mathcal{IW}}(V \otimes V' \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}))$$

induced by our morphism $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda) \hookrightarrow V' \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$. The desired injectivity follows. □

6.6.3. Proof of Theorem 6.6.1. — We are now in a position to prove Theorem 6.6.1. We will first prove that $F_{\mathcal{IW}}$ is fully faithful. For this, we have to check that for any $\mathscr{F}, \mathscr{G}$ in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ the morphism

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(F_{\mathcal{IW}}(\mathscr{F}), F_{\mathcal{IW}}(\mathscr{G})) \tag{6.6.8}$$

induced by $F_{\mathcal{IW}}$ is an isomorphism.

Consider the special case when $\mathscr{F} = \mathscr{O}_{\widetilde{\mathcal{N}}}$ and $\mathscr{G} = V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)[n]$ for some V in $\mathsf{Rep}(G^\vee)$, some $\lambda \in \mathbf{X}_+^\vee$ and some $n \in \mathbb{Z}$. In this case (6.6.8) is injective by

Corollary 6.6.5. We claim that the domain and codomains have the same (finite) dimension, so that this map must be an isomorphism.

First, we consider the right-hand side. By (6.3.18) we have

$$F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}) = \Delta_0^{\mathcal{IW}}, \qquad F_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) \cong \mathscr{Z}^{\mathcal{IW}}(V) \star^I \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell).$$

Hence the space we have to consider is

$$\begin{aligned}
&\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta_0^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V) \star^I \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)[n]) \\
&\qquad\cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta_0^{\mathcal{IW}} \star^I \mathbf{J}_{-\lambda}(\overline{\mathbb{Q}}_\ell), \mathscr{Z}^{\mathcal{IW}}(V)[n]) \\
&\qquad\cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta_0^{\mathcal{IW}} \star^I \Delta^I_{\mathsf{t}(-\lambda)}, \mathscr{Z}^{\mathcal{IW}}(V)[n]) \\
&\qquad\qquad\cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta_{-\lambda}^{\mathcal{IW}}, \mathscr{Z}^{\mathcal{IW}}(V)[n]),
\end{aligned}$$

where the last step uses Lemma 6.4.5. By Theorem 6.5.2 this space vanishes unless $n = 0$, and in this case its dimension is $\dim(V_{-\lambda})$.

Now we consider the left-hand side of (6.6.8) (still in our particular case). As seen in the course of the proof of Corollary 6.6.5, here again the space vanishes unless $n = 0$, and in this case it identifies with

$$\big(V \otimes \Gamma(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda))\big)^{G^\vee},$$

so that its dimension is

$$\sum_{\nu \in \mathbf{X}^\vee_+} [V^* : \mathsf{N}(\nu)] \cdot \Big[\Gamma(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) : \mathsf{N}(\nu)\Big].$$

Now it is known that

$$\Big[\Gamma(\widetilde{\mathcal{N}}, \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)) : \mathsf{N}(\nu)\Big] = \dim\left(\mathsf{N}(\nu)_\lambda\right),$$

see [**Bro**, Proposition 2.1]. We finally deduce that the dimension we are computing equals

$$\dim((V^*)_\lambda) = \dim(V_{-\lambda}),$$

which finishes the proof of our claim.

From this special case, using Lemma 6.2.8 (Case (2)) and the 5-lemma we obtain that (6.6.8) is an isomorphism when $\mathscr{F} = \mathscr{O}_{\widetilde{\mathcal{N}}}$ and $\mathscr{G}$ is any object of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$. Next, if $\mathscr{F} = \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ for some $\lambda \in \mathbf{X}^\vee$, using the isomorphisms

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda), \mathscr{G}) \cong \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(\mathscr{O}_{\widetilde{\mathcal{N}}}, \mathscr{G} \otimes_{\mathscr{O}_{\widetilde{\mathcal{N}}}} \mathscr{O}_{\widetilde{\mathcal{N}}}(-\lambda))$$

and

$$\begin{aligned}
&\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)), F_{\mathcal{IW}}(\mathscr{G})) \\
&\qquad\cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta_0^{\mathcal{IW}} \star^I \mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell), F_{\mathcal{IW}}(\mathscr{G})) \\
&\qquad\cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta_0^{\mathcal{IW}}, F_{\mathcal{IW}}(\mathscr{G}) \star^I \mathbf{J}_{-\lambda}(\overline{\mathbb{Q}}_\ell)) \\
&\qquad\qquad\cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta_0^{\mathcal{IW}}, F_{\mathcal{IW}}(\mathscr{G} \otimes_{\mathscr{O}_{\widetilde{\mathcal{N}}}} \mathscr{O}_{\widetilde{\mathcal{N}}}(-\lambda)))
\end{aligned}$$

one identifies the map (6.6.8) in this case with the analogous map when $\mathscr{F} = \mathscr{O}_{\widetilde{\mathcal{N}}}$ and $\mathscr{G}$ is replaced by $\mathscr{G} \otimes_{\mathscr{O}_{\widetilde{\mathcal{N}}}} \mathscr{O}_{\widetilde{\mathcal{N}}}(-\lambda)$. As explained above this map is known to be an isomorphism. Finally, using Case (1) in Lemma 6.2.8 and again the 5-lemma, we obtain that the map (6.6.8) is an isomorphism for any objects $\mathscr{F}$ and $\mathscr{G}$.

Now that we know that $F_{\mathcal{IW}}$ is fully faithful, we deduce that its essential image is a triangulated subcategory of $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. By (6.6.1), this subcategory contains the objects $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell))$ for $\lambda \in \mathbf{X}^\vee$, which generate $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ by Lemma 6.6.3. Therefore $F_{\mathcal{IW}}$ is essentially surjective, which finishes the proof.

6.6.4. Application: indecomposability of $\mathscr{T}^{\mathcal{IW}}(V)$ when V is simple. — As an immediate application of Theorem 6.6.1, we answer the question raised in Remark 6.5.3.

Proposition 6.6.6. — *If $V \in \mathsf{Rep}(G^\vee)$ is simple, then the tilting perverse sheaf $\mathscr{T}^{\mathcal{IW}}(V)$ is indecomposable.*

Proof. — To prove that $\mathscr{T}^{\mathcal{IW}}(V)$ is indecomposable it suffices to prove that its inverse image under the equivalence of Theorem 6.6.1, i.e. the equivariant coherent sheaf $V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}} \in \mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$, is indecomposable. However, by the same considerations as in the proof of Lemma 6.5.26 we have

$$\mathrm{End}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \cong (\mathrm{End}(V) \otimes \mathscr{O}(\mathcal{N}))^{G^\vee}. \tag{6.6.9}$$

Now, consider the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathfrak{g}^\vee$ for with $z \in \overline{\mathbb{Q}}_\ell^\times$ acts by multiplication by z^{-1}. This action stabilizes $\mathcal{N}$, and therefore defines a grading on the algebra $\mathscr{O}(\mathcal{N})$. By construction the parts in negative degrees vanish, and the degree-0 part consists of the constant functions. This grading induces a grading on the vector space $(\mathrm{End}(V) \otimes \mathscr{O}(\mathcal{N}))^{G^\vee}$, and it is not difficult to check that this endows this space with a graded algebra structure (for the product induced by the product on the left-hand side in (6.6.9)). Now it is clear that a graded $\overline{\mathbb{Q}}_\ell$-algebra concentrated in nonnegative degrees, and whose degree-0 part is $\overline{\mathbb{Q}}_\ell$, cannot contain nontrivial idempotents. Therefore $V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$ is indeed indecomposable, which finishes the proof. $\square$

CHAPTER 7

COMPLEMENTS

In this chapter we present three complements to the results of Chapter 6. We continue with the same setting and notation as in that chapter.

First, in Section 7.1 we describe the t-structure on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ obtained by transporting the perverse t-structure on $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ along the equivalence of Theorem 6.6.1. This t-structure is called the *exotic t-structure*, and is studied in particular in [**Be3**] and in [**MR1**]; it plays an important role in several other works in geometric representation theory; see [**Ac2, Be3, MR2**] for examples. The relation with Theorem 6.6.1 is due to Bezrukavnikov in [**Be3**].

Then, in Section 7.2 we show that the functor considered in §6.5.7 induces an equivalence of monoidal categories between P^0_I and the category of representations of $Z_{G^\vee}(n_0)$. This fact is stated in a remark in [**AB**], but as far as we know no proof appears in the literature. (See however [**BRR**] for closely related considerations in a modular setting.)

Finally, in Section 7.3 we explain how one can deduce from Theorem 6.6.1 a description of the derived category of equivariant coherent sheaves on the nilpotent cone $\mathcal{N}$ in terms of perverse sheaves on Fl_G. Once again this result is due to Bezrukavnikov, see [**Be4**].

7.1. t-structures

In this section we describe the t-structure on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ obtained by transport along the equivalence $F_{\mathcal{IW}}$ (see Theorem 6.6.1) from the perverse t-structure on $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$.

7.1.1. Exceptional collections and associated t-structures. — We start by recalling a general construction of a t-structure on a triangulated category starting from a nice enough family of objects.

In this subsection we let $\Bbbk$ be any field. Let D be a $\Bbbk$-linear triangulated category which is of finite type, i.e. such that for any X, Y in D the $\Bbbk$-vector space $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{D}}(X, Y[n])$ is finite-dimensional. Let $(I, \leq)$ be a partially ordered set. A collection $(\nabla^i : i \in I)$ of objects of D is called an *exceptional collection* (parametrized by $(I, \leq)$) if it satisfies

$$\mathrm{Hom}_{\mathsf{D}}(\nabla^i, \nabla^j[n]) = 0$$

if $i \not\geq j$ or if $i = j$ and $n \neq 0$, and if in addition $\mathrm{Hom}_{\mathsf{D}}(\nabla^i, \nabla^i) = \Bbbk$.

The case which will be the most relevant for us is the case when $(I, \leq) = (\mathbb{Z}_{\geq 0}, \leq)$. Assume we are in this setting, and consider an exceptional collection $(\nabla^i : i \in \mathbb{Z}_{\geq 0})$. There exists a unique collection $(\Delta^i : i \in \mathbb{Z}_{\geq 0})$ of objects of D which satisfy

$$\mathrm{Hom}_{\mathsf{D}}(\Delta^i, \nabla^j) = 0 \text{ if } i > j \qquad \text{and} \qquad \Delta^i \cong \nabla^i \mod \mathsf{D}_{<i}$$

(where $\mathsf{D}_{<i}$ is the full triangulated subcategory of D generated by the objects ∇^j for $j < i$, and the right-hand side means that the images of Δ^i and ∇^i in the Verdier quotient $\mathsf{D}/\mathsf{D}_{<i}$ are isomorphic), see [**Be3**, Proposition 3]. This collection is called the *dual exceptional collection*; it satisfies

$$\mathrm{Hom}_{\mathsf{D}}(\Delta^i, \Delta^j[n]) = 0$$

if $i \geq j$ or if $i = j$ and $n \neq 0$, and in addition $\mathrm{Hom}_{\mathsf{D}}(\Delta^i, \Delta^i) = \Bbbk$. (In other words, this collection is exceptional for the opposite of the usual order on $\mathbb{Z}_{\geq 0}$.) These objects also automatically satisfy the condition

$$\mathrm{Hom}_{\mathsf{D}}(\Delta^i, \nabla^j[n]) = \begin{cases} \Bbbk & \text{if } i = j \text{ and } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Assume now that the objects $(\nabla^j : j \in \mathbb{Z}_{\geq 0})$ generate D as a triangulated category. Define $\mathsf{D}^{\geq 0}$ as the full subcategory of D generated under extensions by the objects $\nabla^i[n]$ with $i \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\leq 0}$, and $\mathsf{D}^{\leq 0}$ as the full subcategory of D generated under extensions by the objects $\Delta^i[n]$ with $i \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$. Then by [**Be3**, Proposition 4], the pair $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ is a bounded t-structure on D, called the t-structure associated with the exceptional collection $(\nabla^i : i \in \mathbb{Z}_{\geq 0})$.

Example 7.1.1. — Consider the order $\leq_{\mathrm{geo}}$ on $\mathbf{X}^\vee$ such that $\lambda \leq_{\mathrm{geo}} \mu$ iff $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda} \subset \overline{\mathrm{Fl}^{\mathcal{IW}}_{G,\mu}}$. Then it is easy to see using adjunction that the collection $(\nabla^{\mathcal{IW}}_\lambda : \lambda \in \mathbf{X}^\vee)$ is exceptional (for the poset $(\mathbf{X}^\vee, \leq_{\mathrm{geo}})$). Choose an arbitrary refinement $\leq'_{\mathrm{geo}}$ of this order to a total order such that $(\mathbf{X}^\vee, \leq'_{\mathrm{geo}})$ is isomorphic to $(\mathbb{Z}_{\geq 0}, \leq)$ as a poset. Then the dual exceptional collection is $(\Delta^{\mathcal{IW}}_\lambda : \lambda \in \mathbf{X}^\vee)$, and the associated t-structure is the perverse t-structure.

Another important operation we can do with exceptional collections is *mutation*. Namely, let us come back to the setting of a general poset $(I, \leq)$, and assume we are given another partial order $\preceq$ on I such that $(I, \preceq)$ is isomorphic to $(\mathbb{Z}_{\geq 0}, \leq)$. Then for any $i \in I$ there exists a unique object $\tilde{\nabla}^i$ in $\mathsf{D}_{\preceq i}$ such that $\mathrm{Hom}_{\mathsf{D}}(X, \tilde{\nabla}^i) = 0$ for any X in $\mathsf{D}_{\prec i}$ and such that $\nabla^i \cong \tilde{\nabla}^i \mod \mathsf{D}_{\prec i}$. (Here $\mathsf{D}_{\preceq i}$, resp. $\mathsf{D}_{\prec i}$, means the triangulated subcategory of D generated by the objects ∇^j with $j \preceq i$, resp. with $j \prec i$.) Moreover, the collection $(\tilde{\nabla}^i : i \in I)$ is an exceptional collection parametrized by $(I, \preceq)$, and it generates D as a triangulated category if $(\nabla^i : i \in I)$ does.

7.1.2. The exotic t-structure. — We come back to the case $\Bbbk = \overline{\mathbb{Q}}_\ell$, and $\mathsf{D} = D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}})$. It is well known that this category is of finite type, see e.g. [**MR1**, Corollary 2.7] for a proof. Consider also the set $\mathbf{X}^\vee$, equipped with the order such that λ is smaller than μ iff $\mu - \lambda$ is a sum of positive roots. The following claim is not difficult to check directly (see e.g. [**Be3**, Lemma 5]), but can also be deduced from the Arkhipov–Bezrukavnikov equivalence.

Lemma 7.1.2. — *The collection $(\mathscr{O}_{\widetilde{\mathscr{N}}}(\lambda) : \lambda \in \mathbf{X})$ is an exceptional collection (parametrized by $\mathbf{X}^\vee$ with the order as above) which generates $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}})$ as a triangulated category.*

Proof. — Recall that for all $\lambda \in \mathbf{X}^\vee$ we have

$$F_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathscr{N}}}(\lambda)) = \mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)).$$

Therefore, using Theorem 6.6.1, the lemma translates into the statement that the collection $(\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)) : \lambda \in \mathbf{X}^\vee)$ is an exceptional collection in $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$, and that it generates this category (as a triangulated category). The latter property is exactly Lemma 6.6.3. For the former, consider $\lambda_1, \lambda_2 \in \mathbf{X}^\vee$ and $n \in \mathbb{Z}$. Then for any $\mu \in \mathbf{X}^\vee$, convolution on the right with $\mathbf{J}_\mu(\overline{\mathbb{Q}}_\ell)$ induces an isomorphism

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(&\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_{\lambda_1}(\overline{\mathbb{Q}}_\ell)), \mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_{\lambda_2}(\overline{\mathbb{Q}}_\ell))[n]\big) \cong \\ &\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_{\lambda_1+\mu}(\overline{\mathbb{Q}}_\ell)), \mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_{\lambda_2+\mu}(\overline{\mathbb{Q}}_\ell))[n]\big),\end{aligned}$$

by Proposition 4.2.3 and Lemma 4.2.6. If we choose μ such that $\lambda_1 + \mu$ is dominant, then we have

$$\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_{\lambda_1+\mu}(\overline{\mathbb{Q}}_\ell)) \cong \mathsf{Av}_{\mathcal{IW}}(\nabla^I_{\mathsf{t}(\lambda_1+\mu)}) \cong \nabla^{\mathcal{IW}}_{\lambda_1+\mu}$$

by Lemma 6.4.5. If we furthermore assume that $\lambda_2 + \mu$ is dominant then we likewise have

$$\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_{\lambda_2+\mu}(\overline{\mathbb{Q}}_\ell)) \cong \nabla^{\mathcal{IW}}_{\lambda_2+\mu}.$$

Hence

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(&\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_{\lambda_1}(\overline{\mathbb{Q}}_\ell)), \mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_{\lambda_2}(\overline{\mathbb{Q}}_\ell))[n]\big) \cong \\ &\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}\big(\nabla^{\mathcal{IW}}_{\lambda_1+\mu}, \nabla^{\mathcal{IW}}_{\lambda_2+\mu}[n]\big).\end{aligned}$$

Here by adjunction the right-hand side vanishes unless $\overline{\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda_2+\mu}} \subset \overline{\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda_1+\mu}}$, or equivalently, unless $(\lambda_1+\mu)-(\lambda_2+\mu) = \lambda_1-\lambda_2$ is a sum of positive roots (see Remark 6.4.1). These considerations also show that in case $\lambda_1 = \lambda_2$ this space vanishes unless $n = 0$, and is 1-dimensional in this case, which finishes the proof. ☐

We now consider the order $\leq_{\mathrm{geo}}$ on $\mathbf{X}^\vee$ from Example 7.1.1, and fix a completion $\leq'_{\mathrm{geo}}$ such that $(\mathbf{X}^\vee, \leq'_{\mathrm{geo}}) \cong (\mathbb{Z}_{\geq 0}, \leq)$. Then we can define the *exotic exceptional collection*

$$(\nabla^{\mathrm{ex}}_\lambda : \lambda \in \mathbf{X}^\vee)$$

as the collection obtained from $(\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda) : \lambda \in \mathbf{X}^\vee)$ by mutation using this new order (see §7.1.1). By the results recalled in §7.1.1, there exists a dual collection, which will be denoted

$$(\Delta_\lambda^{\mathrm{ex}} : \lambda \in \mathbf{X}^\vee),$$

and an associated t-structure on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$, which will be called the *exceptional t-structure*. The heart of this t-structure will be denoted $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$.

Remark 7.1.3. — By Remark 6.4.1, we have $\lambda \leq_{\mathrm{geo}} \mu$ iff the L^+G-orbit containing $\mathrm{Fl}_{G,\mathrm{t}(\lambda)}$ is contained in the closure of the L^+G-orbit containing $\mathrm{Fl}_{G,\mathrm{t}(\mu)}$. In turn, it is known that this condition is equivalent to the property that $w_\lambda \leq_{\mathrm{Bru}} w_\mu$. (See §6.4.3 for the definition of the elements w_ν, and §4.1.1 for the definition of the Bruhat order $\leq_{\mathrm{Bru}}$.) Therefore, the order $\leq_{\mathrm{geo}}$ indeed coincides with the order considered in [**Be3**] or in [**MR1**]. (In these references, one adds some conditions on the completion of the order. However it follows from Proposition 7.1.5 below that the exotic exceptional collection does not depend on the choice of completion, so that these conditions turn out to be unnecessary.)

The exotic t-structure can be defined and studied purely in terms of equivariant coherent sheaves, see [**Be3, MR1**] (see also [**Ac2**] for a survey on this topic and some explicit computations), and for coefficient fields more general than $\overline{\mathbb{Q}}_\ell$; but this requires introducing more notation and constructions. For simplicity, we will not go into these details, and will instead use Theorem 6.6.1 to deduce most of the properties of this t-structure that we will need.

In fact, the only property (to be used in Section 7.2) that we could not prove in this way is the following. Recall the preimage $\widetilde{\mathcal{O}}_{\mathrm{r}}$ of the regular nilpotent orbit under the Springer map, see §6.5.11.

Proposition 7.1.4. — *For any $\lambda \in \mathbf{X}$ and any simple root α such that $\langle \lambda, \alpha^\vee \rangle > 0$, there exist morphisms*

$$\nabla_\lambda^{\mathrm{ex}} \to \nabla_{s_\alpha(\lambda)}^{\mathrm{ex}} \quad \text{and} \quad \Delta_{s_\alpha(\lambda)}^{\mathrm{ex}} \to \Delta_\lambda^{\mathrm{ex}}$$

whose restriction to $\widetilde{\mathcal{O}}_{\mathrm{r}}$ is an isomorphism.

Proof. — The morphisms can be chosen as those appearing in the distinguished triangles of [**Ac2**, Proposition 1.5(4)] (see also [**Be3**, Proposition 7]). Here for each triangle the third object has trivial restriction to $\widetilde{\mathcal{O}}_{\mathrm{r}}$ by definition of the functor denoted Ψ_α, so that these morphisms indeed have the stated property. □

7.1.3. Exotic and perverse t-structures. — The main result of the present section is the following.

Proposition 7.1.5. — *For any $\lambda \in \mathbf{X}^\vee$ we have*

$$F_{\mathcal{IW}}(\nabla_\lambda^{\mathrm{ex}}) \cong \nabla_\lambda^{\mathcal{IW}}, \qquad F_{\mathcal{IW}}(\Delta_\lambda^{\mathrm{ex}}) \cong \Delta_\lambda^{\mathcal{IW}}.$$

As a consequence, the functor $F_{\mathcal{IW}}$ is t-exact with respect to the exotic t-structure on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ and the perverse t-structure on $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. Moreover, the objects $\nabla_\lambda^{\mathrm{ex}}$ and $\Delta_\lambda^{\mathrm{ex}}$ lie in the abelian category $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$.

Proof. — We first observe that if we prove the first isomorphism, the rest of the proposition follows. Indeed, the isomorphisms $F_{\mathcal{IW}}(\Delta^{\mathrm{ex}}_\lambda) \cong \Delta^{\mathcal{IW}}_\lambda$ follow by uniqueness of the dual exceptional collection; t-exactness follows in view of the definition of the t-structure associated with an exceptional collection; and the last assertion follows from the fact that $\nabla^{\mathcal{IW}}_\lambda$ and $\Delta^{\mathcal{IW}}_\lambda$ are perverse (see 6.4.3).

Let us now prove that $F_{\mathcal{IW}}(\nabla^{\mathrm{ex}}_\lambda) \cong \nabla^{\mathcal{IW}}_\lambda$ for all $\lambda \in \mathbf{X}^\vee$. In view of Theorem 6.6.1, this amounts to proving that the collection $(\nabla^{\mathcal{IW}}_\lambda : \lambda \in \mathbf{X}^\vee)$ is the collection obtained from the exceptional collection $(\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)) : \lambda \in \mathbf{X}^\vee)$ in $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ by mutation for the order $\leq'_{\mathrm{geo}}$.

Recall from the proof of Lemma 6.6.3 that for any $\lambda \in \mathbf{X}^\vee$ the perverse sheaf $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell))$ is supported on $\overline{\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}}$, and that its restriction to $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ has rank one. Using these properties it is not difficult to show that for any order ideal $Y \subset \mathbf{X}^\vee$ (i.e. any subset Y such that if $y \in Y$ and $z \leq'_{\mathrm{geo}} y$ then $z \in Y$) the triangulated subcategory of $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ generated by the objects $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell))$ with $\lambda \in Y$ coincides with the subcategory of complexes supported on the closed subset

$$\bigcup_{\lambda \in Y} \overline{\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}}.$$

From this remark it is easily seen (using adjunction and the distinguished triangle of functors associated with a partition of a space into an open subspace and its closed complement) that the collection $(\nabla^{\mathcal{IW}}_\lambda : \lambda \in \mathbf{X}^\vee)$ satisfies the defining property for the collection obtained by mutation, which completes the proof. □

7.1.4. Some consequences. — From the general theory of perverse sheaves (see [**BBDG**]) we know that the simple objects in the category $\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ are naturally parametrized by $\mathbf{X}^\vee$. More precisely, for any $\lambda \in \mathbf{X}^\vee$, the vector space $\mathrm{Hom}_{\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\Delta^{\mathcal{IW}}_\lambda, \nabla^{\mathcal{IW}}_\lambda)$ is 1-dimensional; if we denote (as in §6.4.3) by $\mathscr{IC}^{\mathcal{IW}}_\lambda$ the image of any nonzero morphism in this space, then $\mathscr{IC}^{\mathcal{IW}}_\lambda$ is simple, and the assignment $\lambda \mapsto \mathscr{IC}^{\mathcal{IW}}_\lambda$ induces a bijection between $\mathbf{X}^\vee$ and the set of isomorphism classes of simple objects in $\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$.

From Proposition 7.1.5 we see that $F_{\mathcal{IW}}$ induces an equivalence of abelian categories

$$\mathsf{ExCoh}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} \mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell).$$

Hence from the classification of simple objects in $\mathsf{Perv}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$ one can deduce a similar classification for the category $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$. Namely, for any $\lambda \in \mathbf{X}^\vee$ the vector space $\mathrm{Hom}(\Delta^{\mathrm{ex}}_\lambda, \nabla^{\mathrm{ex}}_\lambda)$ is 1-dimensional. If we denote by $\mathrm{L}^{\mathrm{ex}}_\lambda$ the image of any nonzero morphism in this space, then $\mathrm{L}^{\mathrm{ex}}_\lambda$ is simple, and the assignment $\lambda \mapsto \mathrm{L}^{\mathrm{ex}}_\lambda$ induces a bijection between $\mathbf{X}^\vee$ and the set of isomorphism classes of simple objects in $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$. (Here, the term "image" makes sense because $\Delta^{\mathrm{ex}}_\lambda$ and $\nabla^{\mathrm{ex}}_\lambda$ live in the abelian category $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$: see Proposition 7.1.5.) From these considerations, it is clear that for any $\lambda \in \mathbf{X}^\vee$ we have

$$F_{\mathcal{IW}}(\mathrm{L}^{\mathrm{ex}}_\lambda) \cong \mathscr{IC}^{\mathcal{IW}}_\lambda. \tag{7.1.1}$$

Since $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$ is the heart of a t-structure on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$, there exists a natural "realization functor"

$$D^{\mathrm{b}}\mathsf{ExCoh}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}), \tag{7.1.2}$$

see [**BBDG**, §3.1] (see also [**Beĭ1**] for this construction in a more general context).

Corollary 7.1.6. — *The realization functor* (7.1.2) *is an equivalence of triangulated categories.*

Proof. — Using the equivalence $F_{\mathcal{IW}}$, one translates the statement into the corresponding statement for the perverse t-structure on $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_\ell)$. In this case the claim is well known: as explained in §6.6.1, it follows from the techniques of [**BGS**, §§3.2–3.3]. (Alternatively, the techniques of [**BGS**, §§3.2–3.3] can be applied directly to $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$, using the fact from Proposition 7.1.5 that the objects $\Delta^{\mathrm{ex}}_\lambda$ and $\nabla^{\mathrm{ex}}_\lambda$ lie in this category.) □

7.2. Description of the regular quotient

Our goal in this section is to make Proposition 6.5.18 more precise, by proving that in this proposition we have $\widetilde{\mathsf{P}}^0_I = \mathsf{P}^0_I$, and that the embedding $H \subset Z_{G^\vee}(n_0)$ is an equality. These statements will therefore provide an equivalence of monoidal categories

$$(\mathsf{P}^0_I, \circledast) \xrightarrow{\sim} (\mathsf{Rep}(Z_{G^\vee}(n_0)), \otimes).$$

These facts are stated in [**AB**, Remark 8], with the details of the proof left to the reader.[(1)]

7.2.1. Support of simple exotic sheaves. — First, we need to determine which simple exotic sheaves have a nonzero restriction to $\widetilde{\mathcal{O}}_{\mathrm{r}}$.

Lemma 7.2.1. — *For any $\lambda \in \mathbf{X}^\vee_+$ we have $\nabla^{\mathrm{ex}}_\lambda \cong \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$. Dually, for any $\lambda \in -\mathbf{X}^\vee_+$ we have $\Delta^{\mathrm{ex}}_\lambda \cong \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$.*

Proof. — Assume first that $\lambda \in \mathbf{X}^\vee_+$. Then we have $\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell) = \nabla^I_{\mathsf{t}(\lambda)}$. Using Lemma 6.4.5, it follows that $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)) = \nabla^{\mathcal{IW}}_\lambda$. Comparing (6.6.1) and Proposition 7.1.5, and applying $(F_{\mathcal{IW}})^{-1}$, we deduce the first claim.

Similarly, if $\lambda \in -\mathbf{X}^\vee_+$ we have $\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell) = \Delta^I_{\mathsf{t}(\lambda)}$, so that $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\overline{\mathbb{Q}}_\ell)) = \Delta^{\mathcal{IW}}_\lambda$, and we deduce that $\Delta^{\mathrm{ex}}_\lambda \cong \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$. □

Lemma 7.2.2. — *For any $\lambda \in \mathbf{X}^\vee$, the restriction of $\nabla^{\mathrm{ex}}_\lambda$ to $\widetilde{\mathcal{O}}_{\mathrm{r}}$ is isomorphic to the restriction of $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda^+)$, where λ^+ is the dominant W_{f}-conjugate of λ. Dually, for any $\lambda \in \mathbf{X}^\vee$, the restriction of $\nabla^{\mathrm{ex}}_\lambda$ to $\widetilde{\mathcal{O}}_{\mathrm{r}}$ is isomorphic to the restriction of $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda^-)$, where λ^- is the antidominant W_{f}-conjugate of λ.*

(1) As pointed out to us by a referee, these additional details can also be deduced from [**Be4**, Theorem 4].

Proof. — We prove the claim for the costandard objects; the case of standard objects is similar. The proof proceeds by induction on the length of the minimal element $w \in W_{\mathrm{f}}$ such that $\lambda = w(\lambda^+)$. If $w = e$, then the result follows from Lemma 7.2.1. In general, write $w = sv$ with $s \in S_{\mathrm{f}}$ and $\ell(w) = \ell(v) + 1$. Then by induction we know that the restriction of $\nabla^{\mathrm{ex}}_{s(\lambda)}$ to $\widetilde{\mathcal{O}}_{\mathrm{r}}$ is isomorphic to the restriction of $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda^+)$. We deduce the analogous claim for λ using Proposition 7.1.4. □

Corollary 7.2.3. — *For any $\mathscr{F}$ in $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$, the restriction $\mathscr{F}_{|\widetilde{\mathcal{O}}_{\mathrm{r}}}$ is a $G^\vee$-equivariant vector bundle on $\widetilde{\mathcal{O}}_{\mathrm{r}}$.*

Proof. — In view of (6.5.10), any $G^\vee$-equivariant coherent sheaf on $\widetilde{\mathcal{O}}_{\mathrm{r}}$ is a vector bundle. Hence the statement is equivalent to the claim that the restriction functor

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$$

is t-exact, where the left-hand side is endowed with the exotic t-structure and the right-hand side with the tautological t-structure. This claim in turn follows from Lemma 7.2.2 and the description of the t-structure associated with an exceptional collection in §7.1.1. □

Proposition 7.2.4. — *For $\lambda \in \mathbf{X}^\vee$, the restriction of $\mathrm{L}^{\mathrm{ex}}_\lambda$ to $\widetilde{\mathcal{O}}_{\mathrm{r}}$ coincides with the restriction of $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda^+)$ (where λ^+ is the dominant W_{f}-conjugate of λ) if $\ell(w_\lambda) = 0$, and is 0 otherwise.*

Proof. — First, assume that $\ell(w_\lambda) = 0$. Then the orbit $\mathrm{Fl}^{\mathcal{IW}}_{G,\lambda}$ does not contain any orbit $\mathrm{Fl}^{\mathcal{IW}}_{G,\mu}$ with $\mu \neq \lambda$ in its closure. It follows that $\nabla^{\mathcal{IW}}_\lambda \cong \mathscr{IC}^{\mathcal{IW}}_\lambda$. In view of Proposition 7.1.5 and (7.1.1), this implies that

$$\nabla^{\mathrm{ex}}_\lambda \cong \mathrm{L}^{\mathrm{ex}}_\lambda,$$

so that the claim follows from Lemma 7.2.2 in this case.

Now we consider the general case. Each connected component of Fl_G contains exactly one 0-dimensional I-orbit; in particular there exists $\mu \in \mathbf{X}^\vee$ such that $\ell(w_\mu) = 0$ and Fl_{G,w_μ} and $\mathrm{Fl}_{G,w_\lambda}$ belong to the same connected component. Then by Lemma 4.1.3 $\mathscr{IC}^I_{w_\mu}$ is a composition factor of $\nabla^I_{w_\lambda}$ (with multiplicity 1). Using Lemma 6.4.4, Lemma 6.4.5 and Corollary 6.4.7, it follows that $\mathscr{IC}^{\mathcal{IW}}_{w_\mu}$ is a composition factor of $\nabla^{\mathcal{IW}}_\mu$ (with multiplicity 1). Applying the functor $(F_{\mathcal{IW}})^{-1}$ and using Proposition 7.1.5 and (7.1.1), we deduce that $\mathrm{L}^{\mathrm{ex}}_\mu$ is a composition factor of $\nabla^{\mathrm{ex}}_\lambda$ (with multiplicity 1). Since the restriction of each simple exotic sheaf to $\widetilde{\mathcal{O}}_{\mathrm{r}}$ is a vector bundle (see Corollary 7.2.3) and since the restrictions of both $\nabla^{\mathrm{ex}}_\lambda$ and $\mathrm{L}^{\mathrm{ex}}_\mu$ are of rank 1 (see Lemma 7.2.2 and the case treated above), it follows that all the other composition factors of $\nabla^{\mathrm{ex}}_\lambda$ restrict trivially to $\widetilde{\mathcal{O}}_{\mathrm{r}}$. Since $\mathrm{L}^{\mathrm{ex}}_\lambda$ is a composition factor (in fact, a subobject) of $\nabla^{\mathrm{ex}}_\lambda$, it must in particular restrict trivially to $\widetilde{\mathcal{O}}_{\mathrm{r}}$. □

Remark 7.2.5. — One can check that $\ell(w_\lambda) = 0$ iff $w_\circ(\lambda)$ belongs to the subset of $\mathbf{X}^\vee$ denoted Σ in Remark 4.1.1. This explains the relation between Proposition 7.2.4 and [**Ac2**, Lemmas 5.1 and 5.2].

7.2.2. Induced equivalence for the regular nilpotent orbit. — Recall (see e.g. the proof of Lemma 6.5.25) that the quotient functor $\Pi^0 : \mathsf{P}_I \to \mathsf{P}_I^0$ factors through a functor $\mathsf{P}_I^{\mathrm{asph}} \to \mathsf{P}_I^0$, which will be denoted Π^0_{asph}. (It is easily checked that this functor identifies P_I^0 with the quotient of $\mathsf{P}_I^{\mathrm{asph}}$ by the Serre subcategory generated by the objects of the form $\Pi_{\mathrm{asph}}(\mathscr{IC}_w^I)$ with $w \in {}^{\mathrm{f}}W$ and $\ell(w) > 0$.) Recall also the equivalence F^{asph} defined in (6.6.3). Our goal in this subsection is to prove the following proposition.

Proposition 7.2.6. — *There exists a unique triangulated functor*

$$F^{\mathrm{r}} : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{O}}_{\mathrm{r}}) \to D^{\mathrm{b}}(\mathsf{P}_I^0)$$

which makes the following diagram commutative, where the left vertical arrow is induced by restriction:

$$\begin{array}{ccc} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}}) & \xrightarrow[\sim]{F^{\mathrm{asph}}} & D^{\mathrm{b}}(\mathsf{P}_I^{\mathrm{asph}}) \\ \big\downarrow & & \big\downarrow{\scriptstyle D^{\mathrm{b}}(\Pi^0_{\mathrm{asph}})} \\ D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{O}}_{\mathrm{r}}) & \xrightarrow{F^{\mathrm{r}}} & D^{\mathrm{b}}(\mathsf{P}_I^0). \end{array}$$

Moreover, F^{r} is an equivalence of categories, and it is t-exact with respect to the tautological t-structures on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{O}}_{\mathrm{r}})$ and $D^{\mathrm{b}}(\mathsf{P}_I^0)$.

For the proof of this proposition we will need the following property. We denote by $\mathsf{D}_{\widetilde{\mathscr{N}} \smallsetminus \widetilde{\mathscr{O}}_{\mathrm{r}}}$ the full triangulated subcategory of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}})$ whose objects are the complexes supported (set-theoretically) on the closed subset $\widetilde{\mathscr{N}} \smallsetminus \widetilde{\mathscr{O}}_{\mathrm{r}}$. In these terms, Proposition 7.2.4 implies that this subcategory contains all the objects $\mathrm{L}^{\mathrm{ex}}_\lambda$ with $\lambda \in \mathbf{X}^\vee$ such that $\ell(w_\lambda) > 0$.

Lemma 7.2.7. — *The category $\mathsf{D}_{\widetilde{\mathscr{N}} \smallsetminus \widetilde{\mathscr{O}}_{\mathrm{r}}}$ is generated, as a triangulated category, by the objects $\mathrm{L}^{\mathrm{ex}}_\lambda$ with $\lambda \in \mathbf{X}^\vee$ such that $\ell(w_\lambda) > 0$.*

Proof. — In view of the comments preceding the statement, what remains to be seen is that any complex $\mathscr{F}$ in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}})$ which restricts trivially to $\widetilde{\mathscr{O}}_{\mathrm{r}}$ belongs to the triangulated subcategory generated by the objects $\mathrm{L}^{\mathrm{ex}}_\lambda$ with $\lambda \in \mathbf{X}^\vee$ such that $\ell(w_\lambda) > 0$. Now if $\mathscr{F}_{|\widetilde{\mathscr{O}}_{\mathrm{r}}} = 0$, then using Corollary 7.2.3 we see that each composition factor of each exotic cohomology object of $\mathscr{F}$ also restricts trivially to $\widetilde{\mathscr{O}}_{\mathrm{r}}$. By Proposition 7.2.4 these composition factors must be of the form $\mathrm{L}^{\mathrm{ex}}_\lambda$ with $\ell(w_\lambda) > 0$, so that $\mathscr{F}$ indeed belongs to the triangulated subcategory generated by such objects. ☐

We are now in a position to prove Proposition 7.2.6.

Proof of Proposition 7.2.6. — Let us set

$$\bar{F}^{\mathrm{asph}} := D^{\mathrm{b}}(\Pi^0_{\mathrm{asph}}) \circ F^{\mathrm{asph}} : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}}) \to D^{\mathrm{b}}(\mathsf{P}_I^0).$$

(We have already encountered an abelian-category counterpart of this functor: see (6.5.12).) By (7.1.1) (see also Corollary 6.6.2), for any $\lambda \in \mathbf{X}^\vee$ we have

$$\bar{F}^{\mathrm{asph}}(\mathrm{L}^{\mathrm{ex}}_\lambda) = \Pi^0(\mathscr{I}\mathscr{C}^I_{w_\lambda}). \tag{7.2.1}$$

In particular, if $\ell(w_\lambda) > 0$ the right-hand side vanishes, and from this we deduce that $\bar{F}^{\mathrm{asph}}$ vanishes on the triangulated subcategory of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ generated by the objects $\mathrm{L}^{\mathrm{ex}}_\lambda$ with $\ell(w_\lambda) > 0$, or in other words (see Lemma 7.2.7), on the subcategory $\mathsf{D}_{\widetilde{\mathcal{N}} \smallsetminus \widetilde{\mathcal{O}}_{\mathrm{r}}}$. Now by [**AriB**, Remark after Lemma 2.12] the restriction functor

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$$

identifies $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$ with the Verdier quotient of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ by the subcategory $\mathsf{D}_{\widetilde{\mathcal{N}} \smallsetminus \widetilde{\mathcal{O}}_{\mathrm{r}}}$. The existence and uniqueness of F^{r} follow, by the universal property of the Verdier quotient (see [**SP**, Tag 05RJ]).

To prove that F^{r} is an equivalence, we note that the image of $\mathsf{D}_{\widetilde{\mathcal{N}} \smallsetminus \widetilde{\mathcal{O}}_{\mathrm{r}}}$ under the equivalence F^{asph} is the triangulated subcategory generated by the objects of the form $\Pi_{\mathrm{asph}}(\mathscr{I}\mathscr{C}^I_w)$ with $w \in {}^{\mathrm{f}}W$ such that $\ell(w) > 0$. By [**Miy**, Theorem 3.2] and the remarks at the beginning of the subsection, the functor

$$D^{\mathrm{b}}(\Pi^0_{\mathrm{asph}}) : D^{\mathrm{b}}(\mathsf{P}^{\mathrm{asph}}_I) \to D^{\mathrm{b}}(\mathsf{P}^0_I)$$

identifies $D^{\mathrm{b}}(\mathsf{P}^0_I)$ with the Verdier quotient of $D^{\mathrm{b}}(\mathsf{P}^{\mathrm{asph}}_I)$ by this subcategory, and the invertibility follows.

Finally, it remains to prove that F^{r} is t-exact. For this, observe (as follows e.g. from (6.5.10)) that any object in $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$ has finite length. Since F^{asph} is t-exact (with respect to the exotic t-structure on the domain and the tautological t-structure on the codomain, see Proposition 7.1.5), to conclude it suffices to show that any simple object in $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$ is the restriction of a simple object in $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$. This property can be checked in two different ways as follows.

First, recall from Corollary 7.2.3 that the restriction functor

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$$

is t-exact with respect to the exotic t-structure on the domain and the tautological t-structure on the codomain. On the other hand, it is well known that given an object $\mathscr{F}$ in $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$, there exists an object $\mathscr{G}$ in $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ such that $\mathscr{G}_{|\widetilde{\mathcal{O}}_{\mathrm{r}}} \cong \mathscr{F}$. (See [**AriB**, Lemma 2.12] for a much more general statement.) By exactness the 0-th exotic cohomology object $\mathscr{G}'$ of $\mathscr{G}$ also satisfies $\mathscr{G}'_{|\widetilde{\mathcal{O}}_{\mathrm{r}}} \cong \mathscr{F}$. If we furthermore assume that $\mathscr{F}$ is simple, then $\mathscr{G}'$ must admit a composition factor $\mathscr{G}''$ such that $(\mathscr{G}'')_{|\widetilde{\mathcal{O}}_{\mathrm{r}}} \cong \mathscr{F}$, which finishes the proof.

Alternatively, if $U_0^\vee$ is the unipotent radical of the unique Borel subgroup of $G^\vee$ containing n_0, it is known that multiplication induces a group isomorphism

$$Z(G^\vee) \times Z_{U_0^\vee}(n_0) \xrightarrow{\sim} Z_{G^\vee}(n_0).$$

(This can, for instance, be deduced from the results in [**Hu3**, §4.6].) In particular, since $Z_{U_0^\vee}(n_0)$ is unipotent, restriction induces an isomorphism between the isomorphism classes of simple $Z_{G^\vee}(n_0)$-modules and simple $Z(G^\vee)$-modules. Now $Z(G^\vee)$ is a diagonalizable group, so that any simple $Z(G^\vee)$-module is 1-dimensional; moreover restriction along the embedding $Z(G^\vee) \subset T^\vee$ induces an isomorphism

$$\mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee \xrightarrow{\sim} X^*(Z(G)).$$

For $\lambda \in \mathbf{X}^\vee$, it is not difficult to check that the image of the restriction of $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ to $\widetilde{\mathcal{O}}_{\mathrm{reg}}$ is the character corresponding to the image of λ in $\mathbf{X}^\vee/\mathbb{Z}\mathfrak{R}^\vee$. Then the desired claim follows from Proposition 7.2.4 and Remark 4.1.1. □

7.2.3. Consequence for the equivalence Φ^0. — Proposition 7.2.6 provides an equivalence of categories $\mathsf{P}_I^0 \cong \mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$, and by (6.5.10) we have an equivalence $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}}) \cong \mathsf{Rep}(Z_{G^\vee}(n_0))$. Combining these equivalences we have essentially reached our goal, but with a caveat. Namely, Proposition 6.5.18 is concerned with a certain specific functor, whose comparison with the equivalence obtained from Proposition 7.2.6 is not immediate. The more specific statement we were looking for is the following.

Proposition 7.2.8. — *In Proposition 6.5.18 we have $\widetilde{\mathsf{P}}_I^0 = \mathsf{P}_I^0$, and moreover $H = Z_{G^\vee}(n_0)$.*

To prove the proposition we will need some preliminaries.

Lemma 7.2.9. —
1. *The restriction functor $\mathsf{Coh}_{\mathrm{fr}}^{G^\vee}(\widetilde{\mathcal{N}}) \to \mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$ is fully faithful.*
2. *Every object in $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$ is a quotient of the restriction of some object in $\mathsf{Coh}_{\mathrm{fr}}^{G^\vee}(\widetilde{\mathcal{N}})$.*

Proof. — (1) We have seen in the proof of Lemma 6.5.26 that $\mathscr{O}(\widetilde{\mathcal{N}}) \cong \mathscr{O}(\mathcal{N})$ where $\mathcal{N} \subset \mathfrak{g}^\vee$ is the nilpotent cone, and hence that

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Coh}_{\mathrm{fr}}^{G^\vee}(\widetilde{\mathcal{N}})}(V_1 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}, V_2 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) &\cong \big(V_1^* \otimes V_2 \otimes \mathscr{O}(\widetilde{\mathcal{N}})\big)^{G^\vee} \\ &\cong \big(V_1^* \otimes V_2 \otimes \mathscr{O}(\mathcal{N})\big)^{G^\vee} \cong \mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\mathcal{N})}(V_1 \otimes \mathscr{O}_{\mathcal{N}}, V_2 \otimes \mathscr{O}_{\mathcal{N}}).\end{aligned}$$

Since the Springer map identifies $\widetilde{\mathcal{O}}_{\mathrm{r}}$ with the regular nilpotent orbit $\mathcal{O}_{\mathrm{r}}$ (see §6.5.11), our claim is equivalent to the claim that restriction from $\mathcal{N}$ to $\mathcal{O}_{\mathrm{r}}$ induces an isomorphism

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\mathcal{N})}(V_1 \otimes \mathscr{O}_{\mathcal{N}}, V_2 \otimes \mathscr{O}_{\mathcal{N}}) \cong \mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\mathcal{O}_{\mathrm{r}})}(V_1 \otimes \mathscr{O}_{\mathcal{O}_{\mathrm{r}}}, V_2 \otimes \mathscr{O}_{\mathcal{O}_{\mathrm{r}}}).$$

This claim follows from the fact that $\mathcal{N}$ is a normal variety (see the proof of Lemma 6.5.26) in which the complement of $\mathcal{O}_{\mathrm{r}}$ has codimension 2 (because $\mathcal{N}$ has finitely many $G^\vee$-orbits, each of which is of even dimension; see e.g. [**J2**, §2.8]), so that restriction induces an isomorphism $\mathscr{O}(\mathcal{N}) \xrightarrow{\sim} \mathscr{O}(\mathcal{O}_{\mathrm{r}})$.

(2) As explained in §6.5.11 the Springer map induces an isomorphism $\widetilde{\mathcal{O}}_{\mathrm{r}} \xrightarrow{\sim} \mathcal{O}_{\mathrm{r}}$. If we denote by $\mathscr{F}'$ the $G^\vee$-equivariant coherent sheaf on $\mathcal{O}_{\mathrm{r}}$ corresponding to $\mathscr{F}$ under this identification, then by, say, [**AriB**, Lemma 2.12(a)], $\mathscr{F}'$ is the restriction of some $G^\vee$-equivariant coherent sheaf $\mathscr{G}$ on $\mathcal{N}$. Since $\mathcal{N}$ is an affine variety, $\mathscr{G}$ is a quotient of some equivariant coherent sheaf of the form $V \otimes \mathscr{O}_{\mathcal{N}}$ with V in $\mathsf{Rep}(G^\vee)$.

(In fact, one can take for V any finite-dimensional $G^\vee$-stable subspace of $\Gamma(\mathscr{N}, \mathscr{G})$ which generates this $\mathscr{O}(\mathscr{N})$-module.) Pulling back the surjection $(V \otimes \mathscr{O}_{\mathscr{N}})_{|\mathcal{O}_{\mathrm{r}}} \twoheadrightarrow \mathscr{F}'$ along the isomorphism $\widetilde{\mathcal{O}}_{\mathrm{r}} \xrightarrow{\sim} \mathcal{O}_{\mathrm{r}}$ we obtain a surjection $(V \otimes \mathscr{O}_{\widetilde{\mathscr{N}}})_{|\widetilde{\mathcal{O}}_{\mathrm{r}}} \twoheadrightarrow \mathscr{F}$, which proves the desired claim. □

Corollary 7.2.10. — *For any $\mathscr{F}$ in $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$, there exist V_1, V_2 in $\mathsf{Rep}(G^\vee)$ and $f \in \mathrm{Hom}_{\mathsf{Coh}^{G^\vee}(\widetilde{\mathscr{N}})}(V_1 \otimes \mathscr{O}_{\widetilde{\mathscr{N}}}, V_2 \otimes \mathscr{O}_{\widetilde{\mathscr{N}}})$ such that $\mathscr{F} \cong \mathrm{cok}(f_{|\widetilde{\mathcal{O}}_{\mathrm{r}}})$.*

Proof. — Using Lemma 7.2.9(2) twice, we obtain V_1, V_2 in $\mathsf{Rep}(G^\vee)$ and a morphism $g : (V_1 \otimes \mathscr{O}_{\widetilde{\mathscr{N}}})_{|\widetilde{\mathcal{O}}_{\mathrm{r}}} \to (V_1 \otimes \mathscr{O}_{\widetilde{\mathscr{N}}})_{|\widetilde{\mathcal{O}}_{\mathrm{r}}}$ such that $\mathscr{F} \cong \mathrm{cok}(g)$. Now by Lemma 7.2.9(1) g is the restriction of a morphism $f : V_1 \otimes \mathscr{O}_{\widetilde{\mathscr{N}}} \to V_2 \otimes \mathscr{O}_{\widetilde{\mathscr{N}}}$, which finishes the proof. □

Proof of Proposition 7.2.8. — Consider the following commutative diagram, obtained by restricting the functors appearing in Proposition 7.2.6 to appropriate subcategories:

$$\begin{tikzcd}
\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathscr{N}}) \arrow[r, "F"] \arrow[d] \arrow[rr, bend left, "F^{\mathrm{asph}}"] & \mathsf{P}_I \arrow[r, "\Pi_{\mathrm{asph}}"] \arrow[dr, bend left, "\Pi^0"'] & \mathsf{P}_I^{\mathrm{asph}} \arrow[d, "\Pi^0_{\mathrm{asph}}"] \\
\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}}) \arrow[rr, "F^{\mathrm{r}}", "\sim"'] & & \mathsf{P}_I^0.
\end{tikzcd}$$

As noted in (6.5.12), the composition $\Pi^0 \circ F \cong \Pi^0_{\mathrm{asph}} \circ F^{\mathrm{asph}}$ in this diagram takes values in the subcategory $\widetilde{\mathsf{P}}_I^0 \subset \mathsf{P}_I^0$. By definition, $\widetilde{\mathsf{P}}_I^0$ is closed under subquotients. But if we follow the other path around the diagram, Lemma 7.2.9(2) implies that every object in P_I^0 is a quotient of some object in the image of $\Pi^0_{\mathrm{asph}} \circ F^{\mathrm{asph}}$. We conclude that

$$\widetilde{\mathsf{P}}_I^0 = \mathsf{P}_I^0.$$

In particular, the equivalence Φ^0 from Proposition 6.5.18 has P_I^0 as its domain. Extend the diagram above as follows:

$$\begin{tikzcd}
\mathsf{Coh}^{G^\vee}_{\mathrm{fr}}(\widetilde{\mathscr{N}}) \arrow[r, "F"] \arrow[d] \arrow[rr, bend left, "F^{\mathrm{asph}}"] & \mathsf{P}_I \arrow[r, "\Pi_{\mathrm{asph}}"] \arrow[dr, bend left, "\Pi^0"'] & \mathsf{P}_I^{\mathrm{asph}} \arrow[d, "\Pi^0_{\mathrm{asph}}"] \\
\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}}) \arrow[rr, "F^{\mathrm{r}}", "\sim"'] \arrow[d, "(6.5.10)"', "\wr"] & & \mathsf{P}_I^0 \arrow[d, "\Phi^0", "\wr"'] \\
\mathsf{Rep}(Z_{G^\vee}(n_0)) \arrow[rr, "\mathsf{For}_H^{Z_{G^\vee}(n_0)}"] & & \mathsf{Rep}(H).
\end{tikzcd} \tag{7.2.2}$$

We emphasize that we do *not* claim that this diagram as a whole is commutative. Its upper part is commutative by the considerations above, and if we omit the arrow labelled F^{r} then the remainder of the diagram is commutative by Lemma 6.5.26.

Let us denote by $\Upsilon : \mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}}) \xrightarrow{\sim} \mathsf{Rep}(Z_{G^\vee}(n_0))$ the equivalence from (6.5.10). We claim that for any $\mathscr{F}$ in $\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{O}}_{\mathrm{r}})$ there exists an isomorphism

$$\Phi^0 \circ F^{\mathrm{r}}(\mathscr{F}) \xrightarrow{\sim} \mathsf{For}_H^{Z_{G^\vee}(n_0)} \circ \Upsilon(\mathscr{F}). \tag{7.2.3}$$

In fact, by Corollary 7.2.10, there exist V_1, V_2 in $\mathsf{Rep}(G^\vee)$ and a morphism $f : V_1 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}} \to V_2 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$ such that $\mathscr{F} \cong \mathrm{cok}(f_{|\widetilde{\mathcal{O}}_{\mathrm{r}}})$. Applying the exact functors $\Phi^0 \circ F^{\mathrm{r}}$ and $\mathsf{For}_H^{Z_{G^\vee}(n_0)} \circ \Upsilon$, we deduce isomorphisms

$$\Phi^0 \circ F^{\mathrm{r}}(\mathscr{F}) \cong \mathrm{cok}\big(\Phi^0 \circ F^{\mathrm{r}}(f_{|\widetilde{\mathcal{O}}_{\mathrm{r}}})\big),$$
$$\mathsf{For}_H^{Z_{G^\vee}(n_0)} \circ \Upsilon(\mathscr{F}) \cong \mathrm{cok}\big(\mathsf{For}_H^{Z_{G^\vee}(n_0)} \circ \Upsilon(f_{|\widetilde{\mathcal{O}}_{\mathrm{r}}})\big).$$

Using the commutativity of the upper part of (7.2.2), from the first isomorphism we deduce that

$$\Phi^0 \circ F^{\mathrm{r}}(\mathscr{F}) \cong \mathrm{cok}\big(\Phi^0 \circ \Pi^0 \circ F(f)\big).$$

On the other hand, using the commutativity of the outer part of (7.2.2) we obtain isomorphisms making the following square commute:

$$\begin{tikzcd}
\Phi^0 \circ \Pi^0 \circ F(V_1 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \arrow[r, "\Phi^0\circ\Pi^0\circ F(f)"] \arrow[d, "\wr"'] & \Phi^0 \circ \Pi^0 \circ F(V_2 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \arrow[d, "\wr"'] \\
\mathsf{For}_H^{Z_{G^\vee}(n_0)} \circ \Upsilon((V_1 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}})_{|\widetilde{\mathcal{O}}_{\mathrm{r}}}) \arrow[r, "\mathsf{For}_H^{Z_{G^\vee}(n_0)}\circ\Upsilon(f_{|\widetilde{\mathcal{O}}_{\mathrm{r}}})"] & \mathsf{For}_H^{Z_{G^\vee}(n_0)} \circ \Upsilon((V_2 \otimes \mathscr{O}_{\widetilde{\mathcal{N}}})_{|\widetilde{\mathcal{O}}_{\mathrm{r}}}).
\end{tikzcd}$$

We deduce an isomorphism (7.2.3), finishing the proof of the claim.

Finally, we can conclude the proof as follows. By Tannakian formalism (see e.g. [**DM**, Proposition 2.8]), to prove that $H = Z_{G^\vee}(n_0)$ it suffices to prove that the functor $\mathsf{For}_H^{Z_{G^\vee}(n_0)}$ is an equivalence of categories. This functor is obviously faithful. If V, V' are in $\mathsf{Rep}(Z_{G^\vee}(n_0))$, then we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{Rep}(H)}(\mathsf{For}_H^{Z_{G^\vee}(n_0)}(V), \mathsf{For}_H^{Z_{G^\vee}(n_0)}(V')) \\
&\qquad \cong \mathrm{Hom}_{\mathsf{Rep}(H)}(\Phi^0 F^{\mathrm{r}} \Upsilon^{-1}(V), \Phi^0 F^{\mathrm{r}} \Upsilon^{-1}(V')) \cong \mathrm{Hom}_{\mathsf{Rep}(Z_{G^\vee}(n_0))}(V, V')
\end{aligned}$$

where the first isomorphism uses (7.2.3) (applied to $\Upsilon^{-1}(V)$ and $\Upsilon^{-1}(V')$), and the second one the fact that Φ^0, F^{r} and Υ^{-1} are equivalences. It follows that the injection $\mathrm{Hom}_{\mathsf{Rep}(Z_{G^\vee}(n_0))}(V, V') \hookrightarrow \mathrm{Hom}_{\mathsf{Rep}(H)}(\mathsf{For}_H^{Z_{G^\vee}(n_0)}(V), \mathsf{For}_H^{Z_{G^\vee}(n_0)}(V'))$ must be an isomorphism for dimension reasons, proving that $\mathsf{For}_H^{Z_{G^\vee}(n_0)}$ is fully faithful. On the other hand, for any V in $\mathsf{Rep}(H)$, applying (7.2.3) to $(F^{\mathrm{r}})^{-1} \circ (\Phi^0)^{-1}(V)$ we obtain an isomorphism

$$V \cong \mathsf{For}_H^{Z_{G^\vee}(n_0)}\big(\Upsilon \circ (F^{\mathrm{r}})^{-1} \circ (\Phi^0)^{-1}(V)\big).$$

It follows that $\mathsf{For}_H^{Z_{G^\vee}(n_0)}$ is essentially surjective, and hence an equivalence. □

7.3. A perverse description of equivariant coherent sheaves on $\mathcal{N}$

In this section we describe an equivalence of categories deduced from Theorem 6.6.1 in [**Be4**].

7.3.1. Statement. — We denote by ${}^{\mathrm{f}}W^{\mathrm{f}} \subset W$ the subset of elements w which are minimal (for the Bruhat order) in $W_{\mathrm{f}} w W_{\mathrm{f}}$. Under the bijection $\mathbf{X}^\vee \xrightarrow{\sim} {}^{\mathrm{f}}W$ given by $\lambda \mapsto w_\lambda$ (see §6.4.3), one can easily check that the preimage of the subset ${}^{\mathrm{f}}W^{\mathrm{f}} \subset {}^{\mathrm{f}}W$ is the subset $-\mathbf{X}^\vee_+$ of anti-dominant coweights. We will also denote by ${}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I$ the quotient of the category P_I by the Serre subcategory generated by the simple objects $\mathscr{IC}^I_w$ with $w \in W \smallsetminus {}^{\mathrm{f}}W^{\mathrm{f}}$. The quotient functor ${}^{\mathrm{f}}\Pi^{\mathrm{f}} : \mathsf{P}_I \to {}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I$ factors through a functor ${}^{\mathrm{f}}\Pi^{\mathrm{f}}_{\mathrm{asph}} : \mathsf{P}^{\mathrm{asph}}_I \to {}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I$ which identifies the category ${}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I$ with the quotient of $\mathsf{P}^{\mathrm{asph}}_I$ by the Serre subcategory generated by the objects $\Pi_{\mathrm{asph}}(\mathscr{IC}^I_w)$ with $w \in {}^{\mathrm{f}}W \smallsetminus {}^{\mathrm{f}}W^{\mathrm{f}}$. Recall also the Springer map $p_{\mathrm{Spr}} : \widetilde{\mathcal{N}} \to \mathcal{N}$ introduced in the course of the proof of Lemma 6.5.26, and the equivalence F^{asph} from (6.6.3). In this section we will be interested in the category $\mathsf{Coh}^{G^\vee}(\mathcal{N})$ of $G^\vee$-equivariant coherent sheaves on $\mathcal{N}$.

Theorem 7.3.1. — *There exists a unique triangulated functor*

$$ {}^{\mathrm{f}}F^{\mathrm{f}} : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N}) \to D^{\mathrm{b}}({}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I) $$

which makes the following diagram commutative:

$$\begin{tikzcd} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \arrow[r, "F^{\mathrm{asph}}", "\sim"'] \arrow[d, "R(p_{\mathrm{Spr}})_*"'] & D^{\mathrm{b}}(\mathsf{P}^{\mathrm{asph}}) \arrow[d, "D^{\mathrm{b}}({}^{\mathrm{f}}\Pi^{\mathrm{f}}_{\mathrm{asph}})"] \\ D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N}) \arrow[r, "{}^{\mathrm{f}}F^{\mathrm{f}}"] & D^{\mathrm{b}}({}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I). \end{tikzcd}$$

Moreover, ${}^{\mathrm{f}}F^{\mathrm{f}}$ is an equivalence of categories.

Remark 7.3.2. — Transporting the tautological t-structure on $D^{\mathrm{b}}({}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I)$ along the equivalence ${}^{\mathrm{f}}F^{\mathrm{f}}$ one obtains a nonstandard t-structure on the triangulated category $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})$. This t-structure also has an intrinsic construction: it is a special case of the "perverse coherent" t-structure from [**AriB**]; see [**Be4**, Theorem 2].

The proof of Theorem 7.3.1 will be given in §7.3.4 below, after a number of preliminary results proved in the next two subsections.

7.3.2. Pushforward to the nilpotent cone. — We start with some results regarding the functor $R(p_{\mathrm{Spr}})_*$ and the "free" equivariant coherent sheaves on $\widetilde{\mathcal{N}}$. Recall the simple exotic sheaves $\mathrm{L}^{\mathrm{ex}}_\lambda$ ($\lambda \in \mathbf{X}^\vee$) introduced in §7.1.4.

Lemma 7.3.3. — *If $\lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)$, then we have*

$$ R(p_{\mathrm{Spr}})_* \mathrm{L}^{\mathrm{ex}}_\lambda = 0. $$

Proof. — Let $\lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)$. By the Yoneda lemma, and since the scheme $\mathcal{N}$ is affine, to prove that $R(p_{\mathrm{Spr}})_* \mathrm{L}^{\mathrm{ex}}_\lambda = 0$ it suffices to prove that

$$ \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})}(V \otimes \mathscr{O}_{\mathcal{N}}, R(p_{\mathrm{Spr}})_* \mathrm{L}^{\mathrm{ex}}_\lambda[n]) = 0 $$

for any V in $\mathsf{Rep}(G^\vee)$ and $n \in \mathbb{Z}$, or in other words (using adjunction) that

$$ \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}, \mathrm{L}^{\mathrm{ex}}_\lambda[n]) = 0 $$

for any V in $\mathsf{Rep}(G^\vee)$ and $n \in \mathbb{Z}$. Using the equivalence $F_{\mathcal{IW}}$, in view of (6.6.1) and (7.1.1), what we have to prove translates to

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\overline{\mathbb{Q}}_\ell)}(\mathscr{Z}^{\mathcal{IW}}(V), \mathscr{IC}^{\mathcal{IW}}_\lambda[n]) = 0$$

for all V in $\mathsf{Rep}(G^\vee)$ and $n \in \mathbb{Z}$.

Now the assumption that $\lambda \notin -\mathbf{X}^\vee_+$ means that $w_\lambda \notin {}^{\mathrm{f}}W^{\mathrm{f}}$ (see §7.3.1), and hence that $\mathscr{IC}^{\mathcal{IW}}_\lambda$ is the pullback of a complex (in fact, a shift of a simple perverse sheaf) under the morphism $\pi_s : \mathrm{Fl}_G \to \mathrm{L}G/J_s$ for some $s \in S_{\mathrm{f}}$ (where we use the notation from the proof of Lemma 6.4.4). Thus, by adjunction, to finish the proof it suffices to check that

$$(\pi_s)_* \mathscr{Z}^{\mathcal{IW}}(V) = 0$$

for all V in $\mathsf{Rep}(G^\vee)$. This, in turn, will follow if we prove that

$$(\pi_s)^*(\pi_s)_* \mathscr{Z}^{\mathcal{IW}}(V) = 0$$

for all V in $\mathsf{Rep}(G^\vee)$. Now, using the base change theorem we see that

$$(\pi_s)^*(\pi_s)_* \mathscr{Z}^{\mathcal{IW}}(V) \cong \mathscr{Z}^{\mathcal{IW}}(V) \star^I \mathscr{IC}^I_s[-1].$$

Then, using the centrality of $\mathscr{Z}(V)$ (see Theorem 3.2.3) and Lemma 6.4.4 we see that

$$\mathscr{Z}^{\mathcal{IW}}(V) \star^I \mathscr{IC}^I_s = \Delta^{\mathcal{IW}}_0 \star^I \mathscr{Z}(V) \star^I \mathscr{IC}^I_s \cong \Delta^{\mathcal{IW}}_0 \star^I \mathscr{IC}^I_s \star^I \mathscr{Z}(V) = 0,$$

which concludes the proof. □

Remark 7.3.4. — See [**Be3**, Proposition 8] or [**Ac2**, Proposition 1.6] for proofs of Lemma 7.3.3 which do not rely on the equivalence $F_{\mathcal{IW}}$.

Lemma 7.3.5. — *For any V in $\mathsf{Rep}(G^\vee)$ and any X in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})$, the functor $R(p_{\mathrm{Spr}})_*$ induces an isomorphism*

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})}(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}, X) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})}\big(R(p_{\mathrm{Spr}})_*(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}), R(p_{\mathrm{Spr}})_*(X)\big).$$

Proof. — It follows from (6.6.5) (using the adjunction $(L(p_{\mathrm{Spr}})^*, R(p_{\mathrm{Spr}})_*)$) that the higher derived functors $R^i(p_{\mathrm{Spr}})_*\mathcal{O}_{\widetilde{\mathcal{N}}}$ vanish for $i > 0$. In view of (6.5.14), we have a canonical isomorphism

$$R(p_{\mathrm{Spr}})_*\mathcal{O}_{\widetilde{\mathcal{N}}} \cong \mathcal{O}_{\mathcal{N}}.$$

This implies that $R(p_{\mathrm{Spr}})_*(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}) \cong V \otimes \mathcal{O}_{\mathcal{N}}$, and then the claim follows by adjunction again. □

Lemma 7.3.6. — *For any $\mu \in \mathbf{X}^\vee_+$, the object $\mathsf{N}(\mu) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$ belongs to $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$, and there exists a surjective morphism*

$$\mathsf{N}(\mu) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}} \twoheadrightarrow \nabla^{\mathrm{ex}}_\mu$$

whose kernel belongs to the subcategory generated under extensions by the objects ∇^{ex}_ν ($\nu \in \mathbf{X}^\vee$).

Proof. — The fact that $\mathsf{N}(\mu) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}$ belongs to $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$ follows from (6.6.1) and Proposition 7.1.5, since $\mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\mu))$ is perverse. To prove the second claim, using the same results, it suffices to prove that there exists a surjection $\mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\mu)) \twoheadrightarrow \nabla^{\mathcal{IW}}_{\mu}$ whose kernel admits a filtration with costandard subquotients. In turn, this is easily seen to follow from the fact that $\mathscr{Z}^{\mathcal{IW}}(\mathsf{N}(\mu))$ is tilting (see Theorem 6.5.2) and that $\mathrm{Fl}^{\mathcal{IW}}_{G,\mu}$ is open in its support (see Remark 6.5.5). $\square$

The proof of the following lemma relies on the geometry of nilpotent orbits, and will not be reviewed here. For details, see [**Be1**, Lemma 7] or [**Ac1**, Lemma 5.9].

Lemma 7.3.7. — *The category $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})$ is generated, as a triangulated category, by the essential image of the functor*

$$R(p_{\mathrm{Spr}})_* : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N}).$$

7.3.3. Quotient by some simple exotic shaves. — Let us now denote by $\langle \mathrm{L}^{\mathrm{ex}}_\lambda : \lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)\rangle_{\mathrm{Serre}}$ the Serre subcategory of the abelian category $\mathsf{ExCoh}(\widetilde{\mathcal{N}})$ generated by the simple objects $\mathrm{L}^{\mathrm{ex}}_\lambda$ with $\lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)$. We can then consider the quotient abelian category

$$\mathbf{A} := \mathsf{ExCoh}(\widetilde{\mathcal{N}})/\langle \mathrm{L}^{\mathrm{ex}}_\lambda : \lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)\rangle_{\mathrm{Serre}}.$$

Recall that by Corollary 7.1.6 we have a natural equivalence

$$D^{\mathrm{b}}\mathsf{ExCoh}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}).$$

We deduce a natural triangulated functor

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathbf{A}. \tag{7.3.1}$$

On the other hand, we denote by $\langle \mathrm{L}^{\mathrm{ex}}_\lambda : \lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)\rangle_\Delta$ the triangulated subcategory of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ generated by the objects $\mathrm{L}^{\mathrm{ex}}_\lambda$ with $\lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)$. Then we can also consider the Verdier quotient category

$$\mathbf{D} := D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})/\langle \mathrm{L}^{\mathrm{ex}}_\lambda : \lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)\rangle_\Delta.$$

Most of the statements in this subsection are concerned with the quotient functor

$$\Pi : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \to \mathbf{D}.$$

The universal property of the Verdier quotient ([**SP**, Tag 05RJ]) says that triangulated functors out of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ that kill all $\mathrm{L}^{\mathrm{ex}}_\lambda$ for $\lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+)$ must factor uniquely through Π. This property applies in particular to $R(p_{\mathrm{Spr}})_*$ (by Lemma 7.3.3) and to the functor in (7.3.1). We obtain factorizations

$$\begin{tikzcd} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \arrow[r, "\Pi"'] \arrow[rr, bend left, "R(p_{\mathrm{Spr}})_*"] & \mathbf{D} \arrow[r, "\Phi_{\mathcal{N}}"'] & D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N}) \end{tikzcd} \tag{7.3.2}$$

and

$$\begin{tikzcd} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \arrow[r, "\Pi"'] \arrow[rr, bend left, "(7.3.1)"] & \mathbf{D} \arrow[r, "\sim"'] & D^{\mathrm{b}}\mathbf{A} \end{tikzcd} \tag{7.3.3}$$

In the latter case, the induced functor $\mathbf{D} \to D^{\mathrm{b}}\mathbf{A}$ is an equivalence by [**Miy**, Theorem 3.2].

Lemma 7.3.8. — *For any $X \in \mathbf{D}$, there exists $d \in \mathbb{Z}$ such that*

$$\mathrm{Hom}_{\mathbf{D}}(\Pi(\nabla^{\mathrm{ex}}_{\lambda}), X[i]) = 0$$

for any $i < d$ and any $\lambda \in \mathbf{X}^{\vee}$.

Proof. — Transfer this problem across the equivalence $\mathbf{D} \xrightarrow{\sim} D^{\mathrm{b}}\mathbf{A}$ from (7.3.3). Then the claim becomes clear: since each $\Pi(\nabla^{\mathrm{ex}}_{\lambda})$ belongs to (the preimage of) $\mathbf{A}$, it suffices to choose d such that (the image of) $X[d]$ is concentrated in nonnegative degrees. □

Lemma 7.3.9. — *Let $V \in \mathsf{Rep}(G^{\vee})$.*

1. *For any X in $D^{\mathrm{b}}\mathsf{Coh}^{G^{\vee}}(\widetilde{\mathcal{N}})$, the functor Π induces an isomorphism*

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^{\vee}}(\widetilde{\mathcal{N}})}(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(\Pi(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}), \Pi(X)).$$

2. *For any $\nu \in \mathbf{X}^{\vee}$ we have*

$$\mathrm{Hom}_{\mathbf{D}}(\Pi(V \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}), \Pi(\nabla^{\mathrm{ex}}_{\nu})[1]) = 0.$$

Proof. — (1) By [**Miy**, Lemma 2.1] (or more precisely the dual statement), to prove the claim it suffices to prove that

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^{\vee}}(\widetilde{\mathcal{N}})}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}, X) = 0$$

if X belongs to $\langle \mathrm{L}^{\mathrm{ex}}_{\lambda} : \lambda \in \mathbf{X}^{\vee} \smallsetminus (-\mathbf{X}^{\vee}_{+}) \rangle_{\Delta}$. By standard arguments, it suffices to check this when $X = \mathrm{L}^{\mathrm{ex}}_{\lambda}[n]$ for some $\lambda \in \mathbf{X}^{\vee} \smallsetminus (-\mathbf{X}^{\vee}_{+})$ and $n \in \mathbb{Z}$, and in this case the claim was checked in the course of the proof of Lemma 7.3.3.

(2) In view of (1), to prove the claim it suffices to prove that

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^{\vee}}(\widetilde{\mathcal{N}})}(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}, \nabla^{\mathrm{ex}}_{\nu}[1]) = 0.$$

Using the equivalence $F_{\mathcal{IW}}$, (6.6.1) and Proposition 7.1.5, this assertion is equivalent to the assertion that

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \overline{\mathbb{Q}}_{\ell})}(\mathscr{T}^{\mathcal{IW}}(V), \nabla^{\mathcal{IW}}_{\nu}[1]) = 0.$$

Finally, the latter assertion follows from the fact that $\mathscr{T}^{\mathcal{IW}}(V)$ is tilting (see Theorem 6.5.2) and (6.4.2). □

Lemma 7.3.10. — *For any $\lambda \in \mathbf{X}$ and $w \in W_{\mathrm{f}}$, we have*

$$\Pi(\nabla^{\mathrm{ex}}_{\lambda}) \cong \Pi(\nabla^{\mathrm{ex}}_{w\lambda})$$

in $\mathbf{D}$.

Proof. — Of course, it suffices to prove the claim in case $w = s \in S_{\mathrm{f}}$ and $s\lambda \neq \lambda$. Moreover, replacing λ by $s\lambda$ if necessary, we can assume that $\langle \lambda, \alpha \rangle > 0$, where α is the simple root associated with s. As usual, we transport the statement across the equivalence $F_{\mathcal{IW}}$. Then, in view of Proposition 7.1.5 and (7.1.1), what we have to prove is that $\nabla^{\mathcal{IW}}_{\lambda}$ and $\nabla^{\mathcal{IW}}_{s\lambda}$ have isomorphic images in the Verdier quotient by the full triangulated subcategory generated by the objects $\mathscr{IC}^{\mathcal{IW}}_{\mu}$ with $\mu \notin -\mathbf{X}^{\vee}_{+}$.

From the formula (4.1.1) we see that $\ell(\mathsf{t}(\lambda)s) = \ell(\mathsf{t}(\lambda)) + 1$, so that
$$\nabla^I_{\mathsf{t}(\lambda)s} \cong \nabla^I_{\mathsf{t}(\lambda)} \star^I \nabla^I_s$$
(see Lemma 4.1.4(2)). By Lemma 6.4.5, this implies that
$$\nabla^{\mathcal{IW}}_{s\lambda} \cong \nabla^{\mathcal{IW}}_{\lambda} \star^I \nabla^I_s.$$
Now we have an exact sequence of perverse sheaves $\mathscr{IC}^I_s \hookrightarrow \nabla^I_s \twoheadrightarrow \mathscr{IC}^I_e$; convolving with $\nabla^{\mathcal{IW}}_\lambda$ on the left, we deduce a distinguished triangle
$$\nabla^{\mathcal{IW}}_{\lambda} \star^I \mathscr{IC}^I_s \to \nabla^{\mathcal{IW}}_{s\lambda} \to \nabla^{\mathcal{IW}}_{\lambda} \xrightarrow{[1]} .$$
Hence, to conclude it suffices to prove that $\nabla^{\mathcal{IW}}_{\lambda} \star^I \mathscr{IC}^I_s$ belongs to the full triangulated subcategory generated by the objects $\mathscr{IC}^{\mathcal{IW}}_\mu$ with $\mu \notin -\mathbf{X}^\vee_+$.

As already used above we have $\nabla^{\mathcal{IW}}_{\lambda} \cong \mathsf{Av}_{\mathcal{IW}}(\nabla^I_{\mathsf{t}(\lambda)})$, so that
$$\nabla^{\mathcal{IW}}_{\lambda} \star^I \mathscr{IC}^I_s \cong \mathsf{Av}_{\mathcal{IW}}(\nabla^I_{\mathsf{t}(\lambda)}) \star^I \mathscr{IC}^I_s \cong \mathsf{Av}_{\mathcal{IW}}(\nabla^I_{\mathsf{t}(\lambda)} \star^I \mathscr{IC}^I_s).$$
As in the proof of Lemma 7.3.3, the base change theorem implies that we have
$$\nabla^I_{\mathsf{t}(\lambda)} \star^I \mathscr{IC}^I_s \cong (\pi_s)^*(\pi_s)_* \nabla^I_{\mathsf{t}(\lambda)}[1].$$
Since $\mathsf{t}(\lambda)s >_{\mathrm{Bru}} \mathsf{t}(\lambda)$, π_s induces an isomorphism between $\mathrm{Fl}_{G,\mathsf{t}(\lambda)}$ and its image in $\mathrm{L}G/J_s$. Denote that image (which is an I-orbit) by $\mathrm{Fl}^s_{G,\mathsf{t}(\lambda)}$. Then $(\pi_s)_*\nabla^I_{\mathsf{t}(\lambda)}$ identifies with the !-extension of the constant local system on $\mathrm{Fl}^s_{G,\mathsf{t}(\lambda)}$ shifted by $\ell(\mathsf{t}(\lambda))$. The latter object is a perverse sheaf; since the functor $(\pi_s)^*[1]$ is t-exact and sends simple I-equivariant perverse sheaves to simple perverse sheaves of the form $\mathscr{IC}^I_y$ with $ys < y$ (by [**BBDG**, §§4.2.5–4.2.6]), we deduce that $(\pi_s)^*(\pi_s)_*\nabla^I_{\mathsf{t}(\lambda)}[1]$ is perverse, and that all of its composition factors are of this form. Using Lemma 6.4.4 and Corollary 6.4.7 we deduce that $\nabla^{\mathcal{IW}}_{\lambda} \star^I \mathscr{IC}^I_s$ is perverse, and that all of its composition factors are of the form $\mathscr{IC}^{\mathcal{IW}}_\nu$ where $\nu \in \mathbf{X}^\vee$ satisfies $w_\nu s <_{\mathrm{Bru}} w_\nu$. In particular, these ν's cannot belong to $-\mathbf{X}^\vee_+$, so that indeed $\nabla^{\mathcal{IW}}_{\lambda} \star^I \mathscr{IC}^I_s$ belongs to the triangulated subcategory generated by the objects $\mathscr{IC}^{\mathcal{IW}}_\mu$ with $\mu \notin -\mathbf{X}^\vee_+$. ☐

Lemma 7.3.11. — *For any $\mu \in \mathbf{X}^\vee$, the complex $R(p_{\mathrm{Spr}})_*\nabla^{\mathrm{ex}}_\mu$ is concentrated in degree* 0 *(i.e. is a coherent sheaf).*

Proof. — Recall that we have $R(p_{\mathrm{Spr}})_* = \Phi_{\mathcal{N}} \circ \Pi$. Hence Lemma 7.3.10 implies that it suffices to prove the claim in case $\mu \in \mathbf{X}^\vee_+$. In this case, by Lemma 7.2.1 we have $\nabla^{\mathrm{ex}}_\mu \cong \mathscr{O}_{\widetilde{\mathcal{N}}}(\mu)$. Now, as already used in the proof of Corollary 6.6.5, by [**Bro**, Theorem 2.4] the complex $R(p_{\mathrm{Spr}})_*\mathscr{O}_{\widetilde{\mathcal{N}}}(\mu)$ is concentrated in degree 0, which finishes the proof. ☐

Lemma 7.3.12. — *Let X be an object of $\mathbf{D}$ which admits a filtration (in the sense of triangulated categories) with subquotients of the form $\Pi(\nabla^{\mathrm{ex}}_\mu)$ ($\mu \in \mathbf{X}^\vee$). Then there exist V in $\mathsf{Rep}(G^\vee)$ and a distinguished triangle*
$$Y \to \Pi(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \to X \xrightarrow{[1]}$$
where Y is an object of $\mathbf{D}$ which admits a filtration (again in the sense of triangulated categories) with subquotients of the form $\Pi(\nabla^{\mathrm{ex}}_\mu)$ ($\mu \in \mathbf{X}^\vee$).

(See the proof of Proposition 6.5.7 for the notion of a filtration in the sense of triangulated categories.)

Proof. — We proceed by induction on the length of the filtration of X. If this length is 0, i.e., if $X = 0$, then there is nothing to prove. Otherwise, our object X fits into a distinguished triangle

$$X' \xrightarrow{f} X \xrightarrow{g} \Pi(\nabla^{\mathrm{ex}}_{\mu}) \xrightarrow{[1]}$$

where X' has a shorter filtration, still with subquotients of the form $\Pi(\nabla^{\mathrm{ex}}_{\nu})$. By Lemma 7.3.10, we may assume that $\mu \in \mathbf{X}^{\vee}_{+}$. Assume by induction that the conclusion holds for X': there is a distinguished triangle

$$Y' \to \Pi(V' \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \xrightarrow{h} X' \xrightarrow{[1]}$$

where Y' has a filtration with subquotients of the form $\Pi(\nabla^{\mathrm{ex}}_{\nu})$. On the other hand, since $\mu \in \mathbf{X}^{\vee}_{+}$, Lemma 7.3.6 tells us that there is a short exact sequence of exotic sheaves $K \hookrightarrow \mathsf{N}(\mu) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}} \twoheadrightarrow \nabla^{\mathrm{ex}}_{\mu}$ where K admits a filtration by various $\nabla^{\mathrm{ex}}_{\nu}$. Apply Π to obtain a distinguished triangle

$$\Pi(K) \to \Pi(\mathsf{N}(\mu) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \xrightarrow{j} \Pi(\nabla^{\mathrm{ex}}_{\mu}) \xrightarrow{[1]}$$

By Lemma 7.3.9(2), the composition $\Pi(\mathsf{N}(\mu) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \xrightarrow{j} \Pi(\nabla^{\mathrm{ex}}_{\mu}) \to X'[1]$ vanishes, so j factors through a morphism $j' : \Pi(\mathsf{N}(\mu) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \to X$.

Set $V := V' \oplus \mathsf{N}(\mu)$, and let Y be the cocone of the map $f \circ h + j' : \Pi(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \to X$. We can assemble these objects into the commutative diagram as shown below. In this diagram, the bottom two rows and all three columns are distinguished triangle. (Furthermore, the middle row is split.)

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \Pi(K) \arrow[r, "{[1]}"] \arrow[d] & {} \\
\Pi(V' \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \arrow[r] \arrow[d, "h"'] & \Pi(V \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \arrow[r] \arrow[d, "f\circ h+j'"'] & \Pi(\mathsf{N}(\mu) \otimes \mathscr{O}_{\widetilde{\mathcal{N}}}) \arrow[r, "{[1]}"] \arrow[d, "j"] \arrow[dl, "j'"'] & {} \\
X' \arrow[r, "f"'] \arrow[d, "{[1]}"] & X \arrow[r, "g"'] \arrow[d, "{[1]}"] & \Pi(\nabla^{\mathrm{ex}}_{\mu}) \arrow[r, "{[1]}"] \arrow[d, "{[1]}"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

By the 9-lemma (see [**BBDG**, Proposition 1.1.11]), the top row is a distinguished triangle, and hence Y has a filtration of the desired form. ☐

7.3.4. Proof of Theorem 7.3.1. — We can finally explain the proof of Theorem 7.3.1. It will be based on the following general lemma.

Lemma 7.3.13. — *Let $\mathbf{C}$ be a triangulated category, and consider two cohomological functors $H_1 = (H_1^j : j \in \mathbb{Z})$ and $H_2 = (H_2^j : j \in \mathbb{Z})$ from $\mathbf{C}$ to an abelian category. Also let $\varphi : H_1 \to H_2$ be a morphism of cohomological functors, and let $(X_i : i \in I)$ be a collection of objects in $\mathbf{C}$. Assume that*

1. *there exists $d \in \mathbb{Z}$ such that $H_1^j(X_i) = 0 = H_2^j(X_i)$ for all $j < d$ and all $i \in I$;*

2. *for any $i \in I$ there exist X in $\mathbf{C}$, $i' \in I$, and a distinguished triangle*
$$X_i \to X \to X_{i'} \xrightarrow{[1]}$$
such that $\varphi^j_X : H^j_1(X) \to H^j_2(X)$ is an isomorphism for all $j \in \mathbb{Z}$.

Then $\varphi^j_{X_i} : H^j_1(X_i) \to H^j_2(X_i)$ is an isomorphism for all $i \in I$ and $j \in \mathbb{Z}$.

Proof. — The proof proceeds by induction on j. The claim is obvious if $j < d$ by (1). We then consider some $j \in \mathbb{Z}$, and assume the property is known for $j-1$. Fix $i \in I$ and a triangle as in (2). The associated long exact sequences provide a commutative diagram

$$\begin{tikzcd}
H^{j-1}_1(X) \arrow[r] \arrow[d, "\varphi^{j-1}_X"] & H^{j-1}_1(X_{i'}) \arrow[r] \arrow[d, "\varphi^{j-1}_{X_{i'}}"] & H^j_1(X_i) \arrow[r] \arrow[d, "\varphi^j_{X_i}"] & H^j_1(X) \arrow[r] \arrow[d, "\varphi^j_X"] & H^j_1(X_{i'}) \arrow[d, "\varphi^j_{X_{i'}}"] \\
H^{j-1}_2(X) \arrow[r] & H^{j-1}_2(X_{i'}) \arrow[r] & H^j_2(X_i) \arrow[r] & H^j_2(X) \arrow[r] & H^j_2(X_{i'})
\end{tikzcd}$$

in which the second vertical arrow is an isomorphism by induction, and the first and fourth arrows are isomorphisms by assumption. Ignoring the fifth column and applying one of the four lemmas, this diagram shows that $\varphi^j_{X_i}$ is injective for any i. Then, in the full diagram we know that the fifth vertical arrow is injective, so that by the five lemma the morphism $\varphi^j_{X_i}$ is invertible. $\square$

Proof of Theorem 7.3.1. — Let $\mathbf{D}'$ be the Verdier quotient of $D^{\mathrm{b}}(\mathsf{P}^{\mathrm{asph}})$ by the full triangulated subcategory generated by the objects $\Pi_{\mathrm{asph}}(\mathscr{IC}^I_w)$ with $w \in {}^{\mathrm{f}}W \smallsetminus {}^{\mathrm{f}}W^{\mathrm{f}}$. (The latter subcategory is the image of $\langle \mathrm{L}^{\mathrm{ex}}_\lambda : \lambda \in \mathbf{X}^\vee \smallsetminus (-\mathbf{X}^\vee_+) \rangle_\Delta$ under F^{asph}, thanks to Corollary 6.4.7 and (7.1.1).) Expand the diagram from the statement of the theorem as follows:

$$\begin{tikzcd}
D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}}) \arrow[r, "F^{\mathrm{asph}}", "\sim"'] \arrow[d, "\Pi"] \arrow[dd, bend right, "R(p_{\mathrm{Spr}})_*"'] & D^{\mathrm{b}}(\mathsf{P}^{\mathrm{asph}}) \arrow[d] \arrow[dd, bend left, "D^{\mathrm{b}}({}^{\mathrm{f}}\Pi^{\mathrm{f}}_{\mathrm{asph}})"] \\
\mathbf{D} \arrow[r, "\sim"] \arrow[d, "\Phi_{\mathcal{N}}"] & \mathbf{D}' \arrow[d, "\wr"'] \\
D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N}) \arrow[r, dashed, "{}^{\mathrm{f}}F^{\mathrm{f}}"] & D^{\mathrm{b}}({}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I).
\end{tikzcd}$$

Here, the middle horizontal arrow is an equivalence induced by F^{asph}. In the right-hand column, we have factored $D^{\mathrm{b}}({}^{\mathrm{f}}\Pi^{\mathrm{f}}_{\mathrm{asph}})$ through $\mathbf{D}'$ using the universal property of the latter. As in (7.3.3), [**Miy**, Theorem 3.2] implies that the induced functor $\mathbf{D}' \to D^{\mathrm{b}}({}^{\mathrm{f}}\mathsf{P}^{\mathrm{f}}_I)$ is an equivalence of categories. Thus, the existence of ${}^{\mathrm{f}}F^{\mathrm{f}}$ and the claim that it is an equivalence will both follow if we prove that the functor $\Phi_{\mathcal{N}}$ of (7.3.2) is an equivalence of categories.

First we will prove that $\Phi_{\mathcal{N}}$ is fully faithful. For this we fix X_0 in $\mathbf{D}$, and consider the cohomological functors H_1 and H_2 from $\mathbf{D}^{\mathrm{op}}$ to the category of $\overline{\mathbb{Q}}_\ell$-vector spaces defined by
$$H^j_1(X) := \mathrm{Hom}_{\mathbf{D}}(X, X_0[j]), \quad H^j_2(X) := \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})}(\Phi_{\mathcal{N}}(X), \Phi_{\mathcal{N}}(X_0)[j]).$$
The functor $\Phi_{\mathcal{N}}$ induces a morphism of cohomological functors $H_1 \to H_2$.

We will apply Lemma 7.3.13 to these data, with $(X_i : i \in I)$ being the collection of objects of $\mathbf{D}$ which admit a filtration (in the sense of triangulated categories) with subquotients of the form $\Pi(\nabla^{\mathrm{ex}}_\mu)$ with $\mu \in \mathbf{X}^\vee$. Assumption (1) can be checked independently for H_1 and H_2. For H_1 it follows from Lemma 7.3.8. For H_2 it follows from Lemma 7.3.11. (In fact, one can take for d any integer such that the complex $\Phi_{\mathcal{N}}(X_0)[d]$ is concentrated in nonnegative degrees.)

To check assumption (2), we observe that for any $V \in \mathsf{Rep}(G^\vee)$ the functor $\Phi_{\mathcal{N}}$ induces an isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(\Pi(V \otimes \mathscr{O}_{\mathcal{N}}), X_0[j]) \to \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})}(\Phi_{\mathcal{N}}(V \otimes \mathscr{O}_{\mathcal{N}}), \Phi_{\mathcal{N}}(X_0)[j])$$

for any j, by Lemma 7.3.5 and Lemma 7.3.9(1). (Here we also use the fact that Π is essentially surjective, as a Verdier quotient functor.) Hence the desired property follows from Lemma 7.3.12.

The conclusion of Lemma 7.3.13 implies in particular that the functor $\Phi_{\mathcal{N}}$ induces an isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(\Pi(\nabla^{\mathrm{ex}}_\mu), X_0[j]) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})}(R(p_{\mathrm{Spr}})_* \nabla^{\mathrm{ex}}_\mu, \Phi_{\mathcal{N}}(X_0)[j])$$

for any $j \in \mathbb{Z}$. By construction the objects $(\nabla^{\mathrm{ex}}_\mu : \mu \in \mathbf{X}^\vee)$ generate $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ as a triangulated category. Therefore their images under Π generate $\mathbf{D}$ as a triangulated category, which finally proves that the morphism

$$\mathrm{Hom}_{\mathbf{D}}(X, X_0) \to \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})}(\Phi_{\mathcal{N}}(X), \Phi_{\mathcal{N}}(X_0))$$

induced by $\Phi_{\mathcal{N}}$ is an isomorphism for any X in $\mathbf{D}$, hence that $\Phi_{\mathcal{N}}$ is fully faithful.

This full faithfulness implies that the essential image of $\Phi_{\mathcal{N}}$ is a triangulated subcategory. On the other hand, by Lemma 7.3.7 this essential image generates $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathcal{N})$ as a triangulated category. Thus $\Phi_{\mathcal{N}}$ is essentially surjective, and hence an equivalence. $\square$

CHAPTER 8

A MODULAR ARKHIPOV–BEZRUKAVNIKOV EQUIVALENCE FOR $\mathrm{GL}(n)$

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$, and let $\ell \neq p$ be another prime number. The main result of Chapter 6 is an equivalence of categories

$$F_{\mathcal{IW}} : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{N}}_{\Bbbk}) \to D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk) \qquad \text{for } \Bbbk = \overline{\mathbb{Q}}_\ell$$

where G is a connected reductive algebraic group over $\mathbb{F}$. It is natural to ask for a "modular" analogue of this statement, i.e., one in which $\Bbbk$ is replaced by a field of positive characteristic. In this chapter, we prove such a modular analogue in the following special case:

$$G = \mathrm{GL}(n, \mathbb{F}), \qquad \Bbbk = \begin{array}{c}\text{an algebraic}\\ \text{closure of } \mathbb{F}_\ell,\end{array} \qquad \ell > \frac{1}{2}\binom{n}{\lfloor n/2 \rfloor}. \tag{8.0.1}$$

In this setting, as observed in Example 1.5.4, the group $G^\vee_{\Bbbk}$ is also a general linear group: it identifies canonically with $\mathrm{GL}(E_{\Bbbk})$, where $E_{\Bbbk} := \mathsf{H}^\bullet(\mathbb{P}^{n-1}; \Bbbk)$ (a $\Bbbk$-vector space of dimension n).

The plan of the proof is completely parallel to what we have done in Chapter 6, with only a few additional technical difficulties. Most of these difficulties are encountered in Section 8.1 (the modular analogue of Section 6.2), and are due to the fact that the category $\mathsf{Rep}(G^\vee_{\Bbbk})$ is not semisimple. The standard idea to overcome such difficulties, which we will use, is to work with *tilting* $G^\vee_{\Bbbk}$-modules; in our specific situation there are obstacles to these techniques (in particular, because an exterior power of a tilting module is not necessarily tilting), and our assumptions on G and ℓ are in fact essentially imposed to be able to deal with these obstacles. We will not repeat the statements (nor the proofs) that do not require modifications other than replacing $\overline{\mathbb{Q}}_\ell$ by $\Bbbk$, and will only explain the modifications that are required to work over a coefficient field of positive characteristic.

8.1. Coherent sheaves on the Springer resolution

We consider the setting of (8.0.1).

8.1.1. The basic affine space and its affine completion. — As in Section 6.2 we will consider H-equivariant coherent sheaves on H-equivariant $\Bbbk$-varieties X, for H a $\Bbbk$-algebraic group. Here again, for any $\mathscr{F}, \mathscr{G}$ in $D^{\mathrm{b}}\mathsf{Coh}^H(X)$ the $\Bbbk$-vector space $\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F}, \mathscr{G})$ has a natural structure of an algebraic H-module. But since $\Bbbk$ has positive characteristic, even in the setting when H is reductive the formula (6.2.1) does not hold in general. As a replacement, we will use the following much more specific claim, which uses the notion of *modules with a good filtration*. Recall that if H is a reductive algebraic group, and $B_H \subset H$ is a Borel subgroup, a (not necessarily finite-dimensional) algebraic H-module M is said to *admit a good filtration* if it admits an exhaustive filtration

$$\{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M$$

indexed by $\mathbb{Z}_{\geq 0}$ such that each subquotient M_n/M_{n-1} is of the form $\mathrm{Ind}_{B_H}^H(\lambda_n)$ for some character λ_n of B_H. (It can be easily checked that this notion does not depend on the choice of B_H. For more on good filtrations, the reader might consult [**J1**, §§II.4.16–II.4.24].)

Lemma 8.1.1. — *Assume that H is reductive. Let $\mathscr{F}, \mathscr{G}$ in $D^{\mathrm{b}}\mathsf{Coh}^H(X)$, and assume that*

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F}, \mathscr{G}[n]) = 0$$

for $n \neq 0$, and moreover that the H-module $\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F}, \mathscr{G})$ *admits a good filtration. Then we have*

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^H(X)}(\mathscr{F}, \mathscr{G}[n]) = 0$$

for $n \neq 0$, and the forgetful functor $D^{\mathrm{b}}\mathsf{Coh}^H(X) \to D^{\mathrm{b}}\mathsf{Coh}(X)$ induces an isomorphism

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^H(X)}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \big(\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F}, \mathscr{G})\big)^H.$$

Proof. — Let us denote by $\mathsf{Rep}^\infty(H)$ the category of all (possibly infinite-dimensional) algebraic H-modules, and by $\mathsf{Vect}_\Bbbk^\infty$ the category of all $\Bbbk$-vector spaces. Let

$$\mathbb{I}^H : \mathsf{Rep}^\infty(H) \to \mathsf{Vect}_\Bbbk^\infty$$

be the functor of H-invariants: $\mathbb{I}^H(M) = \{m \in M \mid h \cdot m = m \text{ for all } h \in H\}$. This functor is clearly left exact. It is well known that the category $\mathsf{Rep}^\infty(H)$ has enough injective objects (see [**J1**, Proposition I.3.9]), so $\mathbb{I}^H$ admits a right derived functor

$$R\mathbb{I}^H : D^+\mathsf{Rep}^\infty(H) \to D^+\mathsf{Vect}_\Bbbk^\infty.$$

Now the bifunctor

$$R\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(-,-) : D^{\mathrm{b}}\mathsf{Coh}^H(X) \times D^{\mathrm{b}}\mathsf{Coh}^H(X) \to D^+\mathsf{Vect}_\Bbbk^\infty$$

factors canonically through a bifunctor

$$R\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(-,-) : D^{\mathrm{b}}\mathsf{Coh}^H(X) \times D^{\mathrm{b}}\mathsf{Coh}^H(X) \to D^+\mathsf{Rep}^\infty(H),$$

and by [**MR1**, Proposition A.6], for any $\mathscr{F}, \mathscr{G}$ in $D^{\mathrm{b}}\mathsf{Coh}^H(X)$ and $n \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^H(X)}(\mathscr{F}, \mathscr{G}[n]) \cong \mathsf{H}^n\big(R\mathbb{I}^H(R\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F}, \mathscr{G}))\big).$$

Our assumptions ensure that the complex $R\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F},\mathscr{G})$ is concentrated in degree 0, i.e., it is isomorphic to $\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F},\mathscr{G})$, so that

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^H(X)}(\mathscr{F},\mathscr{G}[n]) \cong \mathsf{H}^n\big(R\mathbb{I}^H(\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F},\mathscr{G}))\big).$$

Finally, since $\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F},\mathscr{G})$ has a good filtration, the complex of vector spaces $R\mathbb{I}^H(\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(X)}(\mathscr{F},\mathscr{G}))$ is concentrated in degree 0 (see [**J1**, Proposition II.4.13 and Lemma I.4.17]), which finishes the proof. □

As in §6.2.1 we can consider the quotient $G^\vee_\Bbbk/U^\vee_\Bbbk$, with its natural action of $G^\vee_\Bbbk \times T^\vee_\Bbbk$, and as in (6.2.3) we have

$$\mathscr{O}(G^\vee_\Bbbk/U^\vee_\Bbbk) \cong \bigoplus_{\lambda\in\mathbf{X}^\vee} \mathsf{N}_\Bbbk(\lambda)\otimes \Bbbk_{T^\vee_\Bbbk}(-\lambda), \tag{8.1.1}$$

where the terms in the right-hand side vanish unless $\lambda \in \mathbf{X}^\vee_+$. Lemma 6.2.1 also remains true, with an identical proof.

It is known in this setting also (although this is much less obvious) that the maps $\mathbf{a}_{\lambda,\mu}$ are surjective for all $\lambda,\mu \in \mathbf{X}^\vee_+$, see [**J1**, Proposition II.14.20]. Therefore the $\Bbbk$-algebra $\mathscr{O}(G^\vee_\Bbbk/U^\vee_\Bbbk)$ is finitely generated, so that it is reasonable to set

$$\mathcal{X}_\Bbbk := \mathrm{Spec}(\mathscr{O}(G^\vee_\Bbbk/U^\vee_\Bbbk)).$$

We again have an open and dense embedding $G^\vee_\Bbbk/U^\vee_\Bbbk \hookrightarrow \mathcal{X}_\Bbbk$, whose complement (with its reduced subscheme structure) will be denoted $\partial\mathcal{X}_\Bbbk$.

8.1.2. The Springer resolution and some variants. — As in §6.2.2 we will consider the Springer resolution

$$\widetilde{\mathcal{N}}_\Bbbk := G^\vee_\Bbbk \times^{B^\vee_\Bbbk} \mathfrak{n}^\vee_\Bbbk$$

(where $\mathfrak{n}^\vee_\Bbbk$ is the Lie algebra of $U^\vee_\Bbbk$) with its natural action of $G^\vee_\Bbbk$, and also the inverse image $\widehat{\mathcal{N}}_\Bbbk$ of $\widetilde{\mathcal{N}}_\Bbbk$ in $\mathfrak{g}^\vee_\Bbbk \times G^\vee_\Bbbk/U^\vee_\Bbbk$, with its natural action of $G^\vee_\Bbbk \times T^\vee_\Bbbk$. (Here, $\mathfrak{g}^\vee_\Bbbk$ is the Lie algebra of $G^\vee_\Bbbk$.)

Now the first difference with Chapter 6 arises: in practice, instead of considering the variety above, we will need to consider a certain "multiplicative analogue" defined as follows. We set

$$\widetilde{\mathcal{U}}_\Bbbk := G^\vee_\Bbbk \times^{B^\vee_\Bbbk} U^\vee_\Bbbk,$$

where $B^\vee_\Bbbk$ acts on its normal subgoup $U^\vee_\Bbbk$ by conjugation. Consider the natural $G^\vee_\Bbbk$-action on this variety. As for $\widetilde{\mathcal{N}}_\Bbbk$ we have a natural closed embedding $\widetilde{\mathcal{U}}_\Bbbk \hookrightarrow G^\vee_\Bbbk \times (G^\vee_\Bbbk/B^\vee_\Bbbk)$ (defined by $[g,u] \mapsto (gug^{-1}, gB^\vee_\Bbbk)$), and we denote by $\widehat{\mathcal{U}}_\Bbbk$ the preimage of $\widetilde{\mathcal{U}}_\Bbbk$ in $G^\vee_\Bbbk \times (G^\vee_\Bbbk/U^\vee_\Bbbk)$. Then $\widehat{\mathcal{U}}_\Bbbk$ admits a natural action of $G^\vee_\Bbbk \times T^\vee_\Bbbk$, and as in (6.2.7) and (6.2.8) we have an equivalence

$$\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk) \xrightarrow{\sim} \mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}(\widehat{\mathcal{U}}_\Bbbk) \tag{8.1.2}$$

and identifications

$$\mathscr{O}(\widehat{\mathcal{U}}_\Bbbk)_{-\lambda} = \Gamma(\widetilde{\mathcal{U}}_\Bbbk, \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)) \tag{8.1.3}$$

for $\lambda \in \mathbf{X}^\vee$. (Here $\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)$ is the pullback to $\widetilde{\mathcal{U}}_\Bbbk$ of the line bundle on $G^\vee_\Bbbk/B^\vee_\Bbbk$ naturally associated with λ.)

The construction of $\widehat{\mathcal{U}}_{\Bbbk}$ can be seen as a particular case of the following general construction. Consider an affine $\Bbbk$-group scheme H, and a separated $\Bbbk$-scheme X endowed with an H-action. Then the *universal stabilizer* for this action is the closed subscheme

$$(H \times X) \times_{X \times X} \Delta X$$

of $H \times X$, where $\Delta X \subset X \times X$ is the diagonal, and the morphism $H \times X \to X \times X$ is given by $(h, x) \mapsto (h \cdot x, x)$. In case X is affine, this closed subscheme can be described more explicitly as follows. The H-action on X defines a right $\mathscr{O}(H)$-comodule structure on $\mathscr{O}(X)$, given by a comultiplication morphism $\Delta : \mathscr{O}(X) \to \mathscr{O}(X) \otimes \mathscr{O}(H)$. Then the ideal in $\mathscr{O}(H \times X) = \mathscr{O}(X) \otimes \mathscr{O}(H)$ defining the universal stabilizer is generated by the image of the map

$$\mathscr{O}(X) \to \mathscr{O}(X) \otimes \mathscr{O}(H)$$

sending f to $\Delta(f) - f \otimes 1$.

Applying this construction in the case of the $G^\vee_{\Bbbk}$-action on $G^\vee_{\Bbbk}/U^\vee_{\Bbbk}$ one recovers the scheme $\widehat{\mathcal{U}}_{\Bbbk}$. The universal stabilizer for the $G^\vee_{\Bbbk}$-action on the affine scheme $\mathcal{X}_{\Bbbk}$ will be denoted $\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}$. As for $\widehat{\mathcal{N}}_{\mathcal{X}}$ in (6.2.10), we have

$$\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk} \cap (G^\vee_{\Bbbk} \times G^\vee_{\Bbbk}/U^\vee_{\Bbbk}) = \widehat{\mathcal{U}}_{\Bbbk}. \tag{8.1.4}$$

Here our group $G^\vee_{\Bbbk}$ is a general linear group; we therefore have a $G^\vee_{\Bbbk}$-equivariant open embedding

$$G^\vee_{\Bbbk} \hookrightarrow \mathfrak{g}^\vee_{\Bbbk} \tag{8.1.5}$$

sending x to $x - \mathrm{Id}$. (Here, the $G^\vee_{\Bbbk}$-actions are given by conjugation.) This morphism restricts to an isomorphism $U^\vee_{\Bbbk} \xrightarrow{\sim} \mathfrak{n}^\vee_{\Bbbk}$, so that (8.1.5) induces an isomorphism

$$\widehat{\mathcal{U}}_{\Bbbk} \xrightarrow{\sim} \widehat{\mathcal{N}}_{\Bbbk}. \tag{8.1.6}$$

The following statement will be our replacement for Lemma 6.2.4.

Lemma 8.1.2. — *There exists $N \in \mathbb{Z}_{\geq 0}$ such that, for any $\lambda \in \mathbf{X}^\vee$ such that $\langle \lambda, \alpha^\vee \rangle \geq N$ for any simple root α, the morphism*

$$\mathscr{O}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})_{-\lambda} \to \mathscr{O}(\widehat{\mathcal{U}}_{\Bbbk})_{-\lambda}$$

induced by restriction is an isomorphism.

Proof. — The proof is the same as for Lemma 6.2.4; in the first step, we replace the complex $\mathscr{F}^\bullet$ by its restriction to $G^\vee_{\Bbbk} \times (G^\vee_{\Bbbk}/B^\vee_{\Bbbk})$ (which provides a resolution of the pushforward of $\mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}$ to $G^\vee_{\Bbbk} \times (G^\vee_{\Bbbk}/B^\vee_{\Bbbk})$). □

8.1.3. Koszul complexes. — The construction of the Koszul complex (6.2.13) associated with a surjection $V \twoheadrightarrow V'$ also applies to $\Bbbk$-vector spaces. In particular, using this construction for the morphism $\mathbf{f}_\lambda$ and using the same procedure as in §6.2.3 we obtain a complex

$$\cdots \to 0 \to \wedge^{d_\lambda}\big(\mathsf{N}_{\Bbbk}(\lambda)\big) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} \to \wedge^{d_\lambda - 1}\big(\mathsf{N}_{\Bbbk}(\lambda)\big) \otimes \Bbbk_{T^\vee}(\lambda) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} \to \cdots$$
$$\to \wedge^1\big(\mathsf{N}_{\Bbbk}(\lambda)\big) \otimes \Bbbk_{T^\vee}((d_\lambda - 1) \cdot \lambda) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} \to \Bbbk_{T^\vee}(d_\lambda \cdot \lambda) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} \to 0 \to \cdots$$

of objects of $\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$ (where $d_\lambda = \dim(\mathsf{N}_\Bbbk(\lambda))$) concentrated in degrees between $-d_\lambda$ and 0, whose restriction to $\widehat{\mathcal{U}}_\Bbbk$ is acyclic. Tensoring this complex by $\Bbbk_{T^\vee_\Bbbk}(-d_\lambda \cdot \lambda)$, forgetting the term in degree 0, and then applying $[-1]$, we obtain a complex

$$\begin{aligned}\cdots \to 0 \to \wedge^{d_\lambda}\big(\mathsf{N}_\Bbbk(\lambda)\big) \otimes \Bbbk_{T^\vee_\Bbbk}(-d_\lambda \cdot \lambda) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}& \\ \to \wedge^{d_\lambda - 1}\big(\mathsf{N}_\Bbbk(\lambda)\big) \otimes \Bbbk_{T^\vee_\Bbbk}((-d_\lambda+1) \cdot \lambda) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} \to \cdots& \\ \to \wedge^{1}\big(\mathsf{N}_\Bbbk(\lambda)\big) \otimes \Bbbk_{T^\vee_\Bbbk}(-\lambda) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} \to 0 \to \cdots&\end{aligned}$$

concentrated in degrees between $-d_\lambda + 1$ and 0, which will be denoted by $\mathscr{G}_\lambda$. By construction there exists a canonical morphism of complexes

$$\mathscr{G}_\lambda \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}, \tag{8.1.7}$$

whose cone is supported set-theoretically on the preimage of $\partial\mathcal{X}_\Bbbk \subset \mathcal{X}_\Bbbk$ in $\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}$.

Later we will also consider the following construction. For any simple root α, we fix a coweight $\varpi^\vee_\alpha$ such that for any simple root β we have

$$\langle \varpi^\vee_\alpha, \beta\rangle = \begin{cases} 1 & \text{if } \beta = \alpha; \\ 0 & \text{otherwise.}\end{cases}$$

Then any $\lambda \in \mathbf{X}^\vee_+$ can be written uniquely in the form

$$\lambda = \sum_\alpha n_\alpha \cdot \varpi^\vee_\alpha + \lambda_0$$

where for each simple root α we have $n_\alpha \in \mathbb{Z}_{\geq 0}$ and $\langle \lambda_0, \alpha\rangle = 0$. (In fact, here we must have $n_\alpha = \langle \lambda, \alpha\rangle$.) We then set

$$\mathscr{G}'_\lambda := \left(\bigotimes_\alpha \mathscr{G}_{\varpi^\vee_\alpha}^{\otimes n_\alpha}\right) \otimes_{\mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}} \mathscr{G}_{\lambda_0}.$$

There exists a canonical morphism of complexes

$$\mathscr{G}'_\lambda \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}, \tag{8.1.8}$$

obtained by taking tensor products of the morphisms $\mathscr{G}_{\varpi^\vee_\alpha} \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ and $\mathscr{G}_{\lambda_0} \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ from (8.1.7).

The first property of the complexes $\mathscr{G}'_\lambda$ that will be important for us is that its components are all direct sums of objects of the form $V \otimes \Bbbk_{T^\vee_\Bbbk}(-\eta) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ where V is in $\mathsf{Rep}(G^\vee_\Bbbk)$ and $\eta \in \mathbf{X}^\vee$ satisfies $\langle \eta, \alpha\rangle \geq \langle \lambda, \alpha\rangle$ for each simple root α. The second important property is given by the following lemma.

Lemma 8.1.3. — *For any $\lambda \in \mathbf{X}^\vee_+$, the cone of the morphism* (8.1.8) *is supported set-theoretically on the preimage of $\partial\mathcal{X}_\Bbbk \subset \mathcal{X}_\Bbbk$ in $\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}$.*

Proof. — The claim is known if $\lambda \in \mathbf{X}^\vee$ satisfies $\langle \lambda, \alpha\rangle = 0$ for each simple root α, since in this case $\mathscr{G}'_\lambda = \mathscr{G}_\lambda$. We therefore only have to check that if it holds for some λ, and if α is a simple root, then the claim holds for $\lambda + \varpi^\vee_\alpha$. Now we have

$\mathscr{G}'_{\lambda+\varpi^\vee_\alpha} = \mathscr{G}'_\lambda \otimes_{\mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}} \mathscr{G}_{\varpi^\vee_\alpha}$, and under this identification the map (8.1.8) for $\lambda + \varpi^\vee_\alpha$ is the composition

$$\mathscr{G}'_\lambda \otimes_{\mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}} \mathscr{G}_{\varpi^\vee_\alpha} \to \mathscr{G}'_\lambda \otimes_{\mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}} \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} = \mathscr{G}'_\lambda \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$$

where the first morphism is induced by the canonical morphism $\mathscr{G}_{\varpi^\vee_\alpha} \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$, and the second one is the morphism (8.1.8) for λ. By induction the cone of the second morphism is supported set-theoretically on the preimage of $\partial\mathcal{X}_\Bbbk$, and since the same property holds for the cone of the morphism $\mathscr{G}_{\varpi^\vee_\alpha} \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$, it also holds for the first morphism. The octahedral axiom implies that the cone of our composition is an extension (in the sense of triangulated categories) of the cones of these two morphisms; therefore it is supported set-theoretically on the preimage of $\partial\mathcal{X}_\Bbbk$, which finishes the proof. □

8.1.4. Equivariant coherent sheaves on $\widetilde{\mathcal{U}}_\Bbbk$. — We now have the following analogue of Lemma 6.2.8. Here we denote by $\mathsf{Tilt}(G^\vee_\Bbbk) \subset \mathsf{Rep}(G^\vee_\Bbbk)$ the subcategory of tilting modules.[(1)]

Lemma 8.1.4. — *The category $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)$ is generated (as a triangulated category) by the following classes of objects:*

1. *the line bundles $\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)$, for $\lambda \in \mathbf{X}^\vee$;*
2. *the objects of the form $V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)$ where $V \in \mathsf{Tilt}(G^\vee_\Bbbk)$ and $\lambda \in \mathbf{X}^\vee_+$.*

Proof. — In view of (8.1.6), the proof is identical to that of Lemma 6.2.8, with the extra observation that the subcategory $\mathsf{Tilt}(G^\vee_\Bbbk)$ generates $D^{\mathrm{b}}\mathsf{Rep}(G^\vee_\Bbbk)$ as a triangulated category (by the general theory of highest weight categories; see e.g. [**Ri**, Proposition 7.17]); therefore in (2) we can equivalently replace the condition $V \in \mathsf{Tilt}(G^\vee_\Bbbk)$ by the condition $V \in \mathsf{Rep}(G^\vee_\Bbbk)$. □

We also have the following analogue of Lemma 6.2.9.

Lemma 8.1.5. — *For any $\lambda \in \mathbf{X}^\vee$ there exists $V \in \mathsf{Tilt}(G^\vee_\Bbbk)$ and an embedding of $G^\vee$-equivariant coherent sheaves $\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda) \hookrightarrow V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}$.*

Proof. — In view of (8.1.6), the same proof as that of Lemma 6.2.9 applies, replacing $\mathsf{N}_\Bbbk(w_\circ(\lambda-\nu))$ by the indecomposable tilting $G^\vee_\Bbbk$-module with highest weight $w_\circ(\lambda-\nu)$ (see [**J1**, Proposition II.E.6]). □

8.1.5. Describing the category of equivariant coherent sheaves on $\widetilde{\mathcal{U}}_\Bbbk$ as a quotient. — We will denote by $\mathsf{Tilt}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$ the full subcategory of $\mathsf{Rep}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$ whose objects are the tilting representations. In more concrete terms, any object V in $\mathsf{Rep}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$ can be written in a canonical way as a direct sum

$$V = \bigoplus_{\lambda \in \mathbf{X}^\vee} V^\lambda \otimes \Bbbk_{T^\vee_\Bbbk}(\lambda)$$

(1)Recall that a finite-dimensional algebraic module M over a reductive algebraic group is called *tilting* if both M and M^* admit good filtrations.

for some representations V^λ in $\mathsf{Rep}(G^\vee_\Bbbk)$ (with only finitely many nonzero terms); then V is tilting iff each V^λ is a tilting $G^\vee_\Bbbk$-module. We will denote by

$$\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$$

the full additive subcategory of $\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$ whose objects are the "free tilting" coherent sheaves, i.e. those of the form $V \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ with V in $\mathsf{Tilt}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$. Since a tensor product of tilting $G^\vee_\Bbbk$-modules is tilting (see [**J1**, Proposition II.E.7]), the subcategory $\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$ is stable under tensor products of coherent sheaves.

As in §6.2.5 we will consider the composition

$$(8.1.9)\quad K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}) \\ \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}(\widehat{\mathcal{U}}_\Bbbk) \xrightarrow[\sim]{(8.1.2)} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk),$$

where the first arrow is the obvious functor, and the second one is pullback under the open embedding $\widehat{\mathcal{U}}_\Bbbk \subset \widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}$ (see (8.1.4)). We will denote by $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})_{\partial\mathcal{X}}$ the kernel of this functor, i.e. the full subcategory of $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$ consisting of objects whose cohomology is supported set-theoretically on the preimage of $\partial\mathcal{X}_\Bbbk$.

Our goal in this subsection is to prove the following analogue of Proposition 6.2.10.

Proposition 8.1.6. — *The functor*

$$K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})/K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})_{\partial\mathcal{X}} \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)$$

induced by (8.1.9) *is an equivalence of triangulated categories.*

Until now, we have not used our assumption on ℓ from (8.0.1), and in fact all the results from the previous subsections hold in all characteristics. The main reason for imposing this assumption is that it guarantees that the following claim holds.

Lemma 8.1.7. — *If λ either is of the form $\varpi^\vee_\alpha$ for α a simple root, or satisfies $\langle\lambda,\alpha\rangle = 0$ for all simple roots α, then for any $r \geq 0$ the $G^\vee_\Bbbk$-module $\wedge^r\big(\mathsf{N}_\Bbbk(\lambda)\big)$ is tilting.*

Proof. — First, if $\langle\lambda,\alpha\rangle = 0$ for all simple roots α, then λ is the restriction of a character of $G^\vee_\Bbbk$, so that $\mathsf{N}_\Bbbk(\lambda) = \Bbbk_{G^\vee_\Bbbk}(\lambda)$. Now any 1-dimensional module is tilting (as a simple induced module), and there are no higher exterior powers to consider.

Next we fix a simple root α, and consider the case $\lambda = \varpi^\vee_\alpha$. In this case also it is well known that $\mathsf{N}_\Bbbk(\lambda)$ is a simple induced module, and therefore tilting. (Indeed, the $T^\vee_\Bbbk$-weights of $\mathsf{N}_\Bbbk(\lambda)$ are exactly the W_{f}-translates of λ, so that this module cannot have a nontrivial submodule.) Moreover, it is known also that

$$d_\lambda := \dim(\mathsf{N}_\Bbbk(\lambda)) \leq \binom{n}{\lfloor n/2 \rfloor}.$$

(More precisely, each $\mathsf{N}_\Bbbk(\varpi^\vee_\alpha)$ is an exterior power of the natural representation of $G^\vee_\Bbbk = \mathrm{GL}(E_\Bbbk)$, tensored with some 1-dimensional module.) If $r \leq d_\lambda/2$, our assumption on ℓ from (8.0.1) guarantees that all $\Bbbk$-representations of the finite group $\mathfrak{S}_r$ are semisimple, so that $\wedge^r\big(\mathsf{N}_{\Bbbk}(\lambda)\big)$ is a direct summand of $\mathsf{N}_{\Bbbk}(\lambda)^{\otimes r}$. The latter module is tilting as a tensor product of tilting modules (see again [**J1**, Proposition II.E.7]), and hence so is $\wedge^r\big(\mathsf{N}_{\Bbbk}(\lambda)\big)$. If instead $r \geq d_\lambda/2$, we observe that

$$\wedge^r\big(\mathsf{N}_{\Bbbk}(\lambda)\big) \cong \wedge^{d_\lambda}\big(\mathsf{N}_{\Bbbk}(\lambda)\big) \otimes \big(\wedge^{d_\lambda - r}\big(\mathsf{N}_{\Bbbk}(\lambda)\big)\big)^*.$$

Here $\wedge^{d_\lambda}\big(\mathsf{N}_{\Bbbk}(\lambda)\big)$ is tilting because it is 1-dimensional, and $\wedge^{d_\lambda - r}\big(\mathsf{N}_{\Bbbk}(\lambda)\big)$ is tilting by the case treated above; so $\wedge^r\big(\mathsf{N}_{\Bbbk}(\lambda)\big)$ is tilting also. $\square$

Lemma 8.1.7 implies that the complexes $\mathscr{G}_\lambda$ from §8.1.3 (with λ as in the lemma) are complexes of objects in $\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$. It follows that the same is true for the complex $\mathscr{G}'_\lambda$, for each $\lambda \in \mathbf{X}^\vee_+$. The images of these complexes in $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$ will play a crucial role in the proof of Proposition 8.1.6.

Proof of Proposition 8.1.6. — Using Lemma 8.1.4, as for Proposition 6.2.10, what we have to prove is that our functor induces, for any $\mu \in \mathbf{X}^\vee$, $n \in \mathbb{Z}$ and V in $\mathsf{Tilt}(G^\vee_{\Bbbk})$, an isomorphism from

$$\mathrm{Hom}_{K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})/K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})_{\partial\mathcal{X}}}(V \otimes \Bbbk_{T^\vee_{\Bbbk}}(\mu) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}, \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}[n])$$

to

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk})}(V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(\mu), \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}[n]).$$

We first prove that our map is injective. A morphism from $V \otimes \Bbbk_{T^\vee_{\Bbbk}}(\mu) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ to $\mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}[n]$ in $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})/K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})_{\partial\mathcal{X}}$ can be represented by a diagram

$$V \otimes \Bbbk_{T^\vee_{\Bbbk}}(\mu) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} \xleftarrow{f} \mathscr{F} \xrightarrow{g} \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}[n] \tag{8.1.10}$$

where $\mathscr{F}$ is an object of $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$, and the cone of f belongs to the subcategory $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})_{\partial\mathcal{X}}$. Saying that the image of this morphism vanishes is equivalent to saying that the image of g under (8.1.9) vanishes.

Now choose $\lambda \in \mathbf{X}^\vee_+$, and consider the complex $\mathscr{G}'_\lambda$ from §8.1.3. Lemma 8.1.3 says that the cone of the canonical morphism $\mathscr{G}'_\lambda \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ is supported on the closed subset $\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk} \smallsetminus \widehat{\mathcal{U}}_{\Bbbk}$; hence the same will be true for the induced morphism $\mathscr{F} \otimes_{\mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}} \mathscr{G}'_\lambda \to \mathscr{F}$. In other words, this morphism is an isomorphism in the quotient category $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})/K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})_{\partial\mathcal{X}}$. This argument shows (after an appropriate choice of λ; see also the comments preceding Lemma 8.1.3) that in the diagram (8.1.10) we can assume that the components of $\mathscr{F}$ are all direct sums of objects of the form $M \otimes \Bbbk_{T^\vee_{\Bbbk}}(-\eta) \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ with M in $\mathsf{Tilt}(G^\vee_{\Bbbk})$ and $\eta \in \mathbf{X}^\vee_+$ which satisfies the condition in Lemma 8.1.2.

For such M and η we have

$$\operatorname{Hom}_{K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})}(M\otimes \Bbbk_{T^\vee_{\Bbbk}}(-\eta)\otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}, \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}[n]) = \begin{cases} \big(M^*\otimes \mathscr{O}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})_{-\eta}\big)^{G^\vee_{\Bbbk}} & \text{if } n=0;\\ 0 & \text{otherwise}\end{cases}$$

and

$$\operatorname{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk})}(M\otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(-\eta), \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}[n]) = \begin{cases} \big(M^*\otimes \Gamma(\widetilde{\mathcal{U}}_{\Bbbk}, \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(\eta))\big)^{G^\vee_{\Bbbk}} & \text{if } n=0;\\ 0 & \text{otherwise.}\end{cases}$$

Here the first equality is clear, but the second one deserves some explanation. First, by [**KLT**, Theorem 2] (see also the isomorphism (8.1.6)), we have

$$\mathsf{H}^n(\widetilde{\mathcal{U}}_{\Bbbk}, \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(\eta)) = 0$$

for $n \neq 0$, which implies that

$$\operatorname{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(\widetilde{\mathcal{U}}_{\Bbbk})}(M\otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(-\eta), \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}[n]) = 0$$

for $n \neq 0$. Next, by [**KLT**, Theorem 7] the $G^\vee_{\Bbbk}$-module $\Gamma(\widetilde{\mathcal{U}}_{\Bbbk}, \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(\eta))$ admits a good filtration, which implies (in view of [**J1**, Proposition II.4.21]) that the same holds for

$$M^*\otimes \Gamma(\widetilde{\mathcal{U}}_{\Bbbk}, \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(\eta)) \cong \operatorname{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(\widetilde{\mathcal{U}}_{\Bbbk})}(M\otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(-\eta), \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}).$$

Hence we are in the setting of Lemma 8.1.1, which implies our second equality.

In view of these isomorphisms, Lemma 8.1.2 says that the functor (8.1.9) induces an isomorphism between these Hom-spaces. By the 5-lemma, it then follows that this functor induces an isomorphism

$$\operatorname{Hom}_{K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})}(\mathscr{H}, \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}[n]) \xrightarrow{\sim} \operatorname{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk})}(\mathscr{H}', \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}[n])$$

for any complex $\mathscr{H}$ whose components are direct sums of objects of this form (where $\mathscr{H}'$ is the image of $\mathscr{H}$). In particular, this property holds for the complex $\mathscr{F}$ considered above, which finishes the proof of injectivity.

The proof of surjectivity will use similar tools. Namely, consider a morphism $f : V\otimes\mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(\mu) \to \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}[n]$ in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk})$. Choose $\lambda \in \mathbf{X}^\vee_+$ such that $\lambda-\mu$ satisfies the condition in Lemma 8.1.2. Then if $\mathscr{G}'_\lambda$ is as in §8.1.3 and $\mathscr{G}''_\lambda$ is its image under (8.1.9), the arguments above show that the composition

$$V\otimes_{\Bbbk} \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(\mu)\otimes_{\mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}} \mathscr{G}''_\lambda \to V\otimes_{\Bbbk} \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}(\mu) \xrightarrow{f} \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}[n]$$

(where the first map is induced by (8.1.8)) is the image of a morphism

$$\tilde{f} : V\otimes \Bbbk_{T^\vee_{\Bbbk}}(\mu)\otimes \mathscr{G}'_\lambda \to \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}[n]$$

in $K^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times T^\vee_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$. Hence f is the image of the morphism represented by the diagram

$$V\otimes \Bbbk_{T^\vee_{\Bbbk}}(\mu)\otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}} \leftarrow V\otimes \Bbbk_{T^\vee_{\Bbbk}}(\mu)\otimes \mathscr{G}'_\lambda \xrightarrow{\tilde{f}} \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}[n],$$

which finishes the proof (in view of Lemma 8.1.3). □

8.2. Construction of the functor

From now on we consider the affine flag variety Fl_G associated with our group $G = \mathrm{GL}(n, \mathbb{F})$, and the I-equivariant derived category of étale $\Bbbk$-sheaves $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$. For brevity, we denote the heart of the perverse t-structure on this category by

$$\mathsf{P}_I := \mathsf{Perv}_I(\mathrm{Fl}_G, \Bbbk) \subset D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk),$$

and we introduce the notation

$$\mathscr{Z} := \mathsf{Z} \circ \mathcal{S}^{-1} : \mathsf{Rep}(G^\vee_\Bbbk) \to \mathsf{P}_I.$$

Our next task will be to construct a functor

$$F^\Bbbk : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathscr{U}}_\Bbbk) \to D^{\mathrm{b}}\mathsf{P}_I,$$

following the strategy of Section 6.3.

8.2.1. Definition of a functor on $\mathsf{Coh}^{G^\vee_\Bbbk \times T^\vee_\Bbbk}_{\mathrm{fr,tilt}}(\widehat{\mathscr{U}}_{\mathcal{X},\Bbbk})$. — The construction of the functor $F^\Bbbk$ will again be based on the considerations of §6.3.2. The only difference is that the role of Example 6.3.1 will now be played by the following variant.

Example 8.2.1. — We consider the special case of the construction of §6.3.2 when $A' = \mathscr{O}(H)$ (for the adjoint action of H on itself). Again in this case the correspondence between extensions of the canonical functor $\mathsf{Rep}(H) \to A\text{-mod}^K$ to $\mathscr{O}(H)\text{-mod}^H_{\mathrm{fr}} \cong \mathsf{Coh}^H_{\mathrm{fr}}(H)$ and K-equivariant algebra morphisms $\mathscr{O}(H) \to A$ admits a reformulation in terms of Tannakian formalism, as follows.

For any V in $\mathsf{Rep}(H)$, the H-equivariant coherent sheaf $V \otimes \mathscr{O}_H$ admits a canonical automorphism $\mathsf{i}^{\mathrm{can}}_V$, described at the level of global sections as the composition

$$V \otimes \mathscr{O}(H) \to V \otimes \mathscr{O}(H) \otimes \mathscr{O}(H) \to V \otimes \mathscr{O}(H)$$

where the first map is induced by the comultiplication morphism $V \to V \otimes \mathscr{O}(H)$, and the second map by the product in $\mathscr{O}(H)$. (Here, $V \otimes \mathscr{O}(H)$ identifies with the space of morphisms $H \to V$, and the automorphism above corresponds to the automorphism sending a function $f : H \to V$ to the function $h \mapsto h \cdot f(h)$.) These automorphisms satisfy

$$\mathsf{i}^{\mathrm{can}}_{V_1 \otimes V_2} = \mathsf{i}^{\mathrm{can}}_{V_1} \otimes \mathsf{i}^{\mathrm{can}}_{V_2}$$

for V_1, V_2 in $\mathsf{Rep}(H)$. Given an extension of our functor $\mathsf{Rep}(H) \to A\text{-mod}^K$ to $\mathsf{Coh}^H_{\mathrm{fr}}(H)$, by taking the images of these automorphisms (and forgetting the K-equivariance) we obtain an automorphism of the functor $V \mapsto V \otimes A$, which determines an A-point of H, or in other words an algebra morphism $\mathscr{O}(H) \to K$. The fact that each automorphism of $V \otimes A$ is K-equivariant translates into the property that this algebra morphism is K-equivariant.

Once again, we leave it to the reader to check that this construction provides the same algebra morphism as the general construction described in §6.3.2 (in this special case). In this way we obtain that the data of the following structures are equivalent:

1. an extension of the functor $V \mapsto V \otimes A$ to a $\Bbbk$-linear (symmetric) monoidal functor $\mathsf{Coh}^H_{\mathrm{fr}}(H) \to A\text{-mod}^K$;
2. a K-equivariant algebra morphism $\mathscr{O}(H) \to A$;

3. a K-invariant A-point of H;
4. for any V in $\mathsf{Rep}(H)$, a K-equivariant automorphism i_V of $V \otimes A$, this collection satisfying
$$i_{V_1 \otimes V_2} = i_{V_1} \otimes i_{V_2}$$
for V_1, V_2 in $\mathsf{Rep}(H)$.

Under this correspondence, the collection $(i_V : V \in \mathsf{Rep}(H))$ attached to an extension $\varphi : \mathsf{Coh}^H_{\mathrm{fr}}(H) \to A\text{-mod}^K$ is given by $i_V = \varphi(\mathsf{i}_V^{\mathrm{can}})$.

We start by considering the full subcategory $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \Bbbk)$ of P_I whose objects are the perverse sheaves admitting a Wakimoto filtration, endowed with its natural monoidal structure, and the natural exact monoidal functor
$$\overline{F}^{\Bbbk} : \mathsf{Rep}(G^\vee_{\Bbbk} \times T^\vee_{\Bbbk}) \to \mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \Bbbk)$$
defined as in the characteristic-0 setting. Then using the same recipe as in §6.3.4 one defines the symmetric monoidal (non-full) subcategory $(\mathscr{C}, \star)$ of $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \Bbbk)$, such that $\overline{F}^{\Bbbk}$ factors through a monoidal functor
$$\underline{F}^{\Bbbk} : \mathsf{Rep}(G^\vee_{\Bbbk} \times T^\vee_{\Bbbk}) \to \mathscr{C}.$$
The unit object $1_{\mathscr{C}}$ in $\mathscr{C}$ corresponds to the perverse sheaf $\delta_{\mathrm{Fl}} := \mathscr{IC}^I_e$.

Next, we consider the commutative $\Bbbk$-algebra
$$A := \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}^{\Bbbk}(\mathscr{O}(G^\vee_{\Bbbk} \times T^\vee_{\Bbbk}))),$$
and the full subcategory
$$A\text{-}\mathsf{mod}_{\mathrm{fr}}^{G^\vee_{\Bbbk} \times T^\vee_{\Bbbk}}$$
of the category of $G^\vee_{\Bbbk} \times T^\vee_{\Bbbk}$-equivariant A-modules whose objects are those of the form $V \otimes A$ with V in $\mathsf{Rep}(G^\vee_{\Bbbk} \times T^\vee_{\Bbbk})$.

The following statement is the analogue of Proposition 6.3.5.

Proposition 8.2.2. — *There is an equivalence of symmetric monoidal categories $H : \mathscr{C} \xrightarrow{\sim} A\text{-}\mathsf{mod}_{\mathrm{fr}}^{G^\vee \times T^\vee}$ given by the formula*
$$H(V) = \mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, \underline{F}^{\Bbbk}(V) \star \underline{F}^{\Bbbk}(\mathscr{O}(G^\vee \times T^\vee))).$$

Proof. — The proof of essential surjectivity and the construction of a monoidal structure on H are identical to their characteristic-0 counterparts, and will not be repeated. However, to check full faithfulness, one needs to argue differently, since in this setting it is not true that any representation is a direct summand of a sum of copies of $\mathscr{O}(G^\vee_{\Bbbk} \times T^\vee_{\Bbbk})$. By adjunction, as in the characteristic-0 setting,what we have to prove is that for V in $\mathsf{Rep}(G^\vee_{\Bbbk} \times T^\vee_{\Bbbk})$, the map
$$\mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, V) \xrightarrow{\sim} (V \otimes A)^{G^\vee_{\Bbbk} \times T^\vee_{\Bbbk}} \tag{8.2.1}$$
induced by our functor is an isomorphism. (Compare with (6.3.12) and the discussion following it.) As in the proof of Proposition 6.3.5, we consider the extension of H to ind-objects. The reasoning given there shows again that (8.2.1) is an isomorphism when V is the ind-object $\mathscr{O}(G^\vee_{\Bbbk} \times T^\vee_{\Bbbk})$, or a finite direct sum of copies of $\mathscr{O}(G^\vee_{\Bbbk} \times T^\vee_{\Bbbk})$.

Before considering (8.2.1) in general, we make the following observation. Let M be an object in $\mathsf{Rep}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$, and let Γ be an abstract group acting on M by $G^\vee_\Bbbk \times T^\vee_\Bbbk$-module automorphisms. Then we can consider the fixed points M^Γ, which is again an object in $\mathsf{Rep}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$, and hence in $\mathscr{C}$. On the other hand, by functoriality, Γ acts by automorphisms on the finite-dimensional $\Bbbk$-vector spaces $\mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, \underline{F}^\Bbbk(M))$ and $\mathrm{Hom}_{\mathsf{P}_I}(\delta_{\mathrm{Fl}}, \overline{F}^\Bbbk(M))$. We claim that the embedding $M^\Gamma \hookrightarrow M$ induces an isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, \underline{F}^\Bbbk(M^\Gamma)) \xrightarrow{\sim} \big(\mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, \underline{F}^\Bbbk(M))\big)^\Gamma. \tag{8.2.2}$$

To see this, we first note that since all the vector spaces under consideration are finite-dimensional, there exists a finite subset $\Gamma_0 \subset \Gamma$ such that

$$M^\Gamma = M^{\Gamma_0},$$
$$\big(\mathrm{Hom}_{\mathsf{P}_I}(\delta_{\mathrm{Fl}}, V)\big)^\Gamma = \big(\mathrm{Hom}_{\mathsf{P}_I}(\delta_{\mathrm{Fl}}, \overline{F}^\Bbbk(M))\big)^{\Gamma_0}.$$

Then M^Γ is the kernel of the morphism $M \to \bigoplus_{\gamma\in\Gamma_0} M$ sending v to $(\gamma\cdot v - v)_{\gamma\in\Gamma_0}$, and by left exactness of the functor $\mathrm{Hom}_{\mathsf{P}_I}(\delta_{\mathrm{Fl}}, -)$ we deduce that

$$\mathrm{Hom}_{\mathsf{P}_I}(\delta_{\mathrm{Fl}}, \overline{F}^\Bbbk(M^\Gamma)) = \big(\mathrm{Hom}_{\mathsf{P}_I}(\delta_{\mathrm{Fl}}, \overline{F}^\Bbbk(M))\big)^\Gamma.$$

It is easy to see (using exactness of $\overline{F}^\Bbbk$ and of convolution in $\mathsf{Perv}_I^{\mathbf{X}^\vee_+}(\mathrm{Fl}_G, \Bbbk)$) that a morphism in $\mathrm{Hom}_{\mathsf{P}_I}(\delta_{\mathrm{Fl}}, \overline{F}^\Bbbk(M^\Gamma))$ belongs to $\mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, \underline{F}^\Bbbk(M^\Gamma))$ iff its composition with the morphism $\overline{F}^\Bbbk(M^\Gamma) \to \overline{F}^\Bbbk(M)$ belongs to $\mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, \underline{F}^\Bbbk(M))$. The isomorphism (8.2.2) follows.

By passing to limits, we see that (8.2.2) also holds when M is an ind-object that is an inductive limit of representations equipped with a compatible action of Γ.

Now, for V in $\mathsf{Rep}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$, consider the vector space $V \otimes \mathscr{O}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$, which can be identified with the space of algebraic functions $G^\vee_\Bbbk \times T^\vee_\Bbbk \to V$. In the proof of Proposition 6.3.5, we introduced four different actions of $G^\vee \times T^\vee$ on this space, including the so-called "left-only" and "mixed" actions. Regard $V \otimes \mathscr{O}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$ as an ind-object of $\mathsf{Rep}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$ using the left-only action, and then make the abstract group $\Gamma = G^\vee_\Bbbk \times T^\vee_\Bbbk$ act on it by the "mixed" action. We then have an isomorphism

$$V \xrightarrow{\sim} \big(V \otimes \mathscr{O}(G^\vee_\Bbbk \times T^\vee_\Bbbk)\big)^\Gamma.$$

in $\mathsf{Rep}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$ given by sending v to the function $g \mapsto g^{-1}\cdot v$.

Since $V \otimes \mathscr{O}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$ under the left-only action is isomorphic to a (finite) direct sum of copies of $\mathscr{O}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$, we have already seen that the corresponding case of (8.2.1), i.e., the map

$$\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(1_{\mathscr{C}}, V \otimes \mathscr{O}(G^\vee_\Bbbk \times T^\vee_\Bbbk)) \xrightarrow{\sim} (V \otimes \mathscr{O}(G^\vee_\Bbbk \times T^\vee_\Bbbk) \otimes A)^{G^\vee_\Bbbk \times T^\vee_\Bbbk},$$

is an isomorphism. The abstract group Γ acts (by the mixed action) on both sides. Taking Γ-invariants and using (8.2.2) (which is permitted because $(V \otimes \mathscr{O}(G^\vee_\Bbbk \times T^\vee_\Bbbk)$ is a limit of finite-dimensional Γ-stable subspaces), we obtain (8.2.1). □

Next, we want to define a $G^\vee_\Bbbk \times T^\vee_\Bbbk$-equivariant algebra morphism

$$\mathscr{O}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}) \to A.$$

First, the same considerations as in the characteristic-0 setting (using the morphisms $\mathfrak{f}_\lambda$) provide an equivariant algebra morphism $\mathscr{O}(\mathcal{X}_\Bbbk) \to A$. Next, we fix a topological generator of the group $\mathbb{Z}_\ell(1)$ associated with $\mathbb{F}$; in this way, in view of Remark 5.3.1 (see also §9.5.2) we obtain (unipotent) monodromy automorphisms $(\mathsf{m}_V : V \in \mathsf{Rep}(G_\Bbbk^\vee))$ (where for simplicity we write m_V for $\mathsf{m}_{\mathscr{Z}(V)}$). By Proposition 3.4.2, these automorphisms satisfy

$$\mathsf{m}_{V_1 \otimes V_2} = \mathsf{m}_{V_1} \star^I \mathsf{m}_{V_2}$$

for V_1, V_2 in $\mathsf{Rep}(G_\Bbbk^\vee)$. As explained in Example 8.2.1, such a datum determines an algebra morphism $\mathscr{O}(G_\Bbbk^\vee) \to A$. Combining these constructions we obtain a $G_\Bbbk^\vee \times T_\Bbbk^\vee$-equivariant algebra morphism

$$\mathscr{O}(G_\Bbbk^\vee \times \mathcal{X}_\Bbbk) \to A. \tag{8.2.3}$$

By construction (see §8.1.2) we have a surjection

$$\mathscr{O}(G_\Bbbk^\vee \times \mathcal{X}_\Bbbk) \to \mathscr{O}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}),$$

whose kernel is generated by the image of a certain morphism $\mathscr{O}(G_\Bbbk^\vee/U_\Bbbk^\vee) \to \mathscr{O}(G_\Bbbk^\vee/U_\Bbbk^\vee) \otimes \mathscr{O}(G_\Bbbk^\vee)$. The restriction of this morphism to $\mathsf{N}_\Bbbk(\lambda) \subset \mathscr{O}(G_\Bbbk^\vee/U_\Bbbk^\vee)$ is constructed from the composition

$$\mathscr{Z}(\mathsf{N}_\Bbbk(\lambda)) \xrightarrow{\mathsf{m}_{\mathsf{N}_\Bbbk(\lambda)}} \mathscr{Z}(\mathsf{N}_\Bbbk(\lambda)) \xrightarrow{\mathfrak{f}_\lambda} \mathbf{J}_\lambda(\Bbbk),$$

which coincides with $\mathfrak{f}_\lambda$ by Lemma 4.6.9. This implies that the morphism (8.2.3) factors through the surjection $\mathscr{O}(G_\Bbbk^\vee \times \mathcal{X}_\Bbbk) \to \mathscr{O}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$, which allows us to define a functor

$$\widetilde{F}^\Bbbk : \mathsf{Coh}_{\mathrm{fr}}^{G_\Bbbk^\vee \times T_\Bbbk^\vee}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}) \to \mathsf{Perv}_I^{\mathbf{X}_+^\vee}(\mathrm{Fl}_G, \Bbbk)$$

as in the characteristic-0 setting, where $\mathsf{Coh}_{\mathrm{fr}}^{G_\Bbbk^\vee \times T_\Bbbk^\vee}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$ is the full subcategory of $\mathsf{Coh}^{G_\Bbbk^\vee \times T_\Bbbk^\vee}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$ whose objects are the coherent sheaves of the form $V \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ with V in $\mathsf{Rep}(G_\Bbbk^\vee \times T_\Bbbk^\vee)$. Below we will in fact mainly consider the restriction of this functor to the full subcategory $\mathsf{Coh}_{\mathrm{fr,tilt}}^{G_\Bbbk^\vee \times T_\Bbbk^\vee}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk})$; this restriction will be denoted $\widetilde{F}^\Bbbk_{\mathrm{tilt}}$.

8.2.2. Factorization through coherent sheaves on $\widetilde{\mathcal{U}}_\Bbbk$. — To finish the construction of the functor $F^\Bbbk$ it only remains to prove the following claim.

Proposition 8.2.3. — *There exists a unique triangulated functor*

$$F^\Bbbk : D^{\mathrm{b}}\mathsf{Coh}^{G_\Bbbk^\vee}(\widetilde{\mathcal{U}}_\Bbbk) \to D^{\mathrm{b}}\mathsf{P}_I$$

such that the following diagram (where the left vertical arrow is induced by restriction to the open subset $\widehat{\mathcal{U}}_\Bbbk$ followed by the equivalence (8.1.2)*, and the right vertical arrow is the obvious functor) commutes up to isomorphism:*

$$\begin{tikzcd}
K^{\mathrm{b}}\mathsf{Coh}_{\mathrm{fr,tilt}}^{G_\Bbbk^\vee \times T_\Bbbk^\vee}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}) \arrow[r, "K^{\mathrm{b}}(\widetilde{F}^\Bbbk_{\mathrm{tilt}})"] \arrow[d] & K^{\mathrm{b}}\mathsf{P}_I \arrow[d] \\
D^{\mathrm{b}}\mathsf{Coh}^{G_\Bbbk^\vee}(\widetilde{\mathcal{U}}_\Bbbk) \arrow[r, "F^\Bbbk"] & D^{\mathrm{b}}\mathsf{P}_I.
\end{tikzcd}$$

Proof. — The proof is very similar to that of Proposition 6.3.9, using Proposition 8.1.6 in place of Proposition 6.2.10. $\square$

In this proof, we had to restrict the functor $\widetilde{F}^{\Bbbk}$ to tilting representations in order to apply Proposition 8.1.6. However, this subtlety can be ignored a posteriori. The following claim is a partial step towards making this idea precise, that will be needed below.

Lemma 8.2.4. — *The composition of $F^{\Bbbk}$ with the canonical functor*

$$\mathsf{Rep}(G^{\vee}_{\Bbbk}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^{\vee}_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk}), \qquad V \mapsto V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}$$

coincides (up to isomorphism) with the composition of the restriction of $\overline{F}^{\Bbbk}$ to $\mathsf{Rep}(G^{\vee}_{\Bbbk})$ with the canonical functor $\mathsf{Perv}_I^{\mathbf{X}^{\vee}_{+}}(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}\mathsf{P}_I$. Moreover, for any V in $\mathsf{Rep}(G^{\vee}_{\Bbbk})$, the image under $F^{\Bbbk}$ of the canonical automorphism of $V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}$ (obtained by pullback from the automorphism $\mathsf{i}_V^{\mathrm{can}}$ of Example 8.2.1) is m_V.

Proof. — We consider the following diagram, where the vertical arrows are all induced by the canonical functor from the homotopy category to the derived category, and the arrows from the first to the second column are given by $V \mapsto V \otimes \mathscr{O}_{\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}}$ and $V \mapsto V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}$ respectively:

$$\begin{tikzcd}
& K^{\mathrm{b}}\mathsf{Coh}^{G^{\vee}_{\Bbbk}\times T^{\vee}_{\Bbbk}}_{\mathrm{fr,tilt}}(\widehat{\mathcal{U}}_{\mathcal{X},\Bbbk}) \arrow[r, "K^{\mathrm{b}}(\widetilde{F}^{\Bbbk}_{\mathrm{tilt}})"] \arrow[dd] & K^{\mathrm{b}}\mathsf{P}_I \arrow[dd] \\
K^{\mathrm{b}}\mathsf{Tilt}(G^{\vee}_{\Bbbk}) \arrow[ur] \arrow[urr, "K^{\mathrm{b}}(\overline{F}^{\Bbbk})"'] \arrow[dd] & & \\
& D^{\mathrm{b}}\mathsf{Coh}^{G^{\vee}_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk}) \arrow[r, "F^{\Bbbk}"] & D^{\mathrm{b}}\mathsf{P}_I. \\
D^{\mathrm{b}}\mathsf{Rep}(G^{\vee}_{\Bbbk}) \arrow[ur] \arrow[urr, "D^{\mathrm{b}}(\overline{F}^{\Bbbk})"'] & &
\end{tikzcd}$$

In this diagram the three squares and the upper triangle all commute. Moreover the leftmost vertical arrow is known to be an equivalence of categories (e.g. by [**Ri**, Proposition 7.17]). It follows that the lower triangle also commutes, which proves the first claim.

For the second claim, since any finite-dimensional algebraic $G^{\vee}_{\Bbbk}$-module is isomorphic to a subquotient of a tilting module, one can assume that V is tilting; then the claim is clear by construction. $\square$

8.3. Antispherical and Iwahori–Whittaker categories

We continue with analogues of the developments in Section 6.4. Define the antispherical category $\mathsf{P}_I^{\mathrm{asph}}$ to be the quotient of the abelian category P_I by the Serre subcategory generated by the simple objects $\mathscr{IC}^I_w$ with $w \notin {}^{\mathrm{f}}W$. The quotient functor $\mathsf{P}_I \to \mathsf{P}_I^{\mathrm{asph}}$ will be denoted Π_{asph}.

Next, it makes sense to consider the Artin–Schreier local system $\mathscr{L}_{\mathrm{AS}}$ over $\Bbbk$ defined exactly as in §6.4.2 (depending on the choice of a primitive p-th root of unity in $\Bbbk$). One can then define the derived category of Iwahori–Whittaker sheaves $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk)$, as well as the abelian subcategory of Iwahori–Whittaker perverse sheaves $\mathsf{P}_{\mathcal{IW}}$. In particular, one can define objects $\Delta^{\mathcal{IW}}_\lambda$ and $\nabla^{\mathcal{IW}}_\lambda$ by the same formulas as in §6.4.3. We can also consider the functor

$$\mathsf{Av}_{\mathcal{IW}} : D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk)$$

defined by

$$\mathsf{Av}_{\mathcal{IW}}(\mathscr{F}) = \Delta^{\mathcal{IW}}_0 \star^I \mathscr{F}.$$

We then have the following analogue of Theorem 6.4.2.

Theorem 8.3.1. — 1. *The functor $\mathsf{Av}_{\mathcal{IW}}$ is t-exact for the perverse t-structures on $D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk)$ and $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk)$.*
2. *The restriction of this functor to the hearts of these t-structures factors through a fully faithful functor $\mathsf{P}^{\mathrm{asph}}_I \to \mathsf{P}_{\mathcal{IW}}$.*

The proof is identical to that of Theorem 6.4.2, and will not be repeated here. (Note in particular that [**BeR2**] allows modular coefficients, and that the results from [**BBM**] that are used apply also in this context, with identical proofs.) Below we will freely use the notation introduced in this proof, simply replacing $\overline{\mathbb{Q}}_\ell$ by $\Bbbk$ when appropriate, as well as the lemmas from §§6.4.4–6.4.5.

8.4. Central sheaves and tilting Iwahori–Whittaker perverse sheaves

8.4.1. Statement. — We now consider the functor

$$\mathscr{Z}^{\mathcal{IW}} := \mathsf{Av}_{\mathcal{IW}} \circ \mathscr{Z} : \mathsf{Rep}(G^\vee_\Bbbk) \to \mathsf{P}_{\mathcal{IW}}.$$

As in the characteristic-0 setting the category $\mathsf{P}_{\mathcal{IW}}$ admits a natural structure of a highest weight category, and we can consider the tilting objects therein.

We have the following counterpart of Theorem 6.5.2.

Theorem 8.4.1. — *For any V in $\mathsf{Tilt}(G^\vee_\Bbbk)$, the perverse sheaf $\mathscr{Z}^{\mathcal{IW}}(V)$ is tilting. Moreover, for any $\lambda \in \mathbf{X}^\vee$ we have*

$$(\mathscr{Z}^{\mathcal{IW}}(V) : \Delta^{\mathcal{IW}}_\lambda) = (\mathscr{Z}^{\mathcal{IW}}(V) : \nabla^{\mathcal{IW}}_\lambda) = \dim(V_\lambda). \tag{8.4.1}$$

8.4.2. Proof of Theorem 8.4.1. — The proof of Theorem 8.4.1 will follow the same pattern as that of Theorem 6.5.2, but will be much simpler since in the present setting $G^\vee_\Bbbk$ is a general linear group.

Namely, the same arguments as for Proposition 6.5.6 show that the formula (8.4.1) will follow if we prove the first claim. Next, the same arguments as in the characteristic-0 setting (see Proposition 6.5.7) prove the following claim.

Proposition 8.4.2. — *If V, V' are objects in $\mathsf{Rep}(G^\vee_\Bbbk)$ such that $\mathscr{Z}^{\mathcal{IW}}(V)$ and $\mathscr{Z}^{\mathcal{IW}}(V')$ are tilting, then $\mathscr{Z}^{\mathcal{IW}}(V \otimes V')$ is tilting.*

For the next step, we choose a basis $(e_1, \dots, e_n)$ in $E_{\Bbbk}$ consisting of eigenvectors for the action of $T^{\vee}_{\Bbbk}$. This lets us identify $G^{\vee}$ with $\mathrm{GL}_n(\Bbbk)$, and $T^{\vee}_{\Bbbk}$ with the subspace of diagonal matrices. Reordering the basis if necessary, we can further assume that $B^{\vee}_{\Bbbk}$ identifies with the subgroup of lower triangular matrices. These identifications provide a canonical isomorphism

$$\mathbf{X}^{\vee} = \mathbb{Z}^n,$$

such that if for $i \in \{1, \dots, n\}$ we set

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ terms}}, 0, \dots, 0),$$

then $(\omega_1, \dots, \omega_n)$ is a $\mathbb{Z}$-basis of $\mathbf{X}^{\vee}$, and $\mathbf{X}^{\vee}_{+}$ consists of the elements of the form

$$k_1\omega_1 + \cdots + k_{n-1}\omega_{n-1} + m\omega_n$$

with $k_i \geq 0$ for $i \in \{1, \dots, n-1\}$.

Lemma 8.4.3. — *For any $k_1, \dots, k_{n-1} \in \mathbb{Z}_{\geq 0}$ and any $m \in \mathbb{Z}$, the $G^{\vee}_{\Bbbk}$-module*

$$\left(\bigotimes_{i=1}^{n-1} \mathsf{N}_{\Bbbk}(\omega_i)^{\otimes k_i} \right) \otimes \mathsf{N}_{\Bbbk}(m \cdot \omega_n)$$

is tilting. Moreover, any (finite-dimensional) tilting $G^{\vee}_{\Bbbk}$-module is isomorphic to a direct sum of direct summands of such modules.

Proof. — For any $i \in \{1, \dots, n-1\}$, the weights of $\mathsf{N}_{\Bbbk}(\omega_i)$ (which are the same as those of the characteristic-0 counterpart of this module) all belong to the W_{f}-orbit of ω_i; therefore this induced module is simple, and hence isomorphic to the Weyl module of highest weight ω_i (in the sense of [**J1**, §II.5.11]); in particular, it is tilting. Similarly, for any $m \in \mathbb{Z}$, the induced module $\mathsf{N}_{\Bbbk}(m \cdot \omega_n)$ is 1-dimensional, with action given by $\det^{\otimes m}$; hence it is simple, and therefore tilting. Since a tensor product of tilting modules is tilting (see [**J1**, §II.E.7]), we deduce the first claim.

For the second claim, recall that any tilting $G^{\vee}_{\Bbbk}$-module is a direct sum of indecomposable tilting modules, and that indecomposable tilting modules are parametrized by $\mathbf{X}^{\vee}_{+}$ (see [**J1**, §II.E.6]). Now if $\lambda = k_1\omega_1 + \cdots + k_{n-1}\omega_{n-1} + m\omega_n$ is dominant (i.e. if $k_i \geq 0$ for $i \in \{1, \dots, n-1\}$), then the tilting module

$$\left(\bigotimes_{i=1}^{n-1} \mathsf{N}_{\Bbbk}(\omega_i)^{\otimes k_i} \right) \otimes \mathsf{N}_{\Bbbk}(m \cdot \omega_n)$$

admits λ as a maximal weight (for the dominance order); therefore it must contain the indecomposable tilting module attached to λ as a direct summand, which implies our claim. $\square$

Proposition 8.4.2 and Lemma 8.4.3 reduce the proof of the first claim in Theorem 8.4.1 to the special case when V is either $\mathsf{N}_{\Bbbk}(\omega_i)$ (for some $i \in \{1, \dots, n-1\}$) or $\mathsf{N}_{\Bbbk}(m \cdot \omega_n)$ (for some $m \in \mathbb{Z}$). In these cases, the claim can be checked by the same considerations as in §6.5.5.

8.4.3. The regular quotient. — To finish this section we consider appropriate analogues of Proposition 6.5.18 and Lemma 6.5.26. (In the present setting these results are not needed for the proof of Theorem 8.4.1. However they will be required in the proof of Lemma 8.5.3 below.) Note that unlike in §6.5.7, our base field is now countable, so one key step from that section (namely, the determination of the division ring in (6.5.4)) does not go through. For this reason we will restrict ourselves to weaker claims, which will still be sufficient for our purposes.

We will denote by P_I^0 the Serre quotient of the category P_I by the Serre subcategory generated by the objects $\mathscr{IC}_w^I$ with $\ell(w) > 0$, and by $\Pi^0 : \mathsf{P}_I \to \mathsf{P}_I^0$ the associated quotient functor. Then, as in the characteristic-0 setting, the functor $(\mathscr{F}, \mathscr{G}) \mapsto \Pi^0({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \star^I \mathscr{G}))$ induces a monoidal structure on P_I^0, with product denoted $\circledast$, and with unit object $\delta^0 := \Pi^0(\mathscr{IC}_e^I)$. The functor $\mathscr{Z}^0 := \Pi^0 \circ \mathscr{Z}$ admits a natural structure of a central functor, and for V in $\mathsf{Rep}(G_{\Bbbk}^\vee)$ we set $\mathsf{m}_V^0 := \Pi^0(\mathsf{m}_V)$. We will also denote by $\widetilde{\mathsf{P}}_I^0$ the full (monoidal) abelian subcategory of P_I^0 whose objects are the subquotients of objects of the form $\mathscr{Z}^0(V)$ for V in $\mathsf{Rep}(G_{\Bbbk}^\vee)$, so that $\mathscr{Z}^0$ factors through a central functor $\widetilde{\mathscr{Z}^0} : \mathsf{Rep}(G_{\Bbbk}^\vee) \to \widetilde{\mathsf{P}}_I^0$.

The (left) regular $G_{\Bbbk}^\vee$-module $\mathscr{O}(G_{\Bbbk}^\vee)$ is the union of its finite-dimensional sub-$G_{\Bbbk}^\vee$-modules; it therefore defines an ind-object $\underline{\mathscr{O}}(G_{\Bbbk}^\vee)$ in $\mathsf{Rep}(G_{\Bbbk}^\vee)$. We consider the image $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee))$, an ind-object in $\widetilde{\mathsf{P}}_I^0$. Since $\underline{\mathscr{O}}(G_{\Bbbk}^\vee)$ is a ring-object, so is $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee))$, and using Theorem 3.5.1 one sees that any left ideal subobject in this ring-object is also a right ideal subobject. Hence, choosing a maximal left ideal subobject $\mathscr{J}$ (whose existence can be checked as in the proof of Lemma 6.5.19) one obtains a ring-object quotient $\underline{\mathscr{O}}(H)$ of $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee))$.

One can then consider the $\Bbbk$-algebra

$$K := \mathrm{Hom}_{\mathrm{Ind}(\widetilde{\mathsf{P}}_I^0)}(\delta^0, \underline{\mathscr{O}}(H)),$$

where the product of $f : \delta^0 \to \underline{\mathscr{O}}(H)$ and $g : \delta^0 \to \underline{\mathscr{O}}(H)$ is the composition

$$\delta^0 = \delta^0 \circledast \delta^0 \xrightarrow{f \circledast g} \underline{\mathscr{O}}(H) \circledast \underline{\mathscr{O}}(H) \to \underline{\mathscr{O}}(H),$$

where the rightmost arrow is the multiplication map.

Lemma 8.4.4. — *The algebra K is a (commutative) field.*

Proof. — Let us first prove commutativity. As part of the central structure on the functor $\widetilde{\mathscr{Z}^0}$, we have a canonical isomorphism

$$\varsigma_{\underline{\mathscr{O}}(G_{\Bbbk}^\vee), \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee))} : \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee)) \circledast \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee)) \xrightarrow{\sim} \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee)) \circledast \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee)).$$

By functoriality this isomorphism sends $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee)) \circledast \mathscr{J}$ to $\mathscr{J} \circledast \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee))$. Moreover, this isomorphism is also (via the appropriate identifications) the image under $\widetilde{\mathscr{Z}^0}$ of the commutativity constraint in $\mathsf{Rep}(G_{\Bbbk}^\vee)$, which is symmetric; therefore it satisfies

$$\varsigma_{\underline{\mathscr{O}}(G_{\Bbbk}^\vee), \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee))} \circ \varsigma_{\underline{\mathscr{O}}(G_{\Bbbk}^\vee), \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee))} = \mathrm{id},$$

and thus also sends $\mathscr{J} \circledast \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee))$ to $\widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee)) \circledast \mathscr{J}$. It follows that this isomorphism sends $\mathscr{J} \circledast \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee)) + \widetilde{\mathscr{Z}^0}(\underline{\mathscr{O}}(G_{\Bbbk}^\vee)) \circledast \mathscr{J}$ to itself, so that it induces a canonical isomorphism

$$\varsigma : \underline{\mathscr{O}}(H) \circledast \underline{\mathscr{O}}(H) \xrightarrow{\sim} \underline{\mathscr{O}}(H) \circledast \underline{\mathscr{O}}(H).$$

Let us denote by $m_{\underline{\mathscr{O}}(H)} : \underline{\mathscr{O}}(H) \circledast \underline{\mathscr{O}}(H) \to \underline{\mathscr{O}}(H)$ the multiplication map. Using the corresponding properties for $\varsigma_{\underline{\mathscr{O}}(G^\vee_\Bbbk), \widetilde{\mathscr{Z}}^0(\underline{\mathscr{O}}(G^\vee_\Bbbk))}$, it is not difficult to check that we have

$$m_{\underline{\mathscr{O}}(H)} \circ \varsigma = m_{\underline{\mathscr{O}}(H)},$$

and that for any morphism $f : \delta^0 \to \underline{\mathscr{O}}(H)$ we have

$$\varsigma \circ (\mathrm{id}_{\underline{\mathscr{O}}(H)} \circledast f) = f \circledast \mathrm{id}_{\underline{\mathscr{O}}(H)}$$

(where we identify $\delta^0 \circledast \underline{\mathscr{O}}(H)$ and $\underline{\mathscr{O}}(H) \circledast \delta^0$ with $\underline{\mathscr{O}}(H)$ using the unit constraint). In particular, for $f, g \in K$, we have a commutative diagram

$$\begin{tikzcd}
 & \delta^0 \circledast \underline{\mathscr{O}}(H) \arrow[r, "f \circledast \mathrm{id}"] & \underline{\mathscr{O}}(H) \circledast \underline{\mathscr{O}}(H) \arrow[dr, "m_{\underline{\mathscr{O}}(H)}"] & \\
\delta^0 \arrow[ur, "g"] \arrow[dr, "g"'] & & & \underline{\mathscr{O}}(H). \\
 & \underline{\mathscr{O}}(H) \circledast \delta^0 \arrow[uu, "\wr"] \arrow[r, "\mathrm{id} \circledast f"] & \underline{\mathscr{O}}(H) \circledast \underline{\mathscr{O}}(H) \arrow[uu, "\varsigma"'] \arrow[ur, "m_{\underline{\mathscr{O}}(H)}"'] &
\end{tikzcd}$$

From this diagram one deduces that $f \cdot g = g \cdot f$, proving that K is commutative.

Next, one sees as in §6.5.7 that K can also be described as the algebra of endomorphisms of $\underline{\mathscr{O}}(H)$ as an $\underline{\mathscr{O}}(H)$-module. Since this module is simple by construction it follows that K is a division algebra, and hence a field. □

As in the proof of Lemma 6.5.20, one sees that for any $\mathscr{F}$ in $\widetilde{\mathsf{P}}^0_I$ the product $\underline{\mathscr{O}}(H) \circledast \mathscr{F}$ is a direct sum of copies of $\underline{\mathscr{O}}(H)$. Using this, one obtains that the functor

$$\mathfrak{v}_K : \widetilde{\mathsf{P}}^0_I \to \mathsf{Vect}_K$$

defined by

$$\mathfrak{v}_K(\mathscr{F}) = \mathrm{Hom}_{\mathsf{Mod}_{\underline{\mathscr{O}}(H)}}(\underline{\mathscr{O}}(H), \underline{\mathscr{O}}(H) \circledast \mathscr{F})$$

is exact and faithful, that it admits a canonical monoidal structure, and finally that its composition with $\widetilde{\mathscr{Z}}^0$ is canonically isomorphic to the functor $\mathsf{Rep}(G^\vee_\Bbbk) \to \mathsf{Vect}_K$ sending V to $K \otimes_\Bbbk V$.

For simplicity, we now choose an algebraic closure $\mathbb{K}$ of K, and consider the functor

$$\mathfrak{v}_\mathbb{K} : \widetilde{\mathsf{P}}^0_I \to \mathsf{Vect}_\mathbb{K}$$

defined by $\mathfrak{v}_\mathbb{K}(\mathscr{F}) = \mathbb{K} \otimes_K \mathfrak{v}_K(\mathscr{F})$. Then $\mathfrak{v}_\mathbb{K}$ admits a canonical monoidal structure, and the composition $\mathfrak{v}_\mathbb{K} \circ \widetilde{\mathscr{Z}}^0$ is isomorphic (as a monoidal functor) to the functor $V \mapsto \mathbb{K} \otimes_\Bbbk V$.

8.4.4. Monodromy and the regular quotient. — For any V in $\mathsf{Rep}(G^\vee_{\Bbbk})$ we set $\mathsf{m}^0_V := \Pi^0(\mathsf{m}_V)$. Then $\mathsf{m}^0_{(-)}$ defines an automorphism of the functor $\widetilde{\mathscr{Z}}^0$, and by Proposition 3.4.2 we have

$$\mathsf{m}^0_{V\otimes V'} = \mathsf{m}^0_V \circledast \mathsf{m}^0_{V'}$$

for any V, V' in $\mathsf{Rep}(G^\vee_{\Bbbk})$. Composing this automorphism with $\mathfrak{v}_{\mathbb{K}}$ we obtain an automorphism of the functor $V \mapsto \mathbb{K} \otimes_{\Bbbk} V$, which defines (by Tannakian formalism, see [**DM**, Proposition 2.8]) a $\mathbb{K}$-point u_0 of $G^\vee_{\Bbbk}$.

The following claim is our modular analogue of Proposition 6.5.23 (in the present special case).

Proposition 8.4.5. — *The element u_0 is unipotent and regular.*

The proof of this proposition will require the following lemma.

Lemma 8.4.6. — *For $m \in \mathbb{Z}_{\geq 1}$, we consider the map*

$$g_m : \mathbb{A}^m_{\mathbb{F}} \to \mathbb{A}^1_{\mathbb{F}}$$

given by $g_m(x_1, \dots, x_m) = x_1 \cdots x_m$. Then the complex $\underline{\Bbbk}_{g_m^{-1}(0)}[m-1] \in D^{\mathrm{b}}_c(\mathbb{A}^m_{\mathbb{F}}, \Bbbk)$ is a perverse sheaf, which satisfies

$$[\underline{\Bbbk}_{g_m^{-1}(0)}[m-1] : \underline{\Bbbk}_{\{0\}}] = 1.$$

Proof. — The proof proceeds by induction on m. If $m = 1$ then the claim is obvious. If $m \geq 2$, we have a closed embedding $i : \mathbb{A}^{m-1} = \mathbb{A}^{m-1} \times \{0\} \hookrightarrow g_m^{-1}(0)$, with open complement $j : g_{m-1}^{-1}(0) \times (\mathbb{A}^1_{\mathbb{F}} \smallsetminus \{0\}) \hookrightarrow g_m^{-1}(0)$. We deduce a distinguished triangle

$$j_! \underline{\Bbbk}_{g_{m-1}^{-1}(0) \times (\mathbb{A}^1_{\mathbb{F}} \smallsetminus \{0\})}[m-1] \to \underline{\Bbbk}_{g_m^{-1}(0)}[m-1] \to i_* \underline{\Bbbk}_{\mathbb{A}^{m-1}}[m-1] \xrightarrow{[1]} .$$

Since the right-hand side is a simple perverse sheaf distinct from $\underline{\Bbbk}_{\{0\}}$ (because $\mathbb{A}^{m-1}_{\mathbb{F}}$ is smooth), to conclude it suffices to prove that $j_! \underline{\Bbbk}_{g_m^{-1}(0) \times (\mathbb{A}^1_{\mathbb{F}} \smallsetminus \{0\})}[m-1]$ is a perverse sheaf which admits $\underline{\Bbbk}_{\{0\}}$ as a composition factor with multiplicity one. Now j factors a composition

$$g_{m-1}^{-1}(0) \times (\mathbb{A}^1_{\mathbb{F}} \smallsetminus \{0\}) \to g_{m-1}^{-1}(0) \times \mathbb{A}^1_{\mathbb{F}} \to g_m^{-1}(0)$$

where the first map is an open embedding and the second one a closed embedding. We deduce an isomorphism

$$j_! \underline{\Bbbk}_{g_{m-1}^{-1}(0) \times (\mathbb{A}^1_{\mathbb{F}} \smallsetminus \{0\})}[m-1] \cong \underline{\Bbbk}_{g_{m-1}^{-1}(0)}[m-2] \boxtimes (j_0)_! \underline{\Bbbk}_{\mathbb{A}^1_{\mathbb{F}} \smallsetminus \{0\}}[1],$$

where $j_0 : \mathbb{A}^1_{\mathbb{F}} \smallsetminus \{0\} \to \mathbb{A}^1_{\mathbb{F}}$ is the embedding. From this we obtain a distinguished triangle

$$\underline{\Bbbk}_{g_{m-1}^{-1}(0)}[m-2] \boxtimes \underline{\Bbbk}_{\{0\}} \to j_! \underline{\Bbbk}_{g_{m-1}^{-1}(0) \times (\mathbb{A}^1_{\mathbb{F}} \smallsetminus \{0\})}[m-1] \\ \to \underline{\Bbbk}_{g_{m-1}^{-1}(0)}[m-2] \boxtimes \underline{\Bbbk}_{\mathbb{A}^1}[1] \xrightarrow{[1]} .$$

Since $\underline{\Bbbk}_{g_{m-1}^{-1}(0)}[m-2]$ is perverse, the first and third terms in this triangle are perverse sheaves, showing that the middle term is also perverse. Moreover, the composition factors of $\underline{\Bbbk}_{g_{m-1}^{-1}(0)}[m-2] \boxtimes \underline{\Bbbk}_{\mathbb{A}^1}[1]$ are all of the form $\mathscr{F} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1}[1]$ with $\mathscr{F}$ a composition factor of $\underline{\Bbbk}_{g_{m-1}^{-1}(0)}[m-2]$ (see e.g. [**BBDG**, §§4.2.5–4.2.6]); in particular, $\underline{\Bbbk}_{\{0\}}$ is not such a composition factor, which finishes the proof. □

Proof of Proposition 8.4.5. — Consider the coweight $\varepsilon_1^\vee$ for T (in the notation of Example 1.2.3). In view of Example 1.5.4, and by construction of this element, to prove that u_0 is unipotent regular it suffices to prove that if m is the monodromy automorphism of $\mathsf{Z}\big(\underline{\Bbbk}_{\mathrm{Gr}_G^{\varepsilon_1^\vee}}[n]\big)$, then m is unipotent and the nilpotent endomorphism $\mathfrak{v}_{\mathbb{K}}(\Pi^0(\mathsf{m})-\mathrm{id})$ of the n-dimensional $\mathbb{K}$-vector space $\mathfrak{v}_{\mathbb{K}}\circ\Pi^0(\mathsf{Z}(\underline{\Bbbk}_{\mathrm{Gr}_G^{\varepsilon_1^\vee}}[n]))$ has a single Jordan block, or in other words a kernel of dimension at most 1. First, m is indeed unipotent by Proposition 2.4.6(1). For the claim about the kernel of $\mathfrak{v}_{\mathbb{K}}(\Pi^0(\mathsf{m})-\mathrm{id})$, denoting by $\omega\in W$ the unique element of length 0 such that $\mathsf{t}(\varepsilon_1^\vee)\in\omega W_{\mathrm{Cox}}$, the object $\Pi^0(\mathsf{Z}(\underline{\Bbbk}_{\mathrm{Gr}_G^{\varepsilon_1^\vee}}[n]))$ has length n, with all composition factors isomorphic to $\Pi^0(\mathscr{IC}_\omega^I)$. (In fact $\mathsf{Z}(\underline{\Bbbk}_{\mathrm{Gr}_G^{\varepsilon_1^\vee}}[n])$ has a Wakimoto filtration of length n, and by Remark 4.2.4 the image in P_I^0 of each of these Wakimoto sheaves is $\Pi^0(\mathscr{IC}_\omega^I)$.) Therefore, to conclude it suffices to prove that the simple perverse sheaf $\mathscr{IC}_\omega^I$ appears at most once as a composition factor of $\ker(\mathsf{m}-\mathrm{id})$.

A description of the geometry involved in the construction of $\mathsf{Z}(\underline{\Bbbk}_{\mathrm{Gr}_G^{\varepsilon_1^\vee}}[n])$ has been given in §2.2.4. From this description we see that what has to be proved is that if g_n is the map considered in Lemma 8.4.6, and if $\mathscr{F}=\underline{\Bbbk}_{g_n^{-1}(\mathbb{A}^1_{\mathbb{F}}\smallsetminus\{0\})}[n]$, then the simple perverse sheaf $\underline{\Bbbk}_{\{0\}}$ appears at most once as a composition factor of $\ker(\mathsf{m}_{\mathscr{F}}-\mathrm{id})\subset\Psi_{g_n}(\mathscr{F})$.

Let us denote by $j:g_n^{-1}(\mathbb{A}^1_{\mathbb{F}}\smallsetminus\{0\})\hookrightarrow\mathbb{A}^n_{\mathbb{F}}$ the embedding. Then in view of [**Re**, Proposition 3.1] there exists a perverse sheaf $\mathscr{G}$ (in fact, the image of $\mathscr{F}$ under Beĭlinson's "maximal extension" functor) and a factorization

$$\begin{tikzcd} \Psi_{g_n}(\mathscr{F}) \arrow[r, hook] \arrow[rr, bend left, "\mathsf{m}_{\mathscr{F}}-\mathrm{id}"] & \mathscr{G} \arrow[r, two heads] & \Psi_{g_n}(\mathscr{F}) \end{tikzcd}$$

such that the kernel of the surjection $\mathscr{G}\twoheadrightarrow\Psi_{g_n}(\mathscr{F})$ is $j_!\mathscr{F}$. Therefore, to conclude it suffices to prove that

$$[j_!\mathscr{F}:\underline{\Bbbk}_{\{0\}}]=1.$$

Now if $i:g_n^{-1}(0)\hookrightarrow\mathbb{A}^n_{\mathbb{F}}$ is the embedding, then we have a canonical distinguished triangle

$$i_*\underline{\Bbbk}_{g_n^{-1}(0)}[n-1]\to j_!\mathscr{F}\to\underline{\Bbbk}_{\mathbb{A}^n_{\mathbb{F}}}[n]\xrightarrow{[1]}.$$

Here the third term is a simple perverse sheaf distinct from $\underline{\Bbbk}_{\{0\}}$ (by smoothness of $\mathbb{A}^n_{\mathbb{F}}$), and by Lemma 8.4.6 the first term is perverse, and admits $\underline{\Bbbk}_{\{0\}}$ as a composition factor with multiplicity 1; it follows that $j_!\mathscr{F}$ also satisfies these properties, which finishes the proof. □

We can now prove the (weak) analogue over $\Bbbk$ of Lemma 6.5.26. We consider the unipotent variety $\mathcal{U}_\Bbbk\subset G_\Bbbk^\vee$, and the (multiplicative) Springer resolution $\widetilde{\mathcal{U}}_\Bbbk\to\mathcal{U}_\Bbbk$. These varieties also have analogues over $\mathbb{K}$, denoted $\mathcal{U}_{\mathbb{K}}$ and $\widetilde{\mathcal{U}}_{\mathbb{K}}$. We have the regular orbit $\mathcal{O}_{\mathrm{r}}^{\mathbb{K}} \subset \mathcal{U}_{\mathbb{K}}$, and the Springer resolution restricts to an isomorphism $\widetilde{\mathcal{O}}_{\mathrm{r}}^{\mathbb{K}} \xrightarrow{\sim} \mathcal{O}_{\mathrm{r}}^{\mathbb{K}}$, where $\widetilde{\mathcal{O}}_{\mathrm{r}}^{\mathbb{K}}$ is the preimage of $\mathcal{O}_{\mathrm{r}}^{\mathbb{K}}$ (see e.g. [**Hu1**, §§6.3–6.4]). Again we have an isomorphism $G_{\mathbb{K}}^{\vee}/Z_{G_{\mathbb{K}}^{\vee}}(u_0) \xrightarrow{\sim} \mathcal{O}_{\mathrm{r}}^{\mathbb{K}}$, where $G_{\mathbb{K}}^{\vee}$ is the base change of $G_{\Bbbk}^{\vee}$ to $\mathbb{K}$ (see the discussion in [**Hu1**, §6.4]), and hence an equivalence of categories

$$\mathsf{Coh}^{G_{\mathbb{K}}^{\vee}}(\widetilde{\mathcal{O}}_{\mathrm{r}}^{\mathbb{K}}) \xrightarrow{\sim} \mathsf{Rep}(Z_{G_{\mathbb{K}}^{\vee}}(u_0)) \tag{8.4.2}$$

as in (6.5.10). If we denote by $\mathsf{Coh}_{\mathrm{fr,tilt}}^{G_{\mathbb{K}}^{\vee}}(\widetilde{\mathcal{U}}_{\mathbb{K}})$, resp. $\mathsf{Coh}_{\mathrm{fr,tilt}}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk})$, the full subcategory of $\mathsf{Coh}^{G_{\mathbb{K}}^{\vee}}(\widetilde{\mathcal{U}}_{\mathbb{K}})$, resp. $\mathsf{Coh}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk})$, whose objects are those of the form $V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\mathbb{K}}}$ for some V in $\mathsf{Tilt}(G_{\mathbb{K}}^{\vee})$, resp. of the form $V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}$ for some V in $\mathsf{Tilt}(G_{\Bbbk}^{\vee})$, we can consider the composition

$$\begin{aligned} \mathsf{Coh}_{\mathrm{fr,tilt}}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk}) \xrightarrow{\mathbb{K}\otimes_{\Bbbk}(-)} \mathsf{Coh}_{\mathrm{fr,tilt}}^{G_{\mathbb{K}}^{\vee}}(\widetilde{\mathcal{U}}_{\mathbb{K}}) \to \mathsf{Coh}^{G_{\mathbb{K}}^{\vee}}(\widetilde{\mathcal{O}}_{\mathrm{r}}^{\mathbb{K}}) \\ \xrightarrow[\sim]{(8.4.2)} \mathsf{Rep}(Z_{G_{\mathbb{K}}^{\vee}}(u_0)) \to \mathsf{Vect}_{\mathbb{K}} \end{aligned} \tag{8.4.3}$$

where the second arrow is given by restriction to the open subset $\widetilde{\mathcal{O}}_{\mathrm{r}}^{\mathbb{K}} \subset \widetilde{\mathcal{U}}_{\mathbb{K}}$, and the fourth one is the natural forgetful functor. (More concretely, the composition of the last three arrows is the pullback under the embedding $\{\tilde{u}_0\} \hookrightarrow \widetilde{\mathcal{U}}_{\mathbb{K}}$, where $\tilde{u}_0$ is the unique preimage of u_0 in $\widetilde{\mathcal{U}}_{\mathbb{K}}$.)

On the other hand, we can also consider the composition

$$\mathsf{Coh}_{\mathrm{fr,tilt}}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk}) \xrightarrow{\Pi^0 \circ F^{\Bbbk}} \widetilde{\mathsf{P}}_I^0 \xrightarrow{\mathfrak{v}_{\mathbb{K}}} \mathsf{Vect}_{\mathbb{K}}, \tag{8.4.4}$$

where the first arrow is defined as follows: the functor $F^{\Bbbk}$ restricts to a functor $\mathsf{Coh}_{\mathrm{fr,tilt}}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk}) \to \mathsf{P}_I$, and then the composition $\Pi^0 \circ F^{\Bbbk} : \mathsf{Coh}_{\mathrm{fr,tilt}}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk}) \to \mathsf{P}_I^0$ factors through $\widetilde{\mathsf{P}}_I^0$. The functor (8.4.4) admits a canonical monoidal structure.

Lemma 8.4.7. — *The functors* (8.4.3) *and* (8.4.4) *are isomorphic as monoidal functors.*

Proof. — The proof is similar to that of Lemma 6.5.26. First we observe that both functors

$$\mathsf{Tilt}(G_{\Bbbk}^{\vee}) \longrightarrow \mathsf{Coh}_{\mathrm{fr,tilt}}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk}) \underset{(8.4.4)}{\overset{(8.4.3)}{\rightrightarrows}} \mathsf{Vect}_{\mathbb{K}}$$

identify (as monoidal functors) with the functor $\mathsf{Tilt}(G_{\Bbbk}^{\vee}) \to \mathsf{Vect}_{\mathbb{K}}$ sending V to $\mathbb{K} \otimes_{\Bbbk} V$. In other words, (8.4.3) and (8.4.4) agree on objects. It remains to check that they agree on morphisms.

For this, we use the fact that both of our functors extend in a natural way to the full subcategory $\mathsf{Coh}_{\mathrm{fr}}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk})$ of $\mathsf{Coh}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk})$ consisting of objects of the form $V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}$ with V in $\mathsf{Rep}(G_{\Bbbk}^{\vee})$. Using Lemma 8.2.4 we see that here again the compositions

$$\mathsf{Rep}(G_{\Bbbk}^{\vee}) \longrightarrow \mathsf{Coh}_{\mathrm{fr}}^{G_{\Bbbk}^{\vee}}(\widetilde{\mathcal{U}}_{\Bbbk}) \underset{(8.4.4)}{\overset{(8.4.3)}{\rightrightarrows}} \mathsf{Vect}_{\mathbb{K}}$$

are both canonically isomorphic (as monoidal functors) to the functor $\mathsf{Rep}(G_{\Bbbk}^{\vee}) \to \mathsf{Vect}_{\mathbb{K}}$ sending V to $\mathbb{K} \otimes_{\Bbbk} V$.

We then consider the composition of these "extended" functors with the pullback functor $\mathsf{Coh}^{G^\vee_\Bbbk}_{\mathrm{fr}}(G^\vee_\Bbbk) \to \mathsf{Coh}^{G^\vee_\Bbbk}_{\mathrm{fr}}(\widetilde{\mathcal{U}}_\Bbbk)$ associated with the composition $\widetilde{\mathcal{U}}_\Bbbk \to \mathcal{U}_\Bbbk \hookrightarrow G^\vee_\Bbbk$. In view of Example 8.2.1, to check that the two functors

$$\mathsf{Coh}^{G^\vee_\Bbbk}_{\mathrm{fr}}(G^\vee_\Bbbk) \xrightarrow{\text{pullback}} \mathsf{Coh}^{G^\vee_\Bbbk}_{\mathrm{fr}}(\widetilde{\mathcal{U}}_\Bbbk) \underset{(8.4.4)}{\overset{(8.4.3)}{\rightrightarrows}} \mathsf{Vect}_{\mathbb{K}} \tag{8.4.5}$$

coincide it suffices to prove that the associated $\mathbb{K}$-points of $G^\vee_\Bbbk$ coincide; however by construction these two points are given by u_0.

This observation shows that, for any V_1, V_2 in $\mathsf{Rep}(G^\vee_\Bbbk)$, the maps

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee_\Bbbk}(G^\vee_\Bbbk)}(V_1 \otimes \mathscr{O}_{G^\vee_\Bbbk}, V_2 \otimes \mathscr{O}_{G^\vee_\Bbbk}) \to \mathrm{Hom}_{\mathbb{K}}(\mathbb{K} \otimes_\Bbbk V_1, \mathbb{K} \otimes_\Bbbk V_2)$$

induced by the two functors in (8.4.5) coincide. To conclude, it therefore only remains to prove that for V_1, V_2 in $\mathsf{Tilt}(G^\vee_\Bbbk)$, the pullback functor induces a surjection

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee_\Bbbk}(G^\vee_\Bbbk)}(V_1 \otimes \mathscr{O}_{G^\vee_\Bbbk}, V_2 \otimes \mathscr{O}_{G^\vee_\Bbbk}) \to \mathrm{Hom}_{\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)}(V_1 \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V_2 \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}).$$

Here the left-hand side identifies with

$$(V_1^* \otimes V_2 \otimes \mathscr{O}(G^\vee_\Bbbk))^{G^\vee_\Bbbk},$$

and the right-hand side with

$$(V_1^* \otimes V_2 \otimes \mathscr{O}(\widetilde{\mathcal{U}}_\Bbbk))^{G^\vee_\Bbbk} = (V_1^* \otimes V_2 \otimes \mathscr{O}(\mathcal{U}_\Bbbk))^{G^\vee_\Bbbk}.$$

(Here we use the fact that $\mathscr{O}(\widetilde{\mathcal{U}}_\Bbbk) = \mathscr{O}(\mathcal{U}_\Bbbk)$, by normality of $\mathcal{U}_\Bbbk$—see [**Hu1**, §4.24]—and Zariski's main theorem.) Therefore, to conclude it suffices to show that the morphism

$$(V_1^* \otimes V_2 \otimes \mathscr{O}(G^\vee_\Bbbk))^{G^\vee_\Bbbk} \to (V_1^* \otimes V_2 \otimes \mathscr{O}(\mathcal{U}_\Bbbk))^{G^\vee_\Bbbk}$$

induced by restriction is surjective,.

It is well known that if $\mathcal{N}_\Bbbk \subset \mathfrak{g}^\vee_\Bbbk$ is the nilpotent cone, we have $\mathscr{O}(\mathcal{N}_\Bbbk) = \mathscr{O}(\mathfrak{g}^\vee_\Bbbk) \otimes_{\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)^{G^\vee_\Bbbk}} \Bbbk$, and that $\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)^{G^\vee_\Bbbk}$ is isomorphic to a polynomial algebra (generated, for instance, by the coefficients of the characteristic polynomial). Moreover, the morphism $\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)^{G^\vee_\Bbbk} \to \mathscr{O}(\mathfrak{g}^\vee_\Bbbk)$ is flat, e.g. by [**BC**, Proposition 4.2.6]. Starting with a finite free resolution of $\Bbbk$ as an $\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)^{G^\vee_\Bbbk}$-module (e.g., the Koszul resolution), and then tensoring with the flat $\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)^{G^\vee_\Bbbk}$-algebra $\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)$, we obtain a $G^\vee$-equivariant finite resolution of $\mathscr{O}(\mathcal{N}_\Bbbk)$ as an $\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)$-module, all of whose terms are direct sums of copies of $\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)$ (as equivariant modules). Then, since the open embedding (8.1.5) restricts to an isomorphism $\mathcal{U}_\Bbbk \xrightarrow{\sim} \mathcal{N}_\Bbbk$, tensoring with the flat $\mathscr{O}(\mathfrak{g}^\vee_\Bbbk)$-algebra $\mathscr{O}(G^\vee_\Bbbk)$ we obtain a $G^\vee_\Bbbk$-equivariant finite free resolution of $\mathscr{O}(\mathcal{U}_\Bbbk)$ as an $\mathscr{O}(G^\vee_\Bbbk)$-module, all of whose terms are direct sums of copies of $\mathscr{O}(G^\vee_\Bbbk)$ (as equivariant modules). Breaking this resolution into short exact sequences and using the same kinds of arguments as in the first step of the proof of Lemma 6.2.4, from the fact that

$$\mathsf{H}^{>0}(G^\vee_\Bbbk, V_1^* \otimes V_2 \otimes \mathscr{O}(G^\vee_\Bbbk)) = 0$$

(because $V_1^* \otimes V_2 \otimes \mathscr{O}(G^\vee_\Bbbk)$ admits a good filtration, see [**J1**, §§II.4.20–II.4.21]) we obtain that the morphism

$$(V_1^* \otimes V_2 \otimes \mathscr{O}(G^\vee_\Bbbk))^{G^\vee_\Bbbk} \to (V_1^* \otimes V_2 \otimes \mathscr{O}(\mathcal{U}_\Bbbk))^{G^\vee_\Bbbk}$$

induced by restriction is surjective, which finishes the proof. □

8.5. Proof of the equivalence

8.5.1. Statement. — As in Section 6.6 we will consider the triangulated functor

$$F^{\Bbbk}_{\mathcal{IW}} : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk}) \to D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk)$$

defined as the composition

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk}) \xrightarrow{F^{\Bbbk}} D^{\mathrm{b}}\mathsf{P}_I \to D^{\mathrm{b}}_I(\mathrm{Fl}_G, \Bbbk) \xrightarrow{\mathsf{Av}_{\mathcal{IW}}} D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk),$$

where the second arrow is the realization functor. Our goal is to prove the following counterpart of Theorem 6.6.1.

Theorem 8.5.1. — *The functor $F^{\Bbbk}_{\mathcal{IW}}$ is an equivalence of categories.*

8.5.2. Preliminaries. — First, the proof of the following lemma is identical to that of Lemma 6.6.3.

Lemma 8.5.2. — *The objects $\mathsf{Av}_{\mathcal{IW}}(\mathbf{J}_\lambda(\Bbbk))$ with $\lambda \in \mathbf{X}^\vee$ generate $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk)$ as a triangulated category.*

Next, we consider the following analogue of Lemma 6.6.4.

Lemma 8.5.3. — *For any V in $\mathsf{Tilt}(G^\vee_{\Bbbk})$, the morphism*

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk})}(\mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}) \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G, \Bbbk)}(F^{\Bbbk}_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}), F^{\Bbbk}_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}))$$

induced by $F^{\Bbbk}_{\mathcal{IW}}$ is injective.

Proof. — The proof is similar to that of Lemma 6.6.4. Namely, since

$$F^{\Bbbk}_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}) = \Delta^{\mathcal{IW}}_0 \quad \text{and} \quad F^{\Bbbk}_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}) = \mathscr{Z}^0(V),$$

the right-hand side identifies (in view of Theorem 8.3.1) with

$$\mathrm{Hom}_{\mathsf{P}^{\mathrm{asph}}_I}(\Pi_{\mathrm{asph}}(\Delta^I_e), \Pi_{\mathrm{asph}}(\mathscr{Z}(V))),$$

so that to conclude it suffices to prove that the induced morphism

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee_{\Bbbk}}(\widetilde{\mathcal{U}}_{\Bbbk})}(\mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}, V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_{\Bbbk}}) \to \mathrm{Hom}_{\mathsf{P}^0_I}(\delta^0, \mathscr{Z}^0(V))$$

is injective. For this we use the functor $\mathfrak{v}_{\mathbb{K}}$ of §8.4.3. Lemma 8.4.7 shows that the desired claim is equivalent to the fact that restriction to u_0 induces an injective map

$$(V \otimes \mathscr{O}(\mathcal{U}_{\Bbbk}))^{G^\vee_{\Bbbk}} \hookrightarrow \mathbb{K} \otimes_{\Bbbk} V.$$

The latter claim can be proved as in the characteristic-0 setting, after remarking that

$$\mathbb{K} \otimes_{\Bbbk} (V \otimes_{\Bbbk} \mathscr{O}(\mathcal{U}_{\Bbbk}))^{G^\vee_{\Bbbk}} \cong \big((\mathbb{K} \otimes_{\Bbbk} V) \otimes_{\mathbb{K}} \mathscr{O}(\mathcal{U}_{\mathbb{K}})\big)^{G^\vee_{\mathbb{K}}}$$

by [**J1**, I.2.6(3)], where $\mathcal{U}_{\mathbb{K}} \subset G^\vee_{\mathbb{K}}$ is the unipotent variety. □

As with Corollary 6.6.5, we next deduce the following corollary.

Corollary 8.5.4. — *For any V in $\mathsf{Tilt}(G^\vee_\Bbbk)$, any $\lambda \in \mathbf{X}^\vee_+$ and any $n \in \mathbb{Z}$, the morphism*

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)[n]) \\ \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\Bbbk)}(F^\Bbbk_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}), F^\Bbbk_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda))[n])\end{aligned}$$

induced by $F^\Bbbk_{\mathcal{IW}}$ is injective.

Proof. — Let us fix V in $\mathsf{Tilt}(G^\vee_\Bbbk)$ and $\lambda \in \mathbf{X}^\vee_+$. By [**KLT**, Theorem 2] (see also (8.1.6)) we have

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)[n]) = V \otimes \mathsf{H}^n(\widetilde{\mathcal{U}}_\Bbbk, \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)) = 0$$

for $n \neq 0$. Moreover, by [**KLT**, Theorem 7] (see again (8.1.6)) and [**J1**, Proposition II.4.21], the $G^\vee_\Bbbk$-module

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)) = V \otimes \mathsf{H}^0(\widetilde{\mathcal{U}}_\Bbbk, \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda))$$

admits a good filtration. In view of Lemma 8.1.1, this implies that

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)[n]) = 0$$

unless $n = 0$, and moreover that we have

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)) = \big(V \otimes \mathsf{H}^n(\widetilde{\mathcal{U}}_\Bbbk, \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda))\big)^{G^\vee_\Bbbk}.$$

In particular, this shows that the only case we have to consider is when $n = 0$.

By Lemma 8.1.5, there exists V' in $\mathsf{Tilt}(G^\vee_\Bbbk)$ and an embedding $\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda) \hookrightarrow V' \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}$ in $\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)$. We deduce an embedding

$$\mathrm{Hom}_{\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)) \hookrightarrow \mathrm{Hom}_{\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V \otimes V' \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}). \tag{8.5.1}$$

By Lemma 8.5.3, the morphism

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}, V \otimes V' \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}) \\ \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\Bbbk)}(F^\Bbbk_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}), F^\Bbbk_{\mathcal{IW}}(V \otimes V' \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}))\end{aligned} \tag{8.5.2}$$

is injective. Now by functoriality the composition of (8.5.1) and (8.5.2) (which is injective by our arguments above) coincides with the composition of the morphism in the statement (for $n = 0$) with the morphism

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\Bbbk)}(F^\Bbbk_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}), F^\Bbbk_{\mathcal{IW}}(V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda))) \\ \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\Bbbk)}(F^\Bbbk_{\mathcal{IW}}(\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}), F^\Bbbk_{\mathcal{IW}}(V \otimes V' \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}))\end{aligned}$$

induced by our embedding $\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda) \hookrightarrow V' \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}$. The desired injectivity follows. $\square$

8.5.3. Proof of Theorem 8.5.1. — We are now in a position to prove Theorem 8.5.1. We will first prove that $F^{\Bbbk}_{\mathcal{IW}}$ is fully faithful. For this, we have to check that for any $\mathscr{F},\mathscr{G}$ in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)$ the morphism

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk}(\widetilde{\mathcal{U}}_\Bbbk)}(\mathscr{F},\mathscr{G}) \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G,\Bbbk)}(F^{\Bbbk}_{\mathcal{IW}}(\mathscr{F}), F^{\Bbbk}_{\mathcal{IW}}(\mathscr{G})) \tag{8.5.3}$$

induced by $F^{\Bbbk}_{\mathcal{IW}}$ is an isomorphism.

Consider the special case when $\mathscr{F} = \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}$ and $\mathscr{G} = V \otimes \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)[n]$ for some V in $\mathsf{Tilt}(G^\vee_\Bbbk)$, some $\lambda \in \mathbf{X}^\vee_+$ and some $n \in \mathbb{Z}$. In this case (8.5.3) is injective by Corollary 8.5.4. We claim that the domain and codomain have the same (finite) dimension, so that this map must be an isomorphism.

For the right-hand side, exactly the same considerations as in the characteristic-0 setting allow us to conclude that this space vanishes unless $n = 0$, and that in this case its dimension is $\dim(V_{-\lambda})$. Next we turn to the left-hand side of (8.5.3) (still in our particular case). As seen in the course of the proof of Corollary 8.5.4, here again the space vanishes unless $n = 0$, and in this case it identifies with

$$\big(V \otimes \Gamma(\widetilde{\mathcal{U}}_\Bbbk, \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda))\big)^{G^\vee_\Bbbk}.$$

Since $\Gamma(\widetilde{\mathcal{U}}_\Bbbk, \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda))$ admits a good filtration (see again the proof of Corollary 8.5.4) and V is tilting, the dimension of this space is

$$\sum_{\nu \in \mathbf{X}^\vee_+} (V^* : \mathsf{N}_\Bbbk(\nu)) \cdot \Big(\Gamma(\widetilde{\mathcal{U}}_\Bbbk, \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)) : \mathsf{N}_\Bbbk(\nu)\Big),$$

where $(M : \mathsf{N}_\Bbbk(\nu))$ denotes the multiplicity of $\mathsf{N}_\Bbbk(\nu)$ in a good filtration of M. Finally, we observe that the proof of [**Bry**, Lemma 6.1] (which assumes that the base field has characteristic 0, but in fact works in arbitrary characteristic) and the higher cohomology vanishing of $\mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)$ imply that

$$\Big(\Gamma(\widetilde{\mathcal{U}}_\Bbbk, \mathscr{O}_{\widetilde{\mathcal{U}}_\Bbbk}(\lambda)) : \mathsf{N}_\Bbbk(\nu)\Big) = \dim\left(\mathsf{N}_\Bbbk(\nu)_\lambda\right),$$

which allows us to conclude as in the characteristic-0 setting.

The rest of the proof is identical to that in the characteristic-0 setting, simply replacing references to Lemma 6.2.8 by references to Lemma 8.1.4.

PART III

APPENDICES

CHAPTER 9

REVIEW OF THE MAIN PROPERTIES OF NEARBY CYCLES

In this chapter we review the standard constructions and properties of the nearby cycles functors. All of the facts stated here are well known (although sometimes treated in the literature in a more limited or slightly different setting), and are recalled only for ease of reference.

In this book we have tried to write everything in such a way that the translation between the "classical" and the "étale" settings is immediate. However when it comes to the definition of nearby cycles, the two settings really require different treatments. We first explain our constructions in the classical setting, and then briefly explain (in Section 9.5) how to treat the étale setting.

9.1. Definition and basic properties

9.1.1. Definition. — Let X be a separated $\mathbb{C}$-scheme of finite type, and let $f : X \to \mathbb{C}$ be an algebraic map. The constructions below will require the consideration of analytic spaces which are not algebraic varieties; so by "X" we usually mean the topological space $X(\mathbb{C})$ with its classical topology; but for simplicity we do not introduce special notation. Let $X_0 = f^{-1}(0)$ and $X^\times = f^{-1}(\mathbb{C}^\times)$. Let $\exp : \mathbb{C} \to \mathbb{C}^\times$ be the exponential map, and set $\widetilde{X}^\times := X^\times \times_{\mathbb{C}^\times} \mathbb{C}$. (Here $\widetilde{X}^\times$ is an analytic space, but not an algebraic variety in general.) We have the following commutative diagram, in which every square is cartesian (as a diagram of topological spaces):

(9.1.1)
$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d, "f_0"'] & X \arrow[d, "f"] & X^\times \arrow[l, "j"'] \arrow[d, "f^\times"] & \widetilde{X}^\times \arrow[l, "\exp_X"'] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] & \mathbb{C}. \arrow[l, "\exp"'] \end{tikzcd}$$

If $\Bbbk$ is a noetherian commutative ring of finite global dimension, the *nearby cycles functor* is the functor
$$\Psi_f : D^{\mathrm{b}}_c(X^\times, \Bbbk) \to D^{\mathrm{b}}(X_0, \Bbbk)$$
given by
$$\Psi_f(\mathscr{F}) = (i^* j_* \exp_{X*} \exp_X^* \mathscr{F})[-1].$$

For simplicity of notation, when the map f is clear from the context, we will sometimes write Ψ_X for Ψ_f.

Note that if $\mathscr{G}$ belongs to $D^{\mathrm{b}}_c(X,\Bbbk)$, then the adjunction morphism $\mathrm{id} \to j_* \exp_{X*} \exp_X^* j^*$ induces a canonical morphism

$$i^*\mathscr{G}[-1] \to \Psi_f(j^*\mathscr{G}). \tag{9.1.2}$$

Remark 9.1.1. — We emphasize that we incorporate a cohomological shift in the definition of the functor Ψ_f; in some references (e.g. [**KS1**]) the nearby cycles functor is defined as $\mathscr{F} \mapsto i^* j_* \exp_{X*} \exp_X^* \mathscr{F}$ instead. This convention is more appropriate when working with perverse sheaves, as shown in Theorem 9.1.3(2) below.

Example 9.1.2. — A surprisingly useful toy example of the setting above is that in which $X = \mathbb{C}$ and $f = \mathrm{id}$. Consider the natural maps

$$\bar\imath, \bar\imath_1 : \mathrm{pt} \to \mathbb{C}, \quad \pi : \mathbb{C} \to \mathrm{pt}, \quad \bar\jmath : \mathbb{C}^\times \to \mathbb{C}$$

where $\bar\imath$ and $\bar\imath_1$ are the embeddings of 0 and 1 respectively, and $\bar\jmath$ is the natural embedding. If $\mathscr{G} \in D^{\mathrm{b}}_c(\mathbb{C},\Bbbk)$ is a constant complex (i.e. a complex which belongs to the essential image of π^*), then adjunction provides canonical isomorphisms

$$\bar\imath_1^*\mathscr{G} = \pi_*(\bar\imath_1)_*\bar\imath_1^*\mathscr{G} \xleftarrow{\sim} \pi_*\mathscr{G} \xrightarrow{\sim} \pi_*\bar\imath_*\bar\imath^*\mathscr{G} = \bar\imath^*\mathscr{G}. \tag{9.1.3}$$

On the other hand, we also have canonical isomorphisms

$$\begin{aligned} \bar\imath_1^*\mathscr{G}[-1] &= \bar\imath^* \exp^* \bar\jmath^*\mathscr{G}[-1] \xleftarrow{\sim} \pi_* \exp^* \bar\jmath^*\mathscr{G}[-1] \\ &= \pi_*\bar\jmath_* \exp_* \exp^* \bar\jmath^*\mathscr{G}[-1] \xrightarrow{\sim} \bar\imath^*\bar\jmath_* \exp_* \exp^* \bar\jmath^*\mathscr{G}[-1] = \Psi_{\mathrm{id}}(\bar\jmath^*\mathscr{G}). \end{aligned} \tag{9.1.4}$$

(For the isomorphism in the second line, see, for instance, [**Ac3**, Proposition B.4.2].) We claim that these two pairs of maps are related by the following commutative diagram, where all maps are isomorphisms, and where the middle and right-hand vertical arrows are induced by the adjunction map $\mathrm{id} \to \bar\jmath_* \exp_* \exp^* \bar\jmath^*$:

$$\begin{tikzcd} \bar\imath_1^*\mathscr{G}[-1] \arrow[d, equal] & \pi_*\mathscr{G}[-1] \arrow[l] \arrow[r] \arrow[d] & \bar\imath^*\mathscr{G}[-1] \arrow[d, "(9.1.2)"] \\ \bar\imath_1^*\mathscr{G}[-1] & \pi_*\bar\jmath_* \exp_* \exp^* \bar\jmath^*\mathscr{G}[-1] \arrow[l] \arrow[r] & \Psi_{\mathrm{id}}(\bar\jmath^*\mathscr{G}). \end{tikzcd}$$

9.1.2. Some basic properties. — The following statement gives some fundamental properties of this functor.

Theorem 9.1.3. — *Let X be a separated $\mathbb{C}$-scheme of finite type, and let $f : X \to \mathbb{C}$ be an algebraic map.*

1. *For all $\mathscr{F} \in D^{\mathrm{b}}_c(X^\times,\Bbbk)$, the object $\Psi_f(\mathscr{F})$ lies in $D^{\mathrm{b}}_c(X_0,\Bbbk)$.*
2. *The functor $\Psi_f : D^{\mathrm{b}}_c(X^\times,\Bbbk) \to D^{\mathrm{b}}_c(X_0,\Bbbk)$ is t-exact for the perverse t-structures.*

For proofs, see [**KS1**, Proposition 8.6.3 and Corollary 10.3.13] or [**Ac3**, Theorems 4.2.3 and 4.2.8]. (Note that since $\exp_X$ is a nonalgebraic map, it is not obvious from the definition that $\Psi_f(\mathscr{F})$ is constructible. Indeed, the intermediate object

$\exp_{X*}\exp_X^*\mathscr{F}$ is never constructible if $\mathscr{F}$ is nonzero: it has stalks that are not finitely generated over $\Bbbk$.)

The following proposition gathers some compatibility properties of nearby cycles with pushforward and pullback functors. Its proof is an easy exercise in applying proper (or smooth) base change (see in particular [**KS1**, Exercise VIII.15] for (1)).

Proposition 9.1.4. — *Let X, Y be separated $\mathbb{C}$-schemes of finite type, and let $f : X \to \mathbb{C}$ and $g : Y \to X$ be algebraic maps.*

1. *For $\mathscr{F} \in D^{\mathrm{b}}_c(Y^\times, \Bbbk)$, there is a natural (functorial) transformation*
$$\Psi_f((g_{|Y^\times})_*\mathscr{F}) \to (g_{|Y_0})_*\Psi_{f\circ g}(\mathscr{F}).$$
If g is proper, this map is an isomorphism.
2. *For $\mathscr{F} \in D^{\mathrm{b}}_c(X^\times, \Bbbk)$, there is a natural (functorial) transformation*
$$(g_{|Y_0})^*\Psi_f(\mathscr{F}) \to \Psi_{f\circ g}((g_{|Y^\times})^*\mathscr{F}).$$
If g is smooth, this map is an isomorphism.

Remark 9.1.5. —

1. Consider the setting of Proposition 9.1.4(1), and let $h : Z \to Y$ be another algebraic map. Then for $\mathscr{G}$ in $D^{\mathrm{b}}_c(Z^\times, \Bbbk)$ one can check that the following diagram commutes, where the horizontal arrows are provided by Proposition 9.1.4(1) and the vertical ones by the compatibility of pushforward with composition:
$$\begin{tikzcd}
\Psi_f((g_{|Y^\times})_*(h_{|Z^\times})_*\mathscr{G}) \arrow[r] \arrow[d, "\wr"'] & (g_{|Y_0})_*\Psi_{f\circ g}((h_{|Z^\times})_*\mathscr{G}) \arrow[r] & (g_{|Y_0})_*(h_{|Z_0})_*\Psi_{f\circ g\circ h}(\mathscr{G}) \arrow[d, "\wr"] \\
\Psi_f(((g\circ h)_{|Z^\times})_*\mathscr{G}) \arrow[rr] & & ((g\circ h)_{|Z_0})_*\Psi_{f\circ g\circ h}(\mathscr{G}).
\end{tikzcd}$$
Of course, a similar claim holds for the morphism in Proposition 9.1.4(2).
2. An important special case of Proposition 9.1.4(1) is when $f = \mathrm{id}_{\mathbb{C}}$. In this case, for g proper we obtain a canonical isomorphism
$$\Psi_{\mathrm{id}}((g_{|Y^\times})_*\mathscr{F}) \xrightarrow{\sim} R\Gamma(Y_0, \Psi_g(\mathscr{F})).$$
If furthermore the complex $(g_{|Y^\times})_*\mathscr{F}$ is constant in the sense of Example 9.1.2 (which will be the case in all the examples we will encounter in practice), we deduce a canonical isomorphism
$$\mathsf{H}^\bullet(g^{-1}(1), \mathscr{F}_{|g^{-1}(1)}) \xrightarrow{\sim} \mathsf{H}^\bullet(Y_0, \Psi_g(\mathscr{F})).$$

We will also require the following compatibility properties of nearby cycles with Verdier duality, change-of-scalars, and external tensor products. Here (1) is stated only for completeness; it will not be used in this book. In (2), given a ring morphism $\varphi : \Bbbk \to \Bbbk'$ (where both rings are noetherian, commutative, of finite global dimension), we consider the change-of-scalars functors
$$\Bbbk' \overset{L}{\otimes}_{\Bbbk} (-) : D^{\mathrm{b}}_c(X, \Bbbk) \to D^{\mathrm{b}}_c(X, \Bbbk'), \qquad \Bbbk' \overset{L}{\otimes}_{\Bbbk} (-) : D^{\mathrm{b}}_c(X_0, \Bbbk) \to D^{\mathrm{b}}_c(X_0, \Bbbk'),$$
see e.g. [**Ac3**, §1.4].

Proposition 9.1.6. — 1. *Let X be a separated $\mathbb{C}$-scheme of finite type, and let $f : X \to \mathbb{C}$ be an algebraic map. For $\mathscr{F} \in D^{\mathrm{b}}_c(X^\times, \Bbbk)$, there is a natural isomorphism*

$$\Psi_f \circ \mathbb{D}_{X^\times}(\mathscr{F}) \xrightarrow{\sim} \mathbb{D}_{X_0} \circ \Psi_f(\mathscr{F}).$$

2. *Let X be a separated $\mathbb{C}$-scheme of finite type, let $f : X \to \mathbb{C}$ be an algebraic map, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For $\mathscr{F} \in D^{\mathrm{b}}_c(X^\times, \Bbbk)$, there is a natural isomorphism*

$$\Bbbk' \overset{L}{\otimes}_{\Bbbk} \Psi_f(\mathscr{F}) \to \Psi_f(\Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathscr{F}).$$

3. *Let X, Y be separated $\mathbb{C}$-schemes of finite type, let $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be algebraic maps, and let $f \times_{\mathbb{C}} g : X \times_{\mathbb{C}} Y \to \mathbb{C}$ be the induced map. For $\mathscr{F} \in D^{\mathrm{b}}_c(X^\times, \Bbbk)$ and $\mathscr{G} \in D^{\mathrm{b}}_c(Y^\times, \Bbbk)$, there is a natural isomorphism*

$$\Psi_f(\mathscr{F}) \overset{L}{\boxtimes} \Psi_g(\mathscr{G}) \xrightarrow{\sim} \Psi_{f \times_{\mathbb{C}} g}(\mathscr{F} \overset{L}{\boxtimes}_{\mathbb{C}} \mathscr{G}),$$

where $\mathscr{F} \boxtimes^L_{\mathbb{C}} \mathscr{G}$ denotes the restriction of $(\mathscr{F} \boxtimes^L \mathscr{G})[-1]$ to $X^\times \times_{\mathbb{C}^\times} Y^\times \subset X^\times \times Y^\times$.

For proofs, see [**Mas**, Corollary 3.2] or [**Ac3**, Proposition 4.2.4] for (1) (this statement has earlier but weaker antecedents; see [**Mas**] for details), see [**Ac3**, Proposition 4.2.5] for (2), and see [**Sc**, Theorem 1.0.4] for (3).

Remark 9.1.7. — Although we will not repeat the proof of Proposition 9.1.6, it will be useful below to have an explicit description of the construction of the map in part (3). Recall that for any continuous map of topological spaces $h : X \to Y$, there is a natural "Künneth" map (not, in general, an isomorphism)

$$h_*\mathscr{F} \overset{L}{\otimes} h_*\mathscr{G} \to h_*(\mathscr{F} \overset{L}{\otimes} \mathscr{G}). \tag{9.1.5}$$

(This map is obtained using the adjunction (h^*, h_*) and the compatibility of h^* with tensor products.) From this construction we deduce, for continuous maps $h_1 : X_1 \to Y_1$ and $h_2 : X_2 \to Y_2$, a canonical morphism

$$(h_1)_*\mathscr{F} \overset{L}{\boxtimes} (h_2)_*\mathscr{G} \to (h_1 \times h_2)_*(\mathscr{F} \overset{L}{\boxtimes} \mathscr{G}); \tag{9.1.6}$$

see e.g. [**Sc**, §1.4].

Briefly, the map in part (3) (following [**Sc**]; see also [**I1**, §4]) is defined by applying (9.1.5) to the map $h = j \circ \exp_{X \times_{\mathbb{C}} Y}$. In more detail, denote by i_X, j_X, $\exp_X$, resp. i_Y, j_Y, $\exp_Y$, the maps involved in the construction of Ψ_f, resp. Ψ_g, and consider the diagram

$$X_0 \times Y_0 \xrightarrow{i^{(2)} = i_X \times i_Y} X \times Y \xleftarrow{j^{(2)} = j_X \times j_Y} X^\times \times Y^\times \xleftarrow{\exp^{(2)} = \exp_X \times \exp_Y} \widetilde{X}^\times \times \widetilde{Y}^\times.$$

Using (9.1.6) we obtain a canonical morphism

$$\Psi_f(\mathscr{F}) \overset{L}{\boxtimes} \Psi_g(\mathscr{G}) \to i^{(2)*} j^{(2)}_* \exp^{(2)}_* \exp^{(2)*}(\mathscr{F} \overset{L}{\boxtimes} \mathscr{G})[-2],$$

and a computation of stalks (see [**Sc**, Proof of Corollary 1.2.1]) shows that this map is an isomorphism. (The idea is that the stalks can be expressed as the cohomology of a *compact* space, so that one can apply the classical Künneth formula.)

Next, consider the commutative diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[dr, "\exp"'] & (\mathbb{C}\times\mathbb{C}) \times_{\mathbb{C}^\times\times\mathbb{C}^\times} \mathbb{C}^\times \arrow[r] \arrow[d] & \mathbb{C}\times\mathbb{C} \arrow[d, "\exp\times\exp"] \\ & \mathbb{C}^\times \arrow[r] & \mathbb{C}^\times\times\mathbb{C}^\times \end{tikzcd}$$

Here, the bottom horizontal map is the diagonal embedding, and the composition of the two maps along the top is also the diagonal embedding. Let k, $\tilde{k}$, e, and d be the maps shown in the diagram below, obtained by taking the fiber product of the diagram above with $X \times Y$:

$$\begin{tikzcd} \widetilde{X}^\times \times_{\mathbb{C}} \widetilde{Y}^\times = (\widetilde{X \times_{\mathbb{C}} Y})^\times \arrow[r, "d"] \arrow[dr, "\exp_{X\times_{\mathbb{C}}Y}"'] & \widetilde{X}^\times \times_{\mathbb{C}^\times} \widetilde{Y}^\times \arrow[r, "\tilde{k}"] \arrow[d, "e"] & \widetilde{X}^\times \times \widetilde{Y}^\times \arrow[d, "\exp^{(2)}"] \\ & X^\times \times_{\mathbb{C}^\times} Y^\times \arrow[r, "k"] & X^\times \times Y^\times. \end{tikzcd}$$

We have a sequence of adjunction and base change maps

$$\begin{aligned} \exp^{(2)}_* \exp^{(2)*}(\mathscr{F} \overset{L}{\boxtimes} \mathscr{G}) &\to k_* k^* \exp^{(2)}_* \exp^{(2)*}(\mathscr{F} \overset{L}{\boxtimes} \mathscr{G}) \\ &\to k_* e_* \tilde{k}^* \exp^{(2)*}(\mathscr{F} \overset{L}{\boxtimes} \mathscr{G}) \cong k_* e_* e^*(\mathscr{F} \overset{L}{\boxtimes}_{\mathbb{C}} \mathscr{G}[1]) \\ &\to k_* e_* d_* d^* e^*(\mathscr{F} \overset{L}{\boxtimes}_{\mathbb{C}} \mathscr{G}[1]) \cong k_* \exp_{X\times_{\mathbb{C}}Y*} \exp^*_{X\times_{\mathbb{C}}Y}(\mathscr{F} \overset{L}{\boxtimes}_{\mathbb{C}} \mathscr{G}[1]). \end{aligned}$$

Applying $i^{(2)*} j^{(2)}_*[-2]$ we deduce a natural map

$$i^{(2)*} j^{(2)}_* \exp^{(2)}_* \exp^{(2)*}(\mathscr{F} \overset{L}{\boxtimes} \mathscr{G})[-2] \to i^{(2)*}(j^{(2)} \circ k)_* \exp_{X\times_{\mathbb{C}}Y*} \exp^*_{X\times_{\mathbb{C}}Y}(\mathscr{F} \overset{L}{\boxtimes}_{\mathbb{C}} \mathscr{G}[-1]).$$

Here, both $i^{(2)}$ and $j^{(2)} \circ k$ factor through the closed immersion $X \times_{\mathbb{C}} Y \to X \times Y$. Using these factorizations and the fact that a composition of push and pull functors associated with a closed immersion is the identity, one sees that the codomain of this morphism identifies with $\Psi_{f\times_{\mathbb{C}}g}(\mathscr{F} \boxtimes^L_{\mathbb{C}} \mathscr{G})$. Once again, a computation of stalks shows that this map is an isomorphism, from which we obtain Proposition 9.1.6(3).

We conclude this subsection with another compatibility property of nearby cycles with external products.

Lemma 9.1.8. — *Let X, Y be separated $\mathbb{C}$-schemes of finite type, and let $f : X \to \mathbb{C}$ be an algebraic map. For any $\mathscr{F}$ in $D^{\mathrm{b}}_c(X^\times, \Bbbk)$ and any $\mathscr{G}$ in $D^{\mathrm{b}}_c(Y, \Bbbk)$, there exists a canonical isomorphism*

$$\Psi_f(\mathscr{F}) \overset{L}{\boxtimes} \mathscr{G} \xrightarrow{\sim} \Psi_{f\circ p_X}(\mathscr{F} \overset{L}{\boxtimes} \mathscr{G})$$

where $p_X : X \times Y \to X$ is the projection morphism.

Proof. — Recall that $*$-pullback functors commute with external products in the natural way, see [**Ac3**, Proposition 1.4.21]. In view of the definition of the nearby cycles functor, it therefore suffices to prove that for any $\mathscr{F}'$ in $D^+(\widetilde{X}^\times, \Bbbk)$ and $\mathscr{G}$ in $D^{\mathrm{b}}_c(Y, \Bbbk)$ the natural morphism

$$(j \circ \exp_X)_* \mathscr{F}' \overset{L}{\boxtimes} \mathscr{G} \to \big((j \circ \exp_X) \times \mathrm{id}_Y\big)_* (\mathscr{F}' \overset{L}{\boxtimes} \mathscr{G})$$

(see (9.1.6)) is an isomorphism. This claim however follows from the proof of [**Ac3**, Proposition 2.9.1]. (This proof shows an isomorphism $h_* \mathscr{F}' \boxtimes^L \mathscr{G} \to (h \times \mathrm{id})_* (\mathscr{F}' \boxtimes^L \mathscr{G})$ under the assumption that h is algebraic and $\mathscr{F}'$ is bounded and constructible, but these assumptions are not used in the arguments; only the fact that $\mathscr{G}$ is bounded and constructible matters.) $\square$

9.1.3. Monodromy. — For $\mathscr{F} \in D^{\mathrm{b}}_c(X^\times, \Bbbk)$, there is a natural action of the fundamental group $\pi_1(\mathbb{C}^\times, 1)$ on the object $\exp_{X*} \exp_X^* \mathscr{F}$ (induced by the action on $\mathbb{C}$), and hence on $\Psi_f(\mathscr{F})$. In particular, one can consider the action of the canonical generator of $\pi_1(\mathbb{C}^\times, 1)$ (i.e. the class of the counterclockwise loop $t \mapsto \exp(2i\pi t)$). The resulting automorphism of $\Psi_f(\mathscr{F})$ is denoted by

$$\mathsf{m}_{\mathscr{F}} : \Psi_f(\mathscr{F}) \xrightarrow{\sim} \Psi_f(\mathscr{F}),$$

and is called the *monodromy automorphism*. The isomorphisms in Proposition 9.1.4 and in Proposition 9.1.6 are all compatible with the monodromy automorphisms in the obvious way; for instance, in the setting of Proposition 9.1.6(3) the map $\mathsf{m}_{\mathscr{F}} \boxtimes^L \mathsf{m}_{\mathscr{G}}$ identifies with $\mathsf{m}_{\mathscr{F} \boxtimes^L_{\mathbb{C}} \mathscr{G}}$. It is well known also that for any $\mathscr{F} \in D^{\mathrm{b}}_c(X^\times, \Bbbk)$ we have a canonical distinguished triangle

$$i^* j_* \mathscr{F}[-1] \to \Psi_f(\mathscr{F}) \xrightarrow{\mathrm{id} - \mathsf{m}_{\mathscr{F}}} \Psi_f(\mathscr{F}) \xrightarrow{[1]}, \tag{9.1.7}$$

see [**Brl**, Proposition 1.1] or [**Sc**, (5.88)].

Using this triangle, the complex $\Psi_f(\mathscr{F})$ can be more explicitly computed in the following case.

Lemma 9.1.9. — *Let X be a separated $\mathbb{C}$-scheme of finite type, let $f : X \to \mathbb{C}$ be an algebraic map, and let $\mathscr{F} \in \mathsf{Perv}(X^\times, \Bbbk)$. Assume that $\mathsf{m}_{\mathscr{F}} = \mathrm{id}$. Then there exists a canonical isomorphism*

$$\Psi_f(\mathscr{F}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_{!*} \mathscr{F}).$$

Proof. — Since $\Psi_f(\mathscr{F})$ is perverse (see Theorem 9.1.3(2)), taking the long exact sequence of perverse cohomology associated with the distinguished triangle (9.1.7) we obtain an isomorphism ${}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_* \mathscr{F}) \xrightarrow{\sim} \Psi_f(\mathscr{F})$. Now, consider the standard distinguished triangle

$$j_! \mathscr{F} \to j_* \mathscr{F} \to i_* i^* j_* \mathscr{F} \xrightarrow{[1]}.$$

Since $j_! \mathscr{F}$ and $j_* \mathscr{F}$ are perverse (by [**Ac3**, Corollary 3.5.9], which is applicable here because j is an affine morphism), using this triangle we obtain that ${}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_* \mathscr{F})$ identifies with the kernel of the canonical map $j_! \mathscr{F} \to j_* \mathscr{F}$. Now, by definition of the functor $j_{!*}$ this kernel is also the kernel of the surjection $j_!\mathscr{F} \twoheadrightarrow j_{!*}\mathscr{F}$, and considerations similar to those above show that the latter kernel is ${}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_{!*}\mathscr{F})$, as desired. □

9.2. Beĭlinson's construction of unipotent nearby cycles

In this section, we describe Beĭlinson's alternative construction [**Beĭ2**] of the "unipotent part" Ψ_f^{un} of Ψ_f, following the exegesis in [**Re**] (see also [**Mo**]). Because the proofs in [**Re**] assume that we are working with field coefficients, we include proofs below of the statements we will need.

9.2.1. The unipotent nearby cycles functor. — If $\mathscr{F} \in \mathsf{Perv}(X^\times, \Bbbk)$, then by Theorem 9.1.3(2) $\Psi_f(\mathscr{F})$ is also a perverse sheaf. We define the *unipotent nearby cycles sheaf* of $\mathscr{F}$, denoted by $\Psi_f^{\mathrm{un}}(\mathscr{F})$, by

$$\Psi_f^{\mathrm{un}}(\mathscr{F}) = \varinjlim_a \ker(1 - \mathsf{m}_{\mathscr{F}})^a. \tag{9.2.1}$$

Let us explain why this limit exists. We clearly have an increasing chain of subobjects

$$\ker(1 - \mathsf{m}_{\mathscr{F}}) \subset \ker(1 - \mathsf{m}_{\mathscr{F}})^2 \subset \ker(1 - \mathsf{m}_{\mathscr{F}})^3 \subset \cdots$$

of $\Psi_f(\mathscr{F})$. Since $\mathsf{Perv}(X_0, \Bbbk)$ is a noetherian category (see [**Ac3**, Theorem 3.4.4]), this sequence is eventually constant, which justifies the existence of the limit in (9.2.1).

By construction, the automorphism $\mathsf{m}_{\mathscr{F}}$ of $\Psi_f(\mathscr{F})$ preserves the subobject $\Psi_f^{\mathrm{un}}(\mathscr{F})$. Moreover, the endomorphism

$$\mathsf{m}_{\mathscr{F}}^{\mathrm{un}} : \Psi_f^{\mathrm{un}}(\mathscr{F}) \to \Psi_f^{\mathrm{un}}(\mathscr{F})$$

induced by $\mathsf{m}_{\mathscr{F}}$ is unipotent (in the sense that $(1 - \mathsf{m}_{\mathscr{F}}^{\mathrm{un}})^a = 0$ for $a \gg 0$), and the endomorphism of $\Psi_f(\mathscr{F})/\Psi_f^{\mathrm{un}}(\mathscr{F})$ induced by $1 - \mathsf{m}_{\mathscr{F}}$ is injective. In fact, $\Psi_f^{\mathrm{un}}(\mathscr{F})$ is characterized (among subobjects of the perverse sheaf $\Psi_f(\mathscr{F})$) by these properties.

In view of the definition and the distinguished triangle (9.1.7), we have canonical isomorphisms

$$\ker(1 - \mathsf{m}_{\mathscr{F}}^{\mathrm{un}}) = \ker(1 - \mathsf{m}_{\mathscr{F}}) \cong {}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_* \mathscr{F}) \tag{9.2.2}$$

and

$$\mathrm{cok}(1 - \mathsf{m}_{\mathscr{F}}) \cong {}^{\mathrm{p}}\mathscr{H}^{0}(i^* j_* \mathscr{F}). \tag{9.2.3}$$

If $\Bbbk$ is an artinian ring (e.g. a field), then the category $\mathsf{Perv}(X_0, \Bbbk)$ is artinian as well (see [**Ac3**, Theorem 3.4.5(3)], whose proof applies to any artinian ring), so that the sequence

$$\mathrm{im}(1 - \mathsf{m}_{\mathscr{F}}) \supset \mathrm{im}(1 - \mathsf{m}_{\mathscr{F}})^2 \supset \mathrm{im}(1 - \mathsf{m}_{\mathscr{F}})^3 \supset \cdots \tag{9.2.4}$$

must be eventually constant. Denoting by $\Psi_f^{\mathrm{n\text{-}un}}(\mathscr{F})$ its limit, it is not difficult to check that we have a canonical decomposition

$$\Psi_f(\mathscr{F}) = \Psi_f^{\mathrm{un}}(\mathscr{F}) \oplus \Psi_f^{\mathrm{n\text{-}un}}(\mathscr{F}) \tag{9.2.5}$$

preserved by $\mathsf{m}_{\mathscr{F}}$. In particular, in this case the functor Ψ_f^{un} is a direct summand of Ψ_f, and then Theorem 9.1.3(2) implies that Ψ_f^{un} is an exact functor.

For general $\Bbbk$, Ψ_f^{un} is at least left exact, but it may fail to be exact.

9.2.2. Beĭlinson's construction. — We identify the fundamental group $\pi_1(\mathbb{C}^\times, 1)$ with $\mathbb{Z}$ in such a way that the canonical generator from §9.1.3 corresponds to $1 \in \mathbb{Z}$. For $a \geq 0$ we set $L_a := \Bbbk[x]/(x^a)$, which we regard as a $\mathbb{Z}$-module on which $1 \in \mathbb{Z}$ acts by multiplication by $1 - x$. Via the identification $\pi_1(\mathbb{C}^\times, 1) \cong \mathbb{Z}$, L_a can be regarded as a $\Bbbk$-representation of $\pi_1(\mathbb{C}^\times, 1)$, and we denote by $\mathscr{L}_a$ the associated $\Bbbk$-local system. (Below, we will sometimes write $\mathscr{L}_a^{\Bbbk}$ instead of $\mathscr{L}_a$ when we want to emphasize the base ring under consideration.). We will denote by $T_a : \mathscr{L}_a \to \mathscr{L}_a$ the automorphism corresponding to multiplication by $1 - x$. Note that $(\mathrm{id} - T_a)^a = 0$. If $a \leq b$, multiplication by x^{b-a} induces an embedding of $\Bbbk$-representations $L_a \hookrightarrow L_b$ whose quotient is L_{b-a}, and hence an embedding of local systems

$$\mathscr{L}_a \hookrightarrow \mathscr{L}_b \tag{9.2.6}$$

whose quotient is $\mathscr{L}_{b-a}$.

Proposition 9.2.1. — *Let X be a separated $\mathbb{C}$-scheme of finite type, let $f : X \to \mathbb{C}$ be an algebraic map, and let $\mathscr{F} \in \mathsf{Perv}(X^\times, \Bbbk)$.*

1. *For any $a \geq 0$, there exists a canonical isomorphism*

$$ {}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) \xrightarrow{\sim} \ker(1 - \mathsf{m}_{\mathscr{F}}^{\mathrm{un}})^a \subset \Psi_f^{\mathrm{un}}(\mathscr{F})$$

that intertwines the map induced by T_a^{-1} with $\mathsf{m}_{\mathscr{F}}^{\mathrm{un}}$. Moreover, under these isomorphisms the map induced by (9.2.6) *corresponds to the embedding* $\ker(1 - \mathsf{m}_{\mathscr{F}}^{\mathrm{un}})^a \hookrightarrow \ker(1 - \mathsf{m}_{\mathscr{F}}^{\mathrm{un}})^b$. *In particular, for $a \gg 0$ we have a canonical isomorphism*

$$ {}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) \xrightarrow{\sim} \Psi_f^{\mathrm{un}}(\mathscr{F}). \tag{9.2.7}$$

2. *For any $a \geq 0$, there exists a canonical isomorphism*

$$ {}^{\mathrm{p}}\mathscr{H}^{0}(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) \xrightarrow{\sim} \mathrm{cok}(1 - \mathsf{m}_{\mathscr{F}})^a$$

that intertwines the map induced by T_a^{-1} with $\mathsf{m}_{\mathscr{F}}$. Moreover, under these isomorphisms the map induced by (9.2.6) *corresponds to the map* $\mathrm{cok}(1 - \mathsf{m}_{\mathscr{F}})^a \to \mathrm{cok}(1 - \mathsf{m}_{\mathscr{F}})^b$ *induced by* $(1 - \mathsf{m}_{\mathscr{F}}^{-1})^{b-a}$.

   In particular, if the sequence (9.2.4) *is eventually constant,*[(1)] *then for $a \gg 0$ we have a canonical isomorphism*

$$ {}^{\mathrm{p}}\mathscr{H}^{0}(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) \xrightarrow{\sim} \Psi_f^{\mathrm{un}}(\mathscr{F}).$$

3. *For any $a \geq 0$ and any $n \in \mathbb{Z} \smallsetminus \{0, -1\}$ we have*

$$ {}^{\mathrm{p}}\mathscr{H}^{n}(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) = 0.$$

---

(1) As noted in §9.2.1, this assumption is automatically satisfied if $\Bbbk$ is artinian. It is also true of course if $\Psi_f^{\mathrm{un}}(\mathscr{F}) = \Psi_f(\mathscr{F})$.

If the sequence (9.2.4) *is eventually constant, then for any* $a \geq 0$ *there exists* $b \geq a$ *such that the map*

$$ {}^{\mathrm{p}}\mathscr{H}^0(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) \to {}^{\mathrm{p}}\mathscr{H}^0(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_b)) \tag{9.2.8}$$

induced by (9.2.6) *vanishes.*

Proof. — (1) It is clear that the complex $\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a$ is a perverse sheaf. By (9.2.2), we have a canonical isomorphism

$$ {}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) \cong \ker(1 - \mathsf{m}^{\mathrm{un}}_{\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a}). \tag{9.2.9}$$

On the other hand, since $\exp^*(\mathscr{L}_a) \cong L_a \otimes_{\Bbbk} \underline{\Bbbk}_{\mathbb{C}}$, we have a natural isomorphism

$$\Psi_f(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a) \cong \Psi_f(\mathscr{F}) \overset{L}{\otimes}_{\Bbbk} L_a \tag{9.2.10}$$

that identifies $\mathsf{m}_{\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a}$ with $\mathsf{m}_{\mathscr{F}} \otimes (1-x)$. The endomorphism $\mathsf{m}_{\mathscr{F}} \otimes (1-x)$ of $\Psi_f(\mathscr{F}) \otimes L_a$ stabilizes $\Psi_f^{\mathrm{un}}(\mathscr{F}) \otimes L_a$, its restriction to this subobject is unipotent, and the endomorphism of $(\Psi_f(\mathscr{F}) \otimes L_a)/(\Psi_f^{\mathrm{un}}(\mathscr{F}) \otimes L_a)$ induced by $1 - (\mathsf{m}_{\mathscr{F}} \otimes (1-x))$ is invertible. Therefore, the isomorphism (9.2.10) restricts to an isomorphism

$$\Psi_f^{\mathrm{un}}(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a) \cong \Psi_f^{\mathrm{un}}(\mathscr{F}) \overset{L}{\otimes}_{\Bbbk} L_a \tag{9.2.11}$$

that identifies $\mathsf{m}^{\mathrm{un}}_{\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a}$ with $\mathsf{m}^{\mathrm{un}}_{\mathscr{F}} \otimes (1-x)$.

In the rest of the proof, to avoid heavy notation it will be convenient to treat $\Psi_f^{\mathrm{un}}(\mathscr{F})$ informally as though it had "elements." (We leave it to interested readers to translate our informal arguments into rigorous ones using the language of abelian categories.) Thus, in view of (9.2.11), any element of $\Psi_f^{\mathrm{un}}(\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a)$ can be written as $\sum_{i=0}^{a-1} m_i \otimes x^i$, where $m_0, \dots, m_{a-1}$ are elements of $\Psi_f^{\mathrm{un}}(\mathscr{F})$. In this language, the operator $1 - \mathsf{m}^{\mathrm{un}}_{\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a}$ is given by

$$\begin{aligned}(1 - \mathsf{m}^{\mathrm{un}}_{\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a})\Big(\sum_{i=0}^{a-1} m_i \otimes x^i\Big) \\ = (m_0 - \mathsf{m}^{\mathrm{un}}_{\mathscr{F}}(m_0)) \otimes 1 + \sum_{i=1}^{a-1} (m_i - \mathsf{m}^{\mathrm{un}}_{\mathscr{F}}(m_i) + \mathsf{m}^{\mathrm{un}}_{\mathscr{F}}(m_{i-1})) \otimes x^i,\end{aligned} \tag{9.2.12}$$

so that our element $\sum_{i=0}^{a-1} m_i \otimes x^i$ lies in the kernel of $1 - \mathsf{m}^{\mathrm{un}}_{\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a}$ if and only if

$$(1 - \mathsf{m}^{\mathrm{un}}_{\mathscr{F}})(m_0) = 0 \quad \text{and} \quad m_i - \mathsf{m}^{\mathrm{un}}_{\mathscr{F}}(m_i) + \mathsf{m}^{\mathrm{un}}_{\mathscr{F}}(m_{i-1}) = 0 \quad \text{for } i = 1, \dots, a-1. \tag{9.2.13}$$

Since $\mathsf{m}^{\mathrm{un}}_{\mathscr{F}}$ is invertible, the latter equation can be rewritten as

$$m_{i-1} = (1 - (\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1}) m_i,$$

and then we deduce that $m_i = (1 - (\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1})^{a-1-i} m_{a-1}$ for all i. In particular, $m_0 = (1 - (\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1})^{a-1} m_{a-1}$. In light of this, the first equation in (9.2.13) is equivalent to $(1-(\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1})^a m_{a-1}=0$. To summarize, $\sum_{i=0}^{a-1} m_i\otimes x^i$ lies in the kernel of $1-\mathsf{m}^{\mathrm{un}}_{\mathscr{F}\otimes^L(f^\times)^*\mathscr{L}_a}$ if and only if

(9.2.14)
$$(1-(\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1})^a m_{a-1}=0 \quad\text{and}\quad m_i=(1-(\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1})^{a-1-i}m_{a-1} \text{ for } i=0,\dots,a-1.$$

Now, consider the surjection

$$\Psi^{\mathrm{un}}_f(\mathscr{F}\overset{L}{\otimes}(f^\times)^*\mathscr{L}_a)\twoheadrightarrow\Psi^{\mathrm{un}}_f(\mathscr{F}) \tag{9.2.15}$$

sending $\sum_{i=0}^{a-1} m_i\otimes x^i$ to m_{a-1}. From (9.2.14) we see that this map is injective on $\ker(1-\mathsf{m}^{\mathrm{un}}_{\mathscr{F}\otimes^L(f^\times)^*\mathscr{L}_a})$, and that the image of this kernel is $\ker(1-(\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1})^a=\ker(1-\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^a$. In view of (9.2.9), this provides the desired isomorphism.

It is clear from the construction that these isomorphisms are compatible with the maps induced by (9.2.6) in the expected way, and an easy computation shows that they intertwine T_a with $(\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1}$.

(2) The construction is "dual" to the one above. Namely, using a nonunipotent version of (9.2.12) we see that the embedding

$$\Psi_f(\mathscr{F})\hookrightarrow\Psi_f(\mathscr{F}\overset{L}{\otimes}(f^\times)^*\mathscr{L}_a)$$

sending m to $m\otimes 1$ (where we tacitly use (9.2.10)) induces an isomorphism

$$\mathrm{cok}(1-\mathsf{m}^{-1}_{\mathscr{F}})^a\xrightarrow{\sim}\mathrm{cok}(1-\mathsf{m}_{\mathscr{F}\otimes^L(f^\times)^*\mathscr{L}_a}).$$

Since $\mathrm{cok}(1-\mathsf{m}^{-1}_{\mathscr{F}})^a=\mathrm{cok}(1-\mathsf{m}_{\mathscr{F}})^a$, using (9.2.3) we deduce the desired isomorphism. We leave it to the reader to check the stated compatibilities with the maps induced by T_a and by (9.2.6).

For the last assertion, we can make use of the decomposition (9.2.5). In this expression, $1-\mathsf{m}_{\mathscr{F}}$ restricts to an automorphism on $\Psi^{\text{n-un}}_f(\mathscr{F})$. It follows that

$$\mathrm{cok}(1-\mathsf{m}_{\mathscr{F}})^a=\mathrm{cok}(1-\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^a. \tag{9.2.16}$$

for any $a\geq 1$. Since $1-\mathsf{m}^{\mathrm{un}}_{\mathscr{F}}$ is nilpotent, the right-hand side above identifies with $\Psi^{\mathrm{un}}_f(\mathscr{F})$ for $a\gg 0$.

(3) Since j is an affine embedding, the complexes $j_!(\mathscr{F}\otimes^L(f^\times)^*\mathscr{L}_a)$ and $j_*(\mathscr{F}\otimes^L(f^\times)^*\mathscr{L}_a)$ are perverse sheaves. Considering the canonical distinguished triangle

$$j_!(\mathscr{F}\overset{L}{\otimes}(f^\times)^*\mathscr{L}_a)\xrightarrow{\alpha}j_*(\mathscr{F}\overset{L}{\otimes}(f^\times)^*\mathscr{L}_a)\to i_*i^*j_*(\mathscr{F}\overset{L}{\otimes}(f^\times)^*\mathscr{L}_a)\xrightarrow{[1]}$$

we deduce that ${}^{\mathrm{p}}\mathscr{H}^n(i^*j_*(\mathscr{F}\otimes^L(f^\times)^*\mathscr{L}_a))=0$ for $n\notin\{-1,0\}$.

Now, consider the case $n=0$. In this case, in view of (2) together with (9.2.16), what we have to prove is that given $a\geq 0$, for sufficiently large $b\geq a$ the map $\mathrm{cok}(1-\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^a\to\mathrm{cok}(1-\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^b$ induced by $(1-(\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1})^{b-a}$ vanishes. But this is obvious, since $1-(\mathsf{m}^{\mathrm{un}}_{\mathscr{F}})^{-1}$ is nilpotent. □

Remark 9.2.2. — Here is an alternative description of Beĭlinson's construction. Let $p_a : L_a\to\Bbbk$ be the $\Bbbk$-linear map given by

$$p_a(x^i)=\begin{cases}0 & \text{if } i=0,1,\dots,a-2,\\ 1 & \text{if } i=a-1.\end{cases}$$

Of course, this is not $\mathbb{Z}$-equivariant, and it does not correspond to any map of local systems on $\mathbb{C}^\times$. However, it *does* correspond to a map of local systems $\exp^* \mathscr{L}_a \to \underline{\Bbbk}_{\mathbb{C}}$ on $\mathbb{C}$. This gives rise to a natural map $\exp_X^*(\mathscr{F} \otimes^L (f^\times)^* \mathscr{L}_a) \to \exp_X^* \mathscr{F}$, and then, by adjunction, to a map

$$\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a \to \exp_{X*} \exp_X^* \mathscr{F}.$$

Applying ${}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_*(-))$ to this map yields a map

$$(9.2.17) \qquad {}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_*(\mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) \to \Psi_f(\mathscr{F}).$$

The map p_a above is closely related to the formula for (9.2.15). Using this observation, it can be checked that (9.2.17) coincides with the map considered in Proposition 9.2.1(1).

The following proposition should be compared with Proposition 9.1.4.

Proposition 9.2.3. — *Let X, Y be separated $\mathbb{C}$-schemes of finite type, and let $f : X \to \mathbb{C}$ and $g : Y \to X$ be algebraic maps.*

1. *Assume that g is proper. Let $\mathscr{F} \in \mathsf{Perv}(Y^\times, \Bbbk)$, and assume that $(g_{|Y^\times})_* \mathscr{F}$ is perverse. Suppose either that $\Bbbk$ is artinian, or that $\Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F}) = \Psi_{f\circ g}(\mathscr{F})$. Then there is a natural isomorphism*

$$\Psi^{\mathrm{un}}_f((g_{|Y^\times})_* \mathscr{F}) \xrightarrow{\sim} (g_{|Y_0})_* \Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F}).$$

2. *Assume that g is smooth of relative dimension r, and let $\mathscr{F} \in \mathsf{Perv}(X^\times, \Bbbk)$. There is a natural isomorphism*

$$(g_{|Y_0})^* \Psi^{\mathrm{un}}_f(\mathscr{F})[r] \xrightarrow{\sim} \Psi^{\mathrm{un}}_{f\circ g}((g_{|Y^\times})^* \mathscr{F}[r]).$$

Proof. — (1) In the case where $\Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F}) = \Psi_{f\circ g}(\mathscr{F})$, the claim is immediate from Proposition 9.1.4. Assume now that $\Bbbk$ is artinian. Then, by Proposition 9.2.1 and the discussion in §9.2.1, for large enough a we have a truncation distinguished triangle

$$\Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F})[1] \to i_Y^* j_{Y*}(\mathscr{F} \overset{L}{\otimes} ((f\circ g)^\times)^* \mathscr{L}_a) \to \Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F}) \xrightarrow{[1]} .$$

Apply $(g_{|Y_0})_*$ to this triangle. By proper base change and the projection formula, we can rewrite the middle term to obtain a distinguished triangle

$$(9.2.18) \quad (g_{|Y_0})_* \Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F})[1] \to i_X^* j_{X*}((g_{|Y^\times})_* \mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a) \\ \to (g_{|Y_0})_* \Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F}) \xrightarrow{[1]} .$$

As in the proof of Proposition 9.2.1(3), the middle term has the property that

$${}^{\mathrm{p}}\mathscr{H}^n(i_X^* j_{X*}((g_{|Y^\times})_* \mathscr{F} \overset{L}{\otimes} (f^\times)^* \mathscr{L}_a)) = 0 \qquad \text{unless } n = -1, 0.$$

We claim that $(g_{|Y_0})_* \Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F})$ is perverse. If this were not the case, then by considering the minimum or maximum k such that ${}^{\mathrm{p}}\mathscr{H}^k((g_{|Y_0})_* \Psi^{\mathrm{un}}_{f\circ g}(\mathscr{F})) \neq 0$ and examining the long exact sequence in perverse cohomology associated with (9.2.18), one arrives at a contradiction.

Another application of Proposition 9.2.1 yields the distinguished triangle

$$(9.2.19)\quad \Psi_f^{\mathrm{un}}((g_{|Y^\times})_*\mathscr{F})[1] \to i_X^* j_{X*}((g_{|Y^\times})_*\mathscr{F} \overset{L}{\otimes} (f^\times)^*\mathscr{L}_a) \to \Psi_f^{\mathrm{un}}((g_{|Y^\times})_*\mathscr{F}) \xrightarrow{[1]} .$$

A standard argument with truncation shows that (9.2.18) and (9.2.19) are canonically isomorphic, which implies the desired claim.

(2) Recall that $g^*[r]$ is t-exact for the perverse t-structures. Using this and the base change theorem we obtain for any $a \geq 0$ an isomorphism

$$(g_{|Y_0})^{*\,\mathrm{p}}\mathscr{H}^{-1}(i_X^* j_{X*}(\mathscr{F} \overset{L}{\otimes} (f^\times)^*\mathscr{L}_a))[r] \cong {}^{\mathrm{p}}\mathscr{H}^{-1}(i_Y^* j_{Y*}((g_{|Y^\times})^*\mathscr{F}[r] \overset{L}{\otimes} ((f \circ g)^\times)^*\mathscr{L}_a)).$$

The result follows from this by taking a large enough and using Proposition 9.2.1. □

As an application of Proposition 9.2.1, we obtain the following generalization of Lemma 9.1.9. (Its proof is based on similar ideas to those in the proof of Lemma 9.1.9, and is therefore left to the reader; this statement is not used in this book.)

Lemma 9.2.4. — *Let X be a separated $\mathbb{C}$-scheme of finite type, let $f : X \to \mathbb{C}$ be an algebraic map, and let $\mathscr{F} \in \mathsf{Perv}(X^\times, \Bbbk)$. Assume that $(\mathsf{m}_{\mathscr{F}}^{\mathrm{un}} - \mathrm{id})^a = 0$ for some $a \geq 0$. Then there exists a canonical isomorphism*

$$\Psi_f^{\mathrm{un}}(\mathscr{F}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathscr{H}^{-1}(i^* j_{!*}(\mathscr{F} \overset{L}{\otimes} (f^\times)^*\mathscr{L}_a)),$$

and an exact sequence of perverse sheaves

$$i_*\Psi_f^{\mathrm{un}}(\mathscr{F}) \hookrightarrow j_!(\mathscr{F} \overset{L}{\otimes} (f^\times)^*\mathscr{L}_a) \twoheadrightarrow j_{!*}(\mathscr{F} \overset{L}{\otimes} (f^\times)^*\mathscr{L}_a).$$

9.3. Monodromic complexes

In this section (which is used in an essential way only for a technical lemma in Part II of the book), we review a rather different notion of "monodromy" for certain complexes of sheaves on a variety acted on by a torus. In a special case, the kind of monodromy discussed here will match the notion that arises in the context of nearby cycles (which we recalled in §9.1.3).

The discussion below is inspired by [**Ve**] as well as [**BeY**, Appendix A], but because we are working with sheaves in the classical topology, we can take advantage of some nonalgebraic constructions to significantly simplify the development.

Remark 9.3.1. — 1. In most of this book, we work only with constructible complexes of sheaves, and thus the term "equivariant derived category" has implicitly been treated as synonymous with "equivariant constructible derived category." In this section, however, because we will work with some nonalgebraic actions and nonalgebraic maps, more care is required. For this section only, equivariant derived categories should *not* be assumed to implicitly consist only of constructible complexes; instead, constructibility will be explicitly indicated in the notation when we wish to impose it.

2. Here we discuss monodromic complexes in the setting of analytic sheaves on complex algebraic varieties, but there are variants of this notion in other sheaf-theoretic contexts. We will more briefly discuss the case of étale sheaves in §9.5.3. In the context of $\mathscr{D}$-modules, the theory that plays this role is that of "weakly equivariant" $\mathscr{D}$-modules; see e.g. [**BeY**, Remark A.1.3] for a discussion.

9.3.1. Definitions. — Let T be a complex algebraic torus. Regarded as a Lie group or a topological group, it admits a universal cover $\exp : \tilde{T} \to T$. In concrete terms, we have $\tilde{T} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}$, and if we identify T with $X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ then $\exp : \tilde{T} \to T$ is induced by the map $\mathbb{C} \to \mathbb{C}^\times$ sending z to $\exp(2i\pi z)$. The group $\tilde{T}$ is contractible, and the kernel of $\exp : \tilde{T} \to T$ is canonically identified with $X_*(T)$ (i.e., with $\pi_1(T, 1)$).

Let X be a separated $\mathbb{C}$-scheme of finite type equipped with an algebraic action of T. Then we can let $\tilde{T}$ act on X via exp. This action is usually not algebraic, but it is at least continuous, so following [**BL**], it makes sense to consider the $\tilde{T}$-equivariant derived category $D^{\mathrm{b}}_{\tilde{T}}(X, \Bbbk)$. Since $\tilde{T}$ is contractible, we know from [**BL**, Theorem 3.7.3] that the forgetful functor $D^{\mathrm{b}}_{\tilde{T}}(X, \Bbbk) \to D^{\mathrm{b}}(X, \Bbbk)$ is fully faithful. We may also consider the constructible $\tilde{T}$-equivariant derived category $D^{\mathrm{b}}_{\tilde{T},c}(X, \Bbbk) := D^{\mathrm{b}}_{\tilde{T}}(X, \Bbbk) \cap D^{\mathrm{b}}_c(X, \Bbbk)$. We call this the *$T$-monodromic category* of X, and we introduce the notation

$$D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk) := D^{\mathrm{b}}_{\tilde{T},c}(X, \Bbbk).$$

This is a full triangulated subcategory of $D^{\mathrm{b}}_c(X, \Bbbk)$.

Next, consider the constructible T-equivariant derived category $D^{\mathrm{b}}_{T,c}(X, \Bbbk)$. The *unipotent T-monodromic category* of X, denoted by

$$D^{\mathrm{b}}_c(X /\!\!/ T, \Bbbk),$$

is defined to be the full triangulated subcategory of $D^{\mathrm{b}}_c(X, \Bbbk)$ generated by the image of the forgetful functor

$$D^{\mathrm{b}}_{T,c}(X, \Bbbk) \to D^{\mathrm{b}}_c(X, \Bbbk). \tag{9.3.1}$$

(For a justification of this terminology, see Corollary 9.3.5 below.) This functor factors through $D^{\mathrm{b}}_{\tilde{T},c}(X, \Bbbk)$, so we have

$$D^{\mathrm{b}}_c(X /\!\!/ T, \Bbbk) \subset D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk).$$

In the case when X is a principal T-bundle over some other variety $Y = X/T$, the category $D^{\mathrm{b}}_{T,c}(X, \Bbbk)$ can be identified with $D^{\mathrm{b}}_c(Y, \Bbbk)$, and the forgetful functor (9.3.1) with pullback along the quotient map $X \to Y$. This observation shows that our definition generalizes that in [**BeY, Ve**].

9.3.2. Monodromy. — The definition of monodromy in this setting will be based on the following construction.

Proposition 9.3.2. — *Let X be a separated $\mathbb{C}$-scheme of finite type equipped with an algebraic action of T. For any object $\mathscr{F}$ in $D^{\mathrm{b}}_{\tilde{T}}(X, \Bbbk)$, there is a canonical group homomorphism*

$$\mu_{\mathscr{F}} : X_*(T) \to \mathrm{Aut}(\mathscr{F}).$$

Moreover, for any morphism $\phi : \mathscr{F} \to \mathscr{G}$ *in* $D^{\mathrm{b}}_{\tilde{T}}(X, \Bbbk)$ *we have* $\phi \circ \mu_{\mathscr{F}} = \mu_{\mathscr{G}} \circ \phi$.

Of course, by restriction, the same result applies to objects and morphisms in $D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk)$ or $D^{\mathrm{b}}_{c}(X /\!\!/ T, \Bbbk)$.

Proof. — Let $a, p : \tilde{T} \times X \to X$ be the action and projection maps, respectively. As in any equivariant derived category (see e.g. [**Ac3**, Lemma 6.4.6]), for every object $\mathscr{F} \in D^{\mathrm{b}}_{\tilde{T}}(X, \Bbbk)$ we have a natural isomorphism $\theta_{\mathscr{F}} : p^*\mathscr{F} \to a^*\mathscr{F}$ in $D^{\mathrm{b}}_{\tilde{T}}(\tilde{T} \times X, \Bbbk)$, where $\tilde{T}$ acts on $\tilde{T} \times X$ via $g \cdot (h, x) = (ghg^{-1}, g \cdot x)$, i.e. (since $\tilde{T}$ is abelian) via the action on the second factor X. Here, "natural" means in particular that for any morphism $\phi : \mathscr{F} \to \mathscr{G}$ in $D^{\mathrm{b}}_{\tilde{T}}(X, \Bbbk)$ we have

$$a^*\phi \circ \theta_{\mathscr{F}} = \theta_{\mathscr{G}} \circ p^*\phi. \tag{9.3.2}$$

Now let $\gamma \in X_*(T)$, regarded as a point in $\tilde{T}$. Because γ acts trivially on X, the restriction to $\{\gamma\} \times X \subset \tilde{T} \times X$ of either $p^*\mathscr{F}$ or $a^*\mathscr{F}$ is canonically identified with $\mathscr{F}$. We set

$$\mu_{\mathscr{F}}(\gamma) := (\theta_{\mathscr{F}})_{|\{\gamma\} \times X} : \mathscr{F} \to \mathscr{F}.$$

It is easy to see that $\mu_{\mathscr{F}}$ is a group homomorphism $X_*(T) \to \mathrm{Aut}(\mathscr{F})$. The fact that it commutes with all morphisms follows from (9.3.2). □

In more pedantic terms, Proposition 9.3.2 says that monodromy defines a $\Bbbk$-algebra morphism from the group algebra $\Bbbk[X_*(T)]$ to the center of the category $D^{\mathrm{b}}_{\tilde{T}}(X, \Bbbk)$ (or $D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk)$, or $D^{\mathrm{b}}_{c}(X /\!\!/ T, \Bbbk)$, by restriction). This construction enjoys various "functoriality" properties. In particular, if X and Y are T-varieties and $f : X \to Y$ is a T-equivariant morphism, then the functors f_*, $f_!$, f^* and $f^!$ send monodromic complexes to monodromic complexes, and moreover, for $\mathscr{F}$ in $D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk)$ and $\mathscr{G}$ in $D^{\mathrm{b}}_{T\text{-mon}}(Y, \Bbbk)$ we have

$$\mu_{f_*\mathscr{F}} = f_*(\mu_{\mathscr{F}}), \quad \mu_{f_!\mathscr{F}} = f_!(\mu_{\mathscr{F}}), \quad \mu_{f^*\mathscr{G}} = f^*(\mu_{\mathscr{G}}), \quad \mu_{f^!\mathscr{G}} = f^!(\mu_{\mathscr{G}}). \tag{9.3.3}$$

Similarly, if X is a T-variety and $\mathscr{F}, \mathscr{F}'$ belong to $D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk)$, then $\mathscr{F} \otimes^L_{\Bbbk} \mathscr{F}'$ and $R\mathscr{H}om(\mathscr{F}, \mathscr{F}')$ are monodromic as well, and we have

$$\begin{gathered} \mu_{\mathscr{F} \otimes^L_{\Bbbk} \mathscr{F}'}(\lambda) = \mu_{\mathscr{F}}(\lambda) \overset{L}{\otimes}_{\Bbbk} \mu_{\mathscr{F}'}(\lambda), \\ \mu_{R\mathscr{H}om(\mathscr{F}, \mathscr{F}')}(\lambda) = R\mathscr{H}om(\mu_{\mathscr{F}}(-\lambda), \mu_{\mathscr{F}'}(\lambda)) \end{gathered} \tag{9.3.4}$$

for any $\lambda \in X_*(T)$.

Remark 9.3.3. — The following observations are immediate from the construction:

1. If $\mathscr{F}$ lies in the image of $D^{\mathrm{b}}_{T}(X, \Bbbk)$, then $\mu_{\mathscr{F}}(\gamma) = \mathrm{id}_{\mathscr{F}}$ for all $\gamma \in X_*(T)$ (because $\theta_{\mathscr{F}}$ is the pullback of a similar isomorphism on $T \times X$).
2. Suppose $X = T$ (with its natural T-action), and let $\mathscr{L}$ be a local system on T. Then $\mathscr{L}$ belongs to $D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk)$. Any such local system corresponds to a

representation of the fundamental group $\varrho : \pi_1(T,1) \to \mathrm{Aut}(\mathscr{L}_1)$, where $\mathscr{L}_1$ is the stalk of $\mathscr{L}$ at the base point $1 \in T$. Then we have a commutative triangle

$$\begin{tikzcd} & \mathrm{Aut}(\mathscr{L}) \arrow[d, "\wr"] \\ \pi_1(T,1) = X_*(T) \arrow[ur, "\mu_{\mathscr{L}}"] \arrow[r, "\varrho"'] & \mathrm{Aut}_{\pi_1(T,1)}(\mathscr{L}_1). \end{tikzcd}$$

The object $\mathscr{L}$ belongs to $D^{\mathrm{b}}_c(X/\!\!/T, \Bbbk)$ iff $X_*(T)$ acts on $\mathscr{L}_1$ by unipotent automorphisms. This observation will be generalized in Corollary 9.3.5 below.

9.3.3. Monodromic perverse sheaves. — Recall that the forgetful functor

$$D^{\mathrm{b}}_{T,c}(X, \Bbbk) \to D^{\mathrm{b}}_c(X, \Bbbk)$$

restricts to a fully faithful functor on perverse sheaves, see e.g. [**Ac3**, Proposition 6.2.15]. Given an object $\mathscr{F}$ in $\mathsf{Perv}_{T\text{-mon}}(X, \Bbbk)$, it can be important to determine whether $\mathscr{F}$ is equivariant or not. The answer to this problem is provided by the following lemma.

Lemma 9.3.4. — *Let X be a separated $\mathbb{C}$-scheme of finite type equipped with an algebraic action of T. A perverse sheaf $\mathscr{F} \in \mathsf{Perv}_{T\text{-mon}}(X, \Bbbk)$ lies in $\mathsf{Perv}_T(X, \Bbbk)$ if and only if $\mu_{\mathscr{F}}(\lambda) = \mathrm{id}$ for all $\lambda \in X_*(T)$.*

Before proceeding to the proof of Lemma 9.3.4, we need to recall a few facts about equivariant perverse sheaves. (For this material, see also [**Ac3**, Theorem 6.4.10].) Consider a separated $\mathbb{C}$-scheme of finite type equipped with an algebraic action of a complex algebraic group H. Following [**BL**, §5], we define the category $\mathsf{Perv}_H(Y, \Bbbk)$ of H-equivariant perverse sheaves to be the full subcategory of $D^{\mathrm{b}}_{H,c}(Y, \Bbbk)$ consisting of objects whose image under the forgetful functor $\mathsf{For}^H : D^{\mathrm{b}}_{H,c}(Y, \Bbbk) \to D^{\mathrm{b}}_c(Y, \Bbbk)$ is perverse.

An alternative, and more elementary, approach to defining equivariant perverse sheaves is to instead follow the pattern for ordinary equivariant sheaves. Let $a, p : H \times Y \to Y$ be the action and projection maps, respectively. Note that since a and p are smooth of relative dimension $\dim(H)$, the functors $a^*[\dim(H)]$ and $p^*[\dim(H)]$ take perverse sheaves to perverse sheaves (see [**Ac3**, Proposition 3.6.1]). Let $\mathsf{Perv}'_H(Y, \Bbbk)$ be the category consisting of pairs $(\mathscr{F}, \theta)$, where $\mathscr{F} \in \mathsf{Perv}(Y, \Bbbk)$ is a (nonequivariant) perverse sheaf, and $\theta : p^*\mathscr{F}[\dim(H)] \xrightarrow{\sim} a^*\mathscr{F}[\dim(H)]$ is an isomorphism in $\mathsf{Perv}(H \times Y, \Bbbk)$ satisfying an appropriate cocycle condition (see [**Ac3**, Definition 6.2.3] for details). Morphisms in $\mathsf{Perv}'_H(Y, \Bbbk)$ are simply morphisms in $\mathsf{Perv}(Y, \Bbbk)$ that "commute with θ" in the appropriate sense.

Since all objects of $D^{\mathrm{b}}_{H,c}(Y, \Bbbk)$ come equipped with such an isomorphism (see again [**Ac3**, Lemma 6.4.6]), there is an obvious functor

$$\mathsf{Perv}_H(Y, \Bbbk) \to \mathsf{Perv}'_H(Y, \Bbbk). \tag{9.3.5}$$

We claim that this functor is an equivalence of categories. To prove this, we must construct a functor $\mathsf{Perv}'_H(Y, \Bbbk) \to \mathsf{Perv}_H(Y, \Bbbk)$ that is inverse to (9.3.5). Fix an integer $N > \dim Y$, so that $\mathsf{Perv}(Y, \Bbbk) \subset D^{\mathrm{b}}_c(Y, \Bbbk)^{[-N,0]}$. Choose an N-acyclic H-resolution $r : P \to Y$ in the sense of [**Ac3**, Definition 6.1.17]. (For existence, see [**Ac3**, Proposition 6.1.23].) By definition r is a smooth morphism (say, of relative dimension d) and P is a principal H-bundle; we therefore have a quotient morphism $q : P \to \bar{P}$ which is smooth (see [**Ac3**, Proposition 6.1.11]). We also have a cartesian square

$$\begin{array}{ccc} H \times P & \xrightarrow{\tilde{a}} & P \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle q} \\ P & \xrightarrow{q} & \bar{P} \end{array} \tag{9.3.6}$$

where $\tilde{a}$ and $\tilde{p}$ are the action and projection maps, respectively. Now let $(\mathscr{F}, \theta)$ be an object of $\mathsf{Perv}'_H(Y, \Bbbk)$, and consider the perverse sheaf $\mathscr{G} = r^*\mathscr{F}[d]$. The map θ gives rise to an isomorphism $\tilde{\theta} : \tilde{p}^*\mathscr{G} \xrightarrow{\sim} \tilde{a}^*\mathscr{G}$ satisfying a suitable cocycle condition. The map $\tilde{\theta}$ can be thought of as a descent datum with respect to the smooth map $q : P \to \bar{P}$. Because perverse sheaves satisfy smooth descent (see [**BBDG**, Théorème 3.2.4] or [**Ac3**, Theorem 3.7.4]), the pair $(\mathscr{G}, \tilde{\theta})$ determines a unique perverse sheaf $\bar{\mathscr{F}} \in \mathsf{Perv}(\bar{P}, \Bbbk)$ together with an isomorphism $\beta : \mathscr{G} \xrightarrow{\sim} q^*\bar{\mathscr{F}}[\dim H]$. The triple $(\mathscr{F}, \bar{\mathscr{F}}[\dim H - d], \beta[-d])$ is an object of $D^{\mathrm{b}}_{H,c}(Y, \Bbbk)$. It clearly lies in $\mathsf{Perv}_H(Y, \Bbbk)$, so this construction gives us a functor $\mathsf{Perv}'_H(Y, \Bbbk) \to \mathsf{Perv}_H(Y, \Bbbk)$, which can be seen to be an inverse to (9.3.5).

There is another setting in which the same considerations apply, which we will require in the proof of Lemma 9.3.4, and which we explain now. Assume that Y is a complex analytic space, that H is a discrete group, and that we are given an action of H on Y which is free (in the sense that each point has a trivial stabilizer) and proper (in the sense that the morphism $H \times Y \to Y \times Y$ given by $(g, y) \mapsto (y, g \cdot y)$ is such that the preimage of any compact set is compact; see [**Lee**, Proposition 12.9] for alternative characterizations). Then by [**Lee**, Theorem 12.11] the quotient Y/H is Hausdorff, and the quotient morphism $Y \to Y/H$ is a covering map. The proof of this theorem shows that this action satisfies the conditions considered in [**Ca**, §4], so that Y/H has a canonical structure of a complex analytic space.

There is a nice theory of perverse sheaves on complex analytic spaces (see e.g. [**Sc**, §6.0.2]), and the constructions of the equivariant derived category (and its perverse t-structure) in [**BL**] also apply in this setting; in fact, our assumptions imply that Y is a free H-space in the sense of [**BL**, §0.3], so that by [**BL**, Proposition 2.2.5] we have $D^{\mathrm{b}}_{H,c}(Y, \Bbbk) \cong D^{\mathrm{b}}_c(Y/H, \Bbbk)$. We deduce an equivalence of categories $\mathsf{Perv}_H(Y, \Bbbk) \cong \mathsf{Perv}(Y/H, \Bbbk)$.

On the other hand we can consider the category $\mathsf{Perv}'_H(Y, \Bbbk)$ defined as above. (In this case, the datum of an isomorphism $a^*\mathscr{F} \cong p^*\mathscr{F}$ is equivalent to the datum of an isomorphism $\alpha_h^*\mathscr{F} \cong \mathscr{F}$ for any $h \in H$, where α_h is the action of h. We leave it to the reader to translate the cocycle condition into these terms.) In this setting each map in the diagram (9.3.6) is a covering map. Now perverse sheaves have a descent property for covering maps (this follows from the fact that they satisfy descent for the analytic topology; see [**BBDG**, Corollaire 2.1.23]), so that the same considerations as above allow us to also identify $\mathsf{Perv}'_H(Y,\Bbbk)$ with $\mathsf{Perv}(Y/H,\Bbbk)$ (and hence also to $\mathsf{Perv}_H(Y,\Bbbk)$).

Proof of Lemma 9.3.4. — The "only if" direction is just a restatement of Remark 9.3.3(1). We now consider the "if" direction. Let $a, p : \tilde{T} \times X \to X$ be the action and projection maps. Since $\mathscr{F}$ is $\tilde{T}$-equivariant, it is equipped with an isomorphism $\theta : p^*\mathscr{F} \to a^*\mathscr{F}$ that satisfies a certain cocycle condition.

We now consider the natural action of the discrete group $X_*(T)$ on the complex analytic space $\tilde{T} \times X$, with quotient $T \times X$. We can forget part of the equivariance, and regard $\mathscr{F}$ as an $X_*(T)$-equivariant perverse sheaf. The maps a and p are both $X_*(T)$-equivariant, so $a^*\mathscr{F}$ and $p^*\mathscr{F}$ are both $X_*(T)$-equivariant complexes as well. In fact, $a^*\mathscr{F}[\dim T]$ and $p^*\mathscr{F}[\dim T]$ are $X_*(T)$-equivariant perverse sheaves. For brevity, we will suppress the shift $[\dim T]$ throughout this proof, while still informally referring to $a^*\mathscr{F}$ and $p^*\mathscr{F}$ as "perverse sheaves."

We will show below that $\theta : p^*\mathscr{F} \to a^*\mathscr{F}$ is an $X_*(T)$-equivariant morphism. Let us first explain how this claim implies the lemma. Let $\bar{a}, \bar{p} : T \times X \to X$ be the action and projection maps, and let $e : \tilde{T} \times X \to T \times X$ be the map induced by the quotient map $\tilde{T} \to T$. As explained above, since $X_*(T)$ acts freely on $\tilde{T} \times X$, we have an equivalence of categories

$$e^* : \mathsf{Perv}(T \times X, \Bbbk) \xrightarrow{\sim} \mathsf{Perv}_{X_*(T)}(\tilde{T} \times X, \Bbbk).$$

Thus, if θ is $X_*(T)$-equivariant, then it must arise by applying e^* to some isomorphism $\bar{\theta} : \bar{p}^*\mathscr{F} \to \bar{a}^*\mathscr{F}$. The map $\bar{\theta}$ makes $\mathscr{F}$ into a T-equivariant perverse sheaf.

To show that θ is $X_*(T)$-equivariant, consider the diagram

$$\begin{tikzcd} X_*(T) \times \tilde{T} \times X \arrow[r, shift left, "\mathrm{id}\times a"] \arrow[r, shift right, "\mathrm{id}\times p"'] \arrow[d, shift right, "q"'] \arrow[d, shift left, "b"] & X_*(T) \times X \arrow[d, "r"'] \\ \tilde{T} \times X \arrow[r, shift left, "a"] \arrow[r, shift right, "p"'] & X \end{tikzcd} \tag{9.3.7}$$

where the maps b, q, and r are given by

$$b(\lambda, t, x) = (\lambda t, x), \qquad q(\lambda, t, x) = (t, x), \qquad r(\lambda, x) = x.$$

Note that $a \circ b = r \circ (\mathrm{id} \times a) = a \circ q$ (because $X_*(T)$ acts trivially on X) and $p \circ b = r \circ (\mathrm{id} \times p) = p \circ q$. These equalities give rise to canonical isomorphisms

$$\psi : b^*a^*\mathscr{F} \xrightarrow{\sim} q^*a^*\mathscr{F}, \qquad \phi : b^*p^*\mathscr{F} \xrightarrow{\sim} q^*p^*\mathscr{F}.$$

Indeed, these are the structure morphisms that give the $X_*(T)$-equivariant structure on $a^*\mathscr{F}$ and $p^*\mathscr{F}$, respectively.

The cocycle condition satisfied by $\theta : p^*\mathscr{F} \to a^*\mathscr{F}$ involves a diagram similar to (9.3.7), but with the upper-left (resp. upper-right) corner replaced by $\tilde{T} \times \tilde{T} \times X$ (resp. $\tilde{T} \times X$). When restricted to the subspace $X_*(T) \times \tilde{T} \times X$, the cocycle equation says that

$$(\mathrm{id} \times a)^*\theta_{|X_*(T)\times X} \circ q^*\theta = b^*\theta.$$

This equation involves some implicit identifications. If we spell those out explicitly, the cocycle condition says that the following diagram commutes:

$$\begin{tikzcd} b^*p^*\mathscr{F} \arrow[rrr, "\phi", "\sim"'] \arrow[d, "b^*\theta"'] & & & q^*p^*\mathscr{F} \arrow[d, "q^*\theta"] \\ b^*a^*\mathscr{F} \arrow[r, "\sim"] & (\mathrm{id}\times a)^*r^*\mathscr{F} \arrow[r, "\left((\mathrm{id}\times a)^*\theta_{|X_*(T)\times X}\right)^{-1}"] & (\mathrm{id}\times a)^*r^*\mathscr{F} \arrow[r, "\sim"] & q^*a^*\mathscr{F} \end{tikzcd} \tag{9.3.8}$$

From the definition of $\mu_{\mathscr{F}}$ in the proof of Proposition 9.3.2, we see that the assumption that $\mu_{\mathscr{F}}(\lambda) = \mathrm{id}$ for all $\lambda \in X_*(T)$ means that $\theta_{|X_*(T)\times X} = \mathrm{id}$. Therefore, the composition of the maps along the bottom of (9.3.8) is ψ, and so we have

$$\psi \circ b^*\theta = q^*\theta \circ \phi.$$

This equation says exactly that θ is $X_*(T)$-equivariant. □

As a consequence of Lemma 9.3.4, we obtain the following characterization of objects in $D^{\mathrm{b}}_c(X/\!\!/T, \Bbbk)$ among those in $D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk)$.

Corollary 9.3.5. — *Let X be a separated $\mathbb{C}$-scheme of finite type equipped with an algebraic action of T. An object $\mathscr{F} \in D^{\mathrm{b}}_{T\text{-mon}}(X, \Bbbk)$ lies in $D^{\mathrm{b}}_c(X/\!\!/T, \Bbbk)$ if and only if $\mu_{\mathscr{F}}(\lambda)$ is a unipotent operator for all $\lambda \in X_*(T)$.*

Proof. — The "only if" follows easily from Remark 9.3.3(1). For the "if" direction, it is enough to prove the claim when $\mathscr{F}$ is a perverse sheaf. Suppose therefore that $\mathscr{F}$ is perverse, and that all $\mu_{\mathscr{F}}(\lambda)$ are unipotent. Choose a basis $\gamma_1, \dots, \gamma_n$ for $X_*(T)$. Using the fact that the operators $\mu_{\mathscr{F}}(\gamma_1), \dots, \mu_{\mathscr{F}}(\gamma_n)$ commute, one can show that $\mathscr{F}$ admits a finite filtration

$$0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_m = \mathscr{F}$$

such that $\mu_{\mathscr{F}_i/\mathscr{F}_{i-1}}(\gamma_k)$ is the identity map for all i and k. It follows from this that $\mu_{\mathscr{F}_i/\mathscr{F}_{i-1}}(\lambda) = \mathrm{id}$ for all $\lambda \in X_*(T)$. By Lemma 9.3.4, each $\mathscr{F}_i/\mathscr{F}_{i-1}$ is T-equivariant, so $\mathscr{F}$ lies in $\mathsf{Perv}(X/\!\!/T, \Bbbk)$, as desired. □

9.3.4. Monodromic complexes and nearby cycles. — We can now finally explain the promised relationship between the monodromy for nearby cycles and that for monodromic complexes.

Proposition 9.3.6. — *Let X be a separated $\mathbb{C}$-scheme of finite type, and let $f : X \to \mathbb{C}$ be an algebraic map. Suppose X admits an (algebraic) action of $\mathbb{C}^\times$ such that f is $\mathbb{C}^\times$-equivariant (with respect to the natural action on $\mathbb{C}$). Then the nearby cycles functor $\Psi_f : D^{\mathrm{b}}_c(X^\times, \Bbbk) \to D^{\mathrm{b}}_c(X_0, \Bbbk)$ can be upgraded to a functor*

$$\Psi_f : D^{\mathrm{b}}_{\mathbb{C}^\times, c}(X^\times, \Bbbk) \to D^{\mathrm{b}}_{\mathbb{C}^\times\text{-mon}}(X_0, \Bbbk),$$

such that for any $\mathscr{F} \in D^{\mathrm{b}}_{\mathbb{C}^\times}(X^\times, \Bbbk)$ we have $\mathsf{m}_{\mathscr{F}} = \mu_{\Psi_f(\mathscr{F})}(-1)$. Similarly, $\Psi^{\mathrm{un}}_f : \mathsf{Perv}(X^\times, \Bbbk) \to \mathsf{Perv}(X_0, \Bbbk)$ can be upgraded to a functor

$$\Psi^{\mathrm{un}}_f : \mathsf{Perv}_{\mathbb{C}^\times}(X^\times, \Bbbk) \to \mathsf{Perv}(X_0/\!\!/\mathbb{C}^\times, \Bbbk).$$

Proof. — The maps i, j, and $\exp_X$ in (9.1.1) are all equivariant for the group $\mathbb{C} = \widetilde{\mathbb{C}^\times}$, so Ψ_f sends any $\mathbb{C}^\times$-equivariant object at least to $D^{\mathrm{b}}_{\mathbb{C}}(X_0, \Bbbk)$. By Theorem 9.1.3(1), it actually takes values in $D^{\mathrm{b}}_{\mathbb{C}^\times\text{-mon}}(X_0, \Bbbk)$.

To compute $\mathsf{m}_{\mathscr{F}}$, we recall an alternative formula for Ψ_f. Using the fact that $\exp_X^* \cong \exp_X^!$, we have

$$\begin{aligned}\Psi_f(\mathscr{F}) = (i^* j_* \exp_{X*} \exp_X^! \mathscr{F})[1] &\cong i^* j_* \exp_{X*} R\mathscr{H}om(\underline{\Bbbk}_{\widetilde{X}^\times}, \exp_X^! \mathscr{F})[1] \\ &\cong i^* j_* R\mathscr{H}om(\exp_{X!} \underline{\Bbbk}_{\widetilde{X}^\times}, \mathscr{F})[1] \cong i^* j_* R\mathscr{H}om((f^\times)^* \exp_! \underline{\Bbbk}_{\mathbb{C}}, \mathscr{F})[1].\end{aligned}$$

By definition, the monodromy map $\mathsf{m}_{\mathscr{F}}$ is the map induced by $\mu_{\exp_! \underline{\Bbbk}_{\mathbb{C}}}(1)$. Since $\mu_{\mathscr{F}}$ is trivial (see Remark 9.3.3(1)), we see from (9.3.3) and (9.3.4) that this coincides with $\mu_{\Psi_f(\mathscr{F})}(-1)$.

Finally, if $\mathscr{F}$ is perverse, by construction the automorphism $\mathsf{m}_{\mathscr{F}}$ restricts to a unipotent automorphism of $\Psi_f^{\mathrm{un}}(\mathscr{F})$. Using Corollary 9.3.5 and the equality $\mathsf{m}_{\mathscr{F}} = \mu_{\Psi_f(\mathscr{F})}(-1)$, this implies that $\Psi_f^{\mathrm{un}}(\mathscr{F})$ belongs to $\mathsf{Perv}(X_0 /\!\!/ \mathbb{C}^\times, \Bbbk)$. □

Remark 9.3.7. — In the setting of Proposition 9.3.6, suppose $\mathscr{F} \in \mathsf{Perv}_{\mathbb{C}^\times}(X^\times, \Bbbk)$ has the property that $\Psi_f(\mathscr{F})$ admits a filtration whose subquotients are $\mathbb{C}^\times$-equivariant perverse sheaves on X_0. Then $\Psi_f(\mathscr{F})$ lies in $D^{\mathrm{b}}_c(X_0 /\!\!/ \mathbb{C}^\times, \Bbbk)$, and then Corollary 9.3.5 implies that $\mathsf{m}_{\mathscr{F}}$ is unipotent, i.e. that $\Psi_f^{\mathrm{un}}(\mathscr{F}) = \Psi_f(\mathscr{F})$.

If the preceding condition holds for all $\mathscr{F} \in \mathsf{Perv}_{\mathbb{C}^\times}(X^\times, \Bbbk)$, then the monodromy operators $\mathsf{m}_{\mathscr{F}}$ are unipotent for all $\mathscr{F} \in D^{\mathrm{b}}_{\mathbb{C}^\times, c}(X^\times, \Bbbk)$. In this case, we may write

$$\Psi_f^{\mathrm{un}} = \Psi_f : D^{\mathrm{b}}_{\mathbb{C}^\times, c}(X^\times, \Bbbk) \to D^{\mathrm{b}}_c(X_0 /\!\!/ \mathbb{C}^\times, \Bbbk).$$

9.4. Nearby cycles over a two-dimensional base

In this section we present a theory of "unipotent nearby cycles over $\mathbb{C}^2$," following Gaitsgory [**G2**]. This theory should not be considered as fully satisfactory and applicable in all situations, but rather as an ad-hoc construction which is sufficient for our purposes in the setting of the construction of central sheaves.

Remark 9.4.1. — After a first version of this book was made public, and at the request of R. Bezrukavnikov and M. Finkelberg who needed a version of these constructions for higher dimensional bases, we studied an extension of the theory explained in the present section in [**AR3**]. This paper contains variants of most of the results considered below, sometimes with slightly different proofs, which we believe might clarify their meaning. We also took this opportunity to explain in detail how to translate the construction of the line bundles $\mathscr{L}_a$ from §9.2.2 in the setting of étale sheaves, cf. §5.3.1 and §9.5, see [**AR3**, §2.6].

On this occasion, A. Salmon also wrote a companion paper that makes a precise connection between these constructions and the general theory of nearby cycles over general bases of [**I2**], see [**Sal**].

9.4.1. Definition. — We will take Proposition 9.2.1 as the motivation for our construction. Given a separated scheme of finite type X over $\mathbb{C}^2$, with structure morphism $f : X \to \mathbb{C}^2$, we let $X^{\times\times}$ and X^{00} denote the preimages of $\mathbb{C}^\times \times \mathbb{C}^\times$ and of $(0,0)$, respectively, and let $\mathbf{j} : X^{\times\times} \hookrightarrow X$ and $\mathbf{i} : X^{00} \hookrightarrow X$ be the inclusion maps. Let $f^{\times\times} := f \circ \mathbf{j} : X^{\times\times} \to \mathbb{C}^\times \times \mathbb{C}^\times$. For $\mathscr{F}$ a perverse sheaf on $X^{\times\times}$ we will say that $\Upsilon_f(\mathscr{F})$ *is well defined* if

1. the object
$$^{\mathrm{p}}\mathscr{H}^{-2}\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))$$
does not depend on a, b (in the sense that the morphisms
$$^{\mathrm{p}}\mathscr{H}^{-2}\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)) \to {}^{\mathrm{p}}\mathscr{H}^{-2}\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_{a'} \boxtimes \mathscr{L}_{b'}))$$
induced by (9.2.6) are isomorphisms for $a \leq a'$ and $b \leq b'$) for $a, b \gg 0$;
2. for $n \neq -2$, for any $a, b \geq 0$ there exist $a' \geq a$ and $b' \geq b$ such that the morphism
$$^{\mathrm{p}}\mathscr{H}^{n}\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)) \to {}^{\mathrm{p}}\mathscr{H}^{n}\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_{a'} \boxtimes \mathscr{L}_{b'}))$$
induced by (9.2.6) vanishes.

If these conditions are satisfied the object $^{\mathrm{p}}\mathscr{H}^{-2}\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \otimes f^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))$ for $a, b \gg 0$ will be denoted $\Upsilon_f(\mathscr{F})$.

The following lemma shows that in condition (2) above, only the cases $n = -1$ and $n = 0$ need to be considered.

Lemma 9.4.2. — *For all $\mathscr{F}$ in $\mathsf{Perv}(X^{\times\times}, \Bbbk)$ and all $a, b \geq 0$ we have*
$$^{\mathrm{p}}\mathscr{H}^{n}(\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \overset{L}{\otimes} f^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))) = 0$$
unless n is one of -2, -1, or 0.

Proof. — We begin with the following general observation: if $h : U \hookrightarrow Y$ is an affine open embedding, and if $k : Z \hookrightarrow Y$ is the complementary closed embedding, then if $\mathscr{G}$ is any perverse sheaf on Y, we have $^{\mathrm{p}}\mathscr{H}^n(k^*\mathscr{G}) = 0$ unless $n \in \{0, -1\}$. This claim follows from the long exact sequence in perverse cohomology associated with the distinguished triangle
$$h_!(\mathscr{G}_{|U}) \to \mathscr{G} \to k_*k^*\mathscr{G} \xrightarrow{[1]},$$
coupled with the fact that $h_!$ is t-exact (see [**Ac3**, Corollary 3.5.9]). More generally, by the same considerations, if $\mathscr{G}$ is a complex that satisfies $^{\mathrm{p}}\mathscr{H}^n(\mathscr{G}) = 0$ unless $a \leq n \leq b$, then $^{\mathrm{p}}\mathscr{H}^n(k^*\mathscr{G}) = 0$ unless $a - 1 \leq n \leq b$.

For brevity, let $\mathscr{F}' = \mathscr{F} \otimes^L f^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)$, and note that $\mathscr{F}'$ is perverse. The map $\mathbf{j}$ is an affine open embedding, so $\mathbf{j}_*\mathscr{F}'$ is perverse. Consider the maps
$$i_2^0 : X^{00} \to X^{0*} \quad \text{and} \quad i_1' : X^{0*} \to X$$
obtained by base change from the obvious embeddings $\{(0,0)\} \to \{0\} \times \mathbb{C}$ and $\{0\} \times \mathbb{C} \to \mathbb{C}^2$, so that $\mathbf{i} = i_1' \circ i_2^0$. Both i_1' and i_2^0 are closed embeddings complementary to affine open embeddings, so by the previous paragraph, $^{\mathrm{p}}\mathscr{H}^n((i_1')^*\mathbf{j}_*\mathscr{F}') = 0$ unless $-1 \leq n \leq 0$, and then $^{\mathrm{p}}\mathscr{H}^n((i_2^0)^*(i_1')^*\mathbf{j}_*\mathscr{F}') = 0$ unless $-2 \leq n \leq 0$, which implies our claim. $\square$

9.4.2. Compatibility with proper pushforward and smooth pullback. — The following proposition shows that the construction of the functor Υ_f considered above behaves reasonably with respect to proper pushforward and smooth pullback, as for ordinary nearby cycles (see Proposition 9.1.4).

Proposition 9.4.3. — *Let X, Y be separated $\mathbb{C}$-schemes of finite type, and let $f : X \to \mathbb{C}^2$ and $g : Y \to X$ be algebraic maps.*

1. *Assume that g is proper, and let $\mathscr{F} \in \mathsf{Perv}(Y^{\times\times}, \Bbbk)$. Suppose that $(g_{|Y^{\times\times}})_*\mathscr{F}$ is perverse, that $\Upsilon_{f\circ g}(\mathscr{F})$ and $\Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F})$ are well defined, and that the complex $(g_{|Y^{00}})_*\Upsilon_{f\circ g}(\mathscr{F})$ is also perverse. Then there is a natural map*
$$(g_{|Y^{00}})_*\Upsilon_{f\circ g}(\mathscr{F}) \to \Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F}).$$
2. *Assume that g is smooth of relative dimension r. Let $\mathscr{F} \in \mathsf{Perv}(X^{\times\times}, \Bbbk)$, and suppose that $\Upsilon_f(\mathscr{F})$ is well defined. Then $\Upsilon_{f\circ g}((g_{|Y^{\times\times}})^*\mathscr{F}[r])$ is well defined, and there is a natural isomorphism*
$$(g_{|Y^{00}})^*\Upsilon_f(\mathscr{F})[r] \xrightarrow{\sim} \Upsilon_{f\circ g}((g_{|Y^{\times\times}})^*\mathscr{F}[r]).$$

As with Proposition 9.1.4, the proof of Proposition 9.4.3 is an easy application of proper or smooth base change. We omit further details. Of course, in favorable situations, one expects the map in part (1) to be an isomorphism. See Corollary 9.4.9 below for an example of such a situation.

9.4.3. Comparison with iterated nearby cycles. — We continue with our separated $\mathbb{C}$-scheme X of finite type, and our algebraic map $f : X \to \mathbb{C}^2$. Consider the diagram
$$\begin{tikzcd}
\mathbb{C}^\times \times \mathbb{C}^\times \arrow[r] \arrow[d] & \mathbb{C} \times \mathbb{C}^\times \arrow[d] \arrow[ddr] & & \{0\} \times \mathbb{C}^\times \arrow[ll] \arrow[d] \\
\mathbb{C}^\times \times \mathbb{C} \arrow[r] \arrow[drr] & \mathbb{C}^2 \smallsetminus \{(0,0)\} \arrow[dr] & & \{0\} \times \mathbb{C} \arrow[dl] \\
\mathbb{C}^\times \times \{0\} \arrow[r] \arrow[u] & \mathbb{C} \times \{0\} \arrow[r] & \mathbb{C}^2 & \{(0,0)\} \arrow[l] \arrow[u]
\end{tikzcd}$$
where all maps are the obvious embeddings. We define a number of subsets of X and inclusion maps between them by pulling back the diagram above along f, as shown in the diagram below:

(9.4.1)
$$\begin{tikzcd}
X^{\times\times} \arrow[r, "j_1"] \arrow[d, "j_2"'] & X^{*\times} \arrow[d, "j_2'"'] \arrow[ddr, "j_2^\times"] & & X^{0\times} \arrow[ll, "i_1"'] \arrow[d, "j_2^0"] \\
X^{\times *} \arrow[r, "j_1'"] \arrow[drr, "j_1^\times"] & X^\circ \arrow[dr, "h"] & & X^{0*} \arrow[dl, "i_1'"'] \\
X^{\times 0} \arrow[r, "j_1^0"] \arrow[u, "i_2"] & X^{*0} \arrow[r, "i_2'"'] & X & X^{00}. \arrow[l, "\mathbf{i}"'] \arrow[u, "i_2^0"']
\end{tikzcd}$$

The top row and the right-hand column of this diagram are both settings where we can apply ordinary nearby cycles. Explicitly, let $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{C}^2 \to \mathbb{C}$ be the two projection maps, and let $f_1 = \mathrm{pr}_1 \circ f_{|X^{*\times}} : X^{*\times} \to \mathbb{C}$ and $f_2 = \mathrm{pr}_2 \circ f_{|X^{0*}} : X^{0*} \to \mathbb{C}$. Also let $f_1^\times = f_1 \circ j_1$ and $f_2^\times = f_2 \circ j_2^0$. If $\mathscr{F}$ is a perverse sheaf on $X^{\times\times}$, then we can consider the "iterated (unipotent) nearby cycles"

$$\Psi_f^{(2)}(\mathscr{F}) := \Psi_{f_2}^{\mathrm{un}} \circ \Psi_{f_1}^{\mathrm{un}}(\mathscr{F}).$$

In view of (9.2.7), for $a, b \gg 0$ we have

$$\Psi_f^{(2)}(\mathscr{F}) = {}^{\mathrm{p}}\mathscr{H}^{-1}\big((i_2^0)^* j_{2*}^0(({}^{\mathrm{p}}\mathscr{H}^{-1}(i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f_1^\times)^* \mathscr{L}_a))) \overset{L}{\otimes} (f_2^\times)^* \mathscr{L}_b)\big).$$

The functors j_{2*}^0 and $(-) \otimes^L (f_2^\times)^* \mathscr{L}_b$ are both t-exact for the perverse t-structure, so we can rewrite this formula as

$$\Psi_f^{(2)}(\mathscr{F}) \cong {}^{\mathrm{p}}\mathscr{H}^{-1}((i_2^0)^* {}^{\mathrm{p}}\mathscr{H}^{-1}(j_{2*}^0(i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f_1^\times)^* \mathscr{L}_a) \overset{L}{\otimes} (f_2^\times)^* \mathscr{L}_b))).$$

Lemma 9.4.4. — *Let $\mathscr{F}$ be a perverse sheaf of $X^{\times\times}$, and assume either that $\Bbbk$ is artinian, or that $\Psi_{f_1}^{\mathrm{un}}(\mathscr{F}) = \Psi_{f_1}(\mathscr{F})$ and $\Psi_{f_2} \circ \Psi_{f_1}(\mathscr{F}) = \Psi_{f_2}^{\mathrm{un}} \circ \Psi_{f_1}(\mathscr{F})$.*

1. *For $a, b \gg 0$ there is a natural isomorphism*

$$\Psi_f^{(2)}(\mathscr{F}) \cong {}^{\mathrm{p}}\mathscr{H}^{-2}((i_2^0)^* j_{2*}^0 i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))).$$

On the other hand, for $n \in \mathbb{Z} \smallsetminus \{-2, -1, 0\}$, for any $a, b \geq 0$ we have

$${}^{\mathrm{p}}\mathscr{H}^n((i_2^0)^* j_{2*}^0 i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))) = 0,$$

and if $n \in \{-1, 0\}$, given $a, b \geq 0$ there exist $a' \geq a$ and $b' \geq b$ such that the natural morphism

$$\begin{aligned} &{}^{\mathrm{p}}\mathscr{H}^n((i_2^0)^* j_{2*}^0 i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))) \\ &\qquad\qquad \to {}^{\mathrm{p}}\mathscr{H}^n((i_2^0)^* j_{2*}^0 i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_{a'} \boxtimes \mathscr{L}_{b'}))) \end{aligned}$$

vanishes.

2. *If $\Upsilon_f(\mathscr{F})$ is well defined, there is a natural transformation*

$$\Upsilon_f(\mathscr{F}) \to \Psi_f^{(2)}(\mathscr{F}).$$

Proof. — Let $\mathscr{L}_b^{*\times} = (f_{|X^{*\times}})^*(\underline{\Bbbk} \boxtimes \mathscr{L}_b)$. We have $(f_2^\times)^* \mathscr{L}_b \cong i_1^* \mathscr{L}_b^{*\times}$, so

$$(i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f_1^\times)^* \mathscr{L}_a)) \overset{L}{\otimes} (f_2^\times)^* \mathscr{L}_b \cong i_1^*(j_{1*}(\mathscr{F} \overset{L}{\otimes} (f_1^\times)^* \mathscr{L}_a) \overset{L}{\otimes} \mathscr{L}_b^{*\times}).$$

Next, observe that (by adjunction) there is a natural map

$$j_{1*}(\mathscr{F} \overset{L}{\otimes} (f_1^\times)^* \mathscr{L}_a) \overset{L}{\otimes} \mathscr{L}_b^{*\times} \to j_{1*}(\mathscr{F} \overset{L}{\otimes} (f_1^\times)^* \mathscr{L}_a \overset{L}{\otimes} (\mathscr{L}_b^{*\times})_{|X^{\times\times}}).$$

In fact, since $\mathscr{L}_b$ is an extension of copies of the constant local system, this map is an isomorphism. Combining the remarks above with the observation that $(f_1^\times)^* \mathscr{L}_a \otimes^L (\mathscr{L}_b^{*\times})_{|X^{\times\times}} \cong (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)$, we conclude that for $r, s \in \mathbb{Z}$ we have

$$\begin{aligned} &{}^{\mathrm{p}}\mathscr{H}^r((i_2^0)^* {}^{\mathrm{p}}\mathscr{H}^s(j_{2*}^0(i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f_1^\times)^* \mathscr{L}_a) \overset{L}{\otimes} (f_2^\times)^* \mathscr{L}_b))) \\ &\qquad \cong {}^{\mathrm{p}}\mathscr{H}^r((i_2^0)^* {}^{\mathrm{p}}\mathscr{H}^s(j_{2*}^0 i_1^* j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)))). \end{aligned} \tag{9.4.2}$$

The arguments in the proof of Lemma 9.4.2 show that the complex $j^0_{2*}i^*_1 j_{1*}(\mathscr{F} \otimes^L (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))$ is concentrated in perverse degrees -1 and 0. We therefore have a truncation triangle

$$\begin{aligned}{}^{\mathrm{p}}\mathscr{H}^{-1}(j^0_{2*}i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)))[1] \to j^0_{2*}i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)) \\ \to {}^{\mathrm{p}}\mathscr{H}^0(j^0_{2*}i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))) \xrightarrow{[1]} .\end{aligned}$$

Applying the functor $(i^0_2)^*$ (which, again as in the proof of Lemma 9.4.2, sends perverse sheaves to objects concentrated in perverse degrees -1 and 0) and taking the long exact sequence in perverse cohomology, we deduce that

$${}^{\mathrm{p}}\mathscr{H}^n((i^0_2)^* j^0_{2*}i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))) = 0$$

unless $n \in \{-2, -1, 0\}$, and that we have isomorphisms

$$\begin{aligned}{}^{\mathrm{p}}\mathscr{H}^{-2}((i^0_2)^* j^0_{2*}i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))) \cong \\ {}^{\mathrm{p}}\mathscr{H}^{-1}((i^0_2)^* {}^{\mathrm{p}}\mathscr{H}^{-1}(j^0_{2*}(i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times}_1)^*\mathscr{L}_a) \overset{L}{\otimes} (f^{\times}_2)^*\mathscr{L}_b))),\end{aligned}$$

$$\begin{aligned}{}^{\mathrm{p}}\mathscr{H}^{0}((i^0_2)^* j^0_{2*}i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))) \cong \\ {}^{\mathrm{p}}\mathscr{H}^{0}((i^0_2)^* {}^{\mathrm{p}}\mathscr{H}^{0}(j^0_{2*}(i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times}_1)^*\mathscr{L}_a) \overset{L}{\otimes} (f^{\times}_2)^*\mathscr{L}_b))),\end{aligned}$$

and a short exact sequence of perverse sheaves

$$\begin{aligned}{}^{\mathrm{p}}\mathscr{H}^{0}((i^0_2)^* {}^{\mathrm{p}}\mathscr{H}^{-1}(j^0_{2*}(i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times}_1)^*\mathscr{L}_a) \overset{L}{\otimes} (f^{\times}_2)^*\mathscr{L}_b))) \\ \hookrightarrow {}^{\mathrm{p}}\mathscr{H}^{-1}((i^0_2)^* j^0_{2*}i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b))) \\ \twoheadrightarrow {}^{\mathrm{p}}\mathscr{H}^{-1}((i^0_2)^* {}^{\mathrm{p}}\mathscr{H}^{0}(j^0_{2*}(i^*_1 j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times}_1)^*\mathscr{L}_a) \overset{L}{\otimes} (f^{\times}_2)^*\mathscr{L}_b))).\end{aligned}$$

Using these, (1) is a consequence of Proposition 9.2.1.

For (2), we observe that the adjunction map $\mathrm{id} \to i_{1*}i^*_1$ gives rise to a natural transformation

$$\mathbf{i}^*\mathbf{j}_* \cong \mathbf{i}^* j^{\times}_{2*} j_{1*} \to \mathbf{i}^* j^{\times}_{2*} i_{1*} i^*_1 j_{1*} \cong \mathbf{i}^* i'_{1*} j^0_{2*} i^*_1 j_{1*} \cong (i^0_2)^* j^0_{2*} i^*_1 j_{1*}.$$

Applying this natural transformation to $\mathscr{F} \otimes^L (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)$ for $a, b \gg 0$ and using (1), we obtain the desired natural map $\Upsilon_f(\mathscr{F}) \to \Psi^{(2)}_f(\mathscr{F})$. □

Remark 9.4.5. — Let us note the following fact for later use. Consider the natural distinguished triangle

$$j_{1!}j^*_1 \to \mathrm{id} \to i_{1*}i^*_1 \xrightarrow{[1]} .$$

A minor variation on the calculation at the end of the preceding proof shows that for any perverse sheaf $\mathscr{F}$ on $X^{\times\times}$, there is a natural distinguished triangle

$$\mathbf{i}^* j^{\times}_{2*} j_{1!}\mathscr{F} \to \mathbf{i}^*\mathbf{j}_*\mathscr{F} \to (i^0_2)^* j^0_{2*} i^*_1 j_{1*}\mathscr{F} \xrightarrow{[1]} .$$

9.4.4. Iterated-clean perverse sheaves. — We will now give a condition that guarantees both that $\Upsilon_f(\mathscr{F})$ is well defined and that the map $\Upsilon_f(\mathscr{F}) \to \Psi_f^{(2)}(\mathscr{F})$ of Lemma 9.4.4(2) is an isomorphism.

By adjunction and base change, there exists a canonical morphism of functors

$$j'_{1!}j_{2*} \to j'_{2*}j_{1!}. \tag{9.4.3}$$

We claim that this morphism is an isomorphism. In fact, by base change it is an isomorphism over $X^{\times *}$ and over $X^{*\times}$. Since those two open sets cover X°, this proves the claim. Applying the natural transformation $h_! \to h_*$, we get a natural map

$$j^{\times}_{1!}j_{2*}\mathscr{F} \to j^{\times}_{2*}j_{1!}\mathscr{F} \tag{9.4.4}$$

for any complex $\mathscr{F}$ on $X^{\times\times}$. The cone of this morphism identifies with $\mathbf{i}_*\mathbf{i}^*j^{\times}_{2*}j_{1!}\mathscr{F}$.

Definition 9.4.6. — A complex $\mathscr{F}$ in $D^{\mathrm{b}}_c(X^{\times\times}, \Bbbk)$ is said to be *iterated-clean* if the map (9.4.4) is an isomorphism, or in other words if

$$\mathbf{i}^*j^{\times}_{2*}j_{1!}\mathscr{F} = 0.$$

For examples of iterated-clean objects, see Lemma 9.4.15 below.

The following lemma is an elementary exercise in using the smooth and proper base change theorems. Its proof will be omitted.

Lemma 9.4.7. — *Let X and Y be separated $\mathbb{C}$-schemes of finite type, and let $f : X \to \mathbb{C}^2$ and $g : Y \to X$ be algebraic maps.*

1. *Assume that g is proper. If $\mathscr{F} \in D^{\mathrm{b}}_c(Y^{\times\times}, \Bbbk)$ is iterated-clean (with respect to $f \circ g$), then so is $(g_{|Y^{\times\times}})_*\mathscr{F}$ (with respect to f).*
2. *Assume that g is smooth. If $\mathscr{F} \in D^{\mathrm{b}}_c(X^{\times\times}, \Bbbk)$ is iterated-clean (with respect to f), then so is $(g_{|Y^{\times\times}})^*\mathscr{F}$ (with respect to $f \circ g$).*
3. *Assume that g is smooth and surjective. For $\mathscr{F} \in D^{\mathrm{b}}_c(X^{\times\times}, \Bbbk)$, we have that $\mathscr{F}$ is iterated-clean if and only if $(g_{|Y^{\times\times}})^*\mathscr{F}$ is iterated-clean.*

The following proposition states that iterated-cleanness is the condition we were looking for.

Proposition 9.4.8. — *Let $\mathscr{F}$ be a perverse sheaf on $X^{\times\times}$, and assume either that $\Bbbk$ is artinian, or that $\Psi^{\mathrm{un}}_{f_1}(\mathscr{F}) = \Psi_{f_1}(\mathscr{F})$ and $\Psi_{f_2} \circ \Psi_{f_1}(\mathscr{F}) = \Psi^{\mathrm{un}}_{f_2} \circ \Psi_{f_1}(\mathscr{F})$. If $\mathscr{F}$ is iterated-clean, then $\Upsilon_f(\mathscr{F})$ is well defined, and the natural map $\Upsilon_f(\mathscr{F}) \to \Psi_f^{(2)}(\mathscr{F})$ from Lemma 9.4.4 is an isomorphism.*

Proof. — Observe first that $\mathscr{F} \otimes^L (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)$ is iterated-clean for all a and b. This claim holds by assumption when $a = b = 1$; the general case follows by induction on a and b. Applying the distinguished triangle from Remark 9.4.5 to $\mathscr{F} \otimes^L (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)$, we obtain a natural isomorphism

$$\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)) \xrightarrow{\sim} (i^0_2)^*j^0_{2*}i^*_1j_{1*}(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)).$$

The result then follows from Lemma 9.4.4. □

Corollary 9.4.9. — *Let X and Y be separated $\mathbb{C}$-schemes of finite type, and let $f : X \to \mathbb{C}^2$ and $g : Y \to X$ be algebraic maps, with g proper. Let $\mathscr{F} \in \mathsf{Perv}(Y^{\times\times}, \Bbbk)$. Assume that $\mathscr{F}$ is iterated-clean, and that $(g_{|Y^{\times\times}})_*\mathscr{F}$ is perverse. Assume also either that $\Bbbk$ is artinian, or that $\Psi^{\mathrm{un}}_{(f\circ g)_1}(\mathscr{F}) = \Psi_{(f\circ g)_1}(\mathscr{F})$ and $\Psi_{(f\circ g)_2} \circ \Psi_{(f\circ g)_1}(\mathscr{F}) = \Psi^{\mathrm{un}}_{(f\circ g)_2} \circ \Psi_{(f\circ g)_1}(\mathscr{F})$. Then $\Upsilon_{f\circ g}(\mathscr{F})$ and $\Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F})$ are well defined, and the natural map*

$$(g_{|Y^{00}})_*\Upsilon_{f\circ g}(\mathscr{F}) \to \Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F})$$

from Proposition 9.4.3 is an isomorphism.

Proof. — Before proving the claim, let us first check that both uses of Υ are defined. For $\Upsilon_{f\circ g}(\mathscr{F})$, this is immediate from our assumptions and Proposition 9.4.8. For $\Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F})$, note first that $(g_{|Y^{\times\times}})_*\mathscr{F}$ is also iterated-clean (by Lemma 9.4.7). If $\Bbbk$ is artinian, we are done. Otherwise, consider the objects

$$\Psi_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}) \cong (g_{|X^{0\times}})_*\Psi_{(f\circ g)_1}(\mathscr{F}),$$
$$\Psi_{f_2} \circ \Psi_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}) \cong (g_{|Y^{00}})_*\Psi_{(f\circ g)_2} \circ \Psi_{(f\circ g)_1}(\mathscr{F}),$$

where both isomorphisms come from Proposition 9.1.4. These isomorphisms are compatible with the monodromy automorphisms, so the monodromy of both left-hand sides must be unipotent: we have

$$\Psi^{\mathrm{un}}_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}) = \Psi_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}),$$
$$\Psi^{\mathrm{un}}_{f_2} \circ \Psi_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}) = \Psi_{f_2} \circ \Psi_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}).$$

Now Proposition 9.4.8 tells us that $\Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F})$ is defined. It is straightforward to see that we have a commutative diagram

$$\begin{tikzcd}
(g_{|Y^{00}})_*\Upsilon_{f\circ g}(\mathscr{F}) \arrow[r, "\sim"] \arrow[d] & (g_{|Y^{00}})_*\Psi^{(2)}_{f\circ g}(\mathscr{F}) \arrow[d, "\wr"] \\
\Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F}) \arrow[r, "\sim"] & \Psi^{(2)}_f((g_{|Y^{\times\times}})_*\mathscr{F}).
\end{tikzcd}$$

Here, the horizontal maps are isomorphisms by Proposition 9.4.8, and the right-hand vertical map is an isomorphism by Proposition 9.2.3. We conclude that the left-hand vertical map is an isomorphism as well. □

9.4.5. Comparison with diagonal nearby cycles. — We continue with our separated $\mathbb{C}$-scheme X of finite type, and our algebraic map $f : X \to \mathbb{C}^2$. Consider the diagram

$$\begin{tikzcd}
(\mathbb{C}^\times \times \mathbb{C}^\times) \smallsetminus \Delta\mathbb{C}^\times \arrow[r] \arrow[d] & \mathbb{C}^\times \times \mathbb{C}^\times \arrow[d] \arrow[ddr] & & \Delta\mathbb{C}^\times \arrow[ll] \arrow[d] \\
\mathbb{C}^2 \smallsetminus \Delta\mathbb{C} \arrow[r] & \mathbb{C}^2 \smallsetminus \{(0,0)\} \arrow[dr] & & \Delta\mathbb{C} \arrow[dl] \\
& & \mathbb{C}^2 & \{(0,0)\} \arrow[l] \arrow[u]
\end{tikzcd}$$

where a symbol Δ means the diagonal copy, and all maps are the obvious embeddings. We define a number of subsets of X and inclusion maps between them by pulling back the diagram above along f, as shown in the diagram below:

$$\begin{tikzcd}
\check{X}^{\times\times} \arrow[r, "u"] \arrow[d, "\check{\jmath}_{12}"] & X^{\times\times} \arrow[d, "j_{12}"'] \arrow[drr, "\mathbf{j}"] & & & X^{\Delta\times} \arrow[lll, "k"'] \arrow[d, "j^{\Delta}"] \\
\check{X}^{\circ} \arrow[r, "u'"] & X^{\circ} \arrow[drr, "h"] & & & X^{\Delta} \arrow[dl, "k'"'] \\
& & & X & X^{00} \arrow[l, "\mathbf{i}"'] \arrow[u, "i^{\Delta}"']
\end{tikzcd}$$

The right-hand column of this diagram is a setting in which one can apply ordinary nearby cycles. Explicitly, there is a unique map $f_\Delta : X^\Delta \to \mathbb{C}$ whose composition with the diagonal embedding $\mathbb{C} = \Delta\mathbb{C} \hookrightarrow \mathbb{C}^2$ coincides with $f_{|X^\Delta}$, and we set $f_\Delta^\times = f_\Delta \circ j^\Delta$.

We will need the following lemma on unipotent $\mathbb{Z}$-representations. The last part of this lemma involves the map $p_a : L_a \to \Bbbk$ defined in Remark 9.2.2.

Lemma 9.4.10. — *For $a, b \geq 1$, there is a natural map*

$$L_a \otimes_{\Bbbk} L_b \to L_{a+b-1}$$

with the following properties.

1. *It intertwines $T_a \otimes T_b$ with T_{a+b-1}.*
2. *It is compatible with the inclusion maps from* (9.2.6).
3. *Its image contains $L_{\max\{a,b\}}$.*
4. *The following diagram commutes:*

$$\begin{tikzcd}
L_a \otimes L_b \arrow[r] \arrow[d, "p_a \otimes p_b"'] & L_{a+b-1} \arrow[d, "p_{a+b-1}"] \\
\Bbbk \otimes \Bbbk \arrow[r] & \Bbbk.
\end{tikzcd}$$

Of course, since the map in the lemma intertwines $T_a \otimes T_b$ with T_{a+b-1}, it gives rise to a map of local systems

$$\mathscr{L}_a \overset{L}{\otimes} \mathscr{L}_b \to \mathscr{L}_{a+b-1}$$

whose image contains $\mathscr{L}_{\max\{a,b\}}$.

Proof. — Consider the ring of Laurent polynomials $\Bbbk[z, z^{-1}]$, i.e., the function algebra of the multiplicative group $\mathbb{G}_{\mathrm{m}}$ over $\Bbbk$. Let D be the distribution algebra of this group scheme. This distribution algebra is described explicitly in [**J1**, §I.7.8]: it is the (free) $\Bbbk$-submodule of $\mathrm{Hom}_{\Bbbk}(\Bbbk[z, z^{-1}], \Bbbk)$ with basis

$$\delta_0, \delta_1, \ldots : \Bbbk[z, z^{-1}] \to \Bbbk \qquad \text{given by} \qquad \delta_r(z^n) = \tbinom{n}{r}.^{(2)}$$

(2) We use the convention that $\binom{n}{r} = \frac{n(n-1)\cdots(n-k+1)}{r(r-1)\cdots 1}$ for any $n \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq 0}$, with an empty product interpreted as 1.

The coalgebra structure on $\Bbbk[z, z^{-1}]$ (which is given by $\Delta(z) = z \otimes z$) induces an algebra structure on D, also described in [**J1**, §I.7.8]:

$$\delta_r \delta_s = \sum_{i=0}^{\min\{r,s\}} \frac{(r+s-i)!}{(r-i)!(s-i)!i!} \delta_{r+s-i}. \tag{9.4.5}$$

In particular, δ_0 is the unit in this ring.

Let $T : \Bbbk[z, z^{-1}] \to \Bbbk[z, z^{-1}]$ be the $\Bbbk$-linear map given by $T(z^n) = z^{n+1}$. This is a coalgebra automorphism (but not a ring homomorphism). The induced ring automorphism $T : D \to D$ (defined so that $T(f) = f \circ T$ for $f \in D$) satisfies

$$T(\delta_r) = \delta_r + \delta_{r-1},$$

with the convention that $\delta_{-1} = 0$. (To see this, observe that both sides applied to z^n yield $\binom{n+1}{r}$.)

Recall that $L_a = \Bbbk[x]/(x^a)$. Define a $\Bbbk$-linear map

$$L_a \to D \qquad \text{by} \qquad x^i \mapsto (-1)^i \delta_{a-i-1} \quad (i \in \{0, \ldots, a-1\}).$$

Since $T_a x^i = x^i - x^{i+1}$, this map intertwines $T_a : L_a \to L_a$ with $T : D \to D$. Note also that for any $a \leq b$ the following diagram commutes, where the vertical arrow is the map used in the definition of (9.2.6):

$$\begin{tikzcd} L_a \arrow[d] \arrow[dr] & \\ L_b \arrow[r] & D. \end{tikzcd}$$

For the remainder of this proof, we identify L_a with its image under this map:

$$L_a = \operatorname{span}\{\delta_0, \ldots, \delta_{a-1}\}.$$

It is immediate from (9.4.5) that the multiplication map $D \otimes_\Bbbk D \to D$ restricts to a map

$$L_a \otimes_\Bbbk L_b \to L_{a+b-1}$$

that intertwines $T_a \otimes T_b$ with T_{a+b-1}, and that is compatible with the inclusions (9.2.6). It sends $\delta_i \otimes \delta_0$ to δ_i for any i, so the image contains L_a. By the same reasoning, it also contains L_b, and hence $L_{\max\{a,b\}}$.

Finally, consider the $\Bbbk$-linear map $p : D \to \Bbbk$ given by

$$p(\delta_r) = \begin{cases} 0 & \text{if } r \geq 1, \\ 1 & \text{if } r = 0. \end{cases}$$

In particular, the restriction of p to $L_a \subset D$ agrees with the map $p_a : L_a \to \Bbbk$ defined in Remark 9.2.2. It follows from (9.4.5) that the span of $\delta_1, \delta_2, \ldots$ is an ideal in D, and hence that $p : D \to \Bbbk$ is a ring homomorphism. (In fact, p is the restriction to D of the augmentation map for the algebra dual to the Hopf algebra $\Bbbk[z, z^{-1}]$.) This fact implies the commutativity of the diagram in part (4). $\square$

Remark 9.4.11. — If $\Bbbk$ is a field of characteristic 0, the map in Lemma 9.4.10 is surjective, but it need not be surjective in general. For instance, if $\Bbbk$ is a field of characteristic 2, the image of $\mathscr{L}_2 \otimes^L \mathscr{L}_2 \to \mathscr{L}_3$ is just $\mathscr{L}_2$.

Lemma 9.4.12. — *Let $\mathscr{F}$ be a perverse sheaf on $X^{\times\times}$ with the property that $k^*\mathscr{F}[-1]$ is a perverse sheaf on $X^{\Delta\times}$. If $\Upsilon_f(\mathscr{F})$ is well defined, then there is a natural morphism*

$$\Upsilon_f(\mathscr{F}) \to \Psi^{\mathrm{un}}_{f_\Delta}(k^*\mathscr{F}[-1]).$$

Proof. — Under our assumptions, for any $a, b \geq 1$, we have

$$k^*(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)) \cong k^*\mathscr{F} \overset{L}{\otimes} (f^{\times}_{\Delta})^*(\mathscr{L}_a \overset{L}{\otimes} \mathscr{L}_b).$$

The adjunction map $\mathrm{id} \to k_*k^*$ gives rise to a natural transformation

$$\mathbf{i}^*\mathbf{j}_* \to \mathbf{i}^*\mathbf{j}_*k_*k^* \cong \mathbf{i}^*k'_*j^{\Delta}_*k^* \cong (i^{\Delta})^*j^{\Delta}_*k^*.$$

Combining this with the map from Lemma 9.4.10, we obtain a natural map

$$\mathbf{i}^*\mathbf{j}_*(\mathscr{F} \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \boxtimes \mathscr{L}_b)) \to (i^{\Delta})^*j^{\Delta}_*(k^*\mathscr{F} \overset{L}{\otimes} (f^{\times}_{\Delta})^*\mathscr{L}_{a+b-1}).$$

Now apply ${}^{\mathrm{p}}\mathscr{H}^{-2}$ to this map. When a and b are sufficiently large, in view of Proposition 9.2.1(1) this yields our desired map $\Upsilon_f(\mathscr{F}) \to \Psi^{\mathrm{un}}_{f_\Delta}(k^*\mathscr{F}[-1])$. $\square$

In the following statement we consider the constructions above for two different schemes X and Y over $\mathbb{C}^2$. To distinguish the two cases, we write k_X and k_Y for the maps "k" in these two settings.

Proposition 9.4.13. — *Let X and Y be separated $\mathbb{C}$-schemes of finite type, and let $f : X \to \mathbb{C}^2$ and $g : Y \to X$ be algebraic maps.*

1. *Assume that g is proper, and let $\mathscr{F} \in \mathsf{Perv}(Y^{\times\times}, \Bbbk)$. Assume that $\mathscr{F}$ is iterated-clean, and that the complexes*
$$k_Y^*\mathscr{F}[-1], \quad (g_{|Y^{\times\times}})_*\mathscr{F} \quad \text{and} \quad k_X^*(g_{|Y^{\times\times}})_*\mathscr{F}[-1]$$
are perverse. Assume also either that $\Bbbk$ is artinian, or that $\Psi^{\mathrm{un}}_{(f\circ g)_1}(\mathscr{F}) = \Psi_{(f\circ g)_1}(\mathscr{F})$ and $\Psi_{(f\circ g)_2} \circ \Psi_{(f\circ g)_1}(\mathscr{F}) = \Psi^{\mathrm{un}}_{(f\circ g)_2} \circ \Psi_{(f\circ g)_1}(\mathscr{F})$. If the natural map
$$\Upsilon_{f\circ g}(\mathscr{F}) \to \Psi^{\mathrm{un}}_{(f\circ g)_\Delta}(k_Y^*\mathscr{F}[-1])$$
is an isomorphism, then
$$\Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F}) \to \Psi^{\mathrm{un}}_{f_\Delta}(k_X^*(g_{|Y^{\times\times}})_*\mathscr{F}[-1])$$
is as well.
2. *Assume that g is smooth of relative dimension r. Let $\mathscr{F} \in \mathsf{Perv}(X^{\times\times}, \Bbbk)$ be such that $k_X^*\mathscr{F}[-1]$ is perverse, and suppose that $\Upsilon_f(\mathscr{F})$ is well defined. Then $k_Y^*(g_{|Y^{\times\times}})^*\mathscr{F}[r-1]$ is perverse, and if the natural map*
$$\Upsilon_f(\mathscr{F}) \to \Psi^{\mathrm{un}}_{f_\Delta}(k_X^*\mathscr{F}[-1])$$
is an isomorphism, then
$$\Upsilon_{f\circ g}((g_{|Y^{\times\times}})^*\mathscr{F}[r]) \to \Psi^{\mathrm{un}}_{(f\circ g)_\Delta}(k_Y^*(g_{|Y^{\times\times}})^*\mathscr{F}[r-1])$$

is as well. If g is surjective, the opposite implication also holds.

Proof. — (1) Note that $\Upsilon_{f\circ g}(\mathscr{F})$ and $\Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F})$ are well defined thanks to Corollary 9.4.9. We have a commutative diagram

$$\begin{array}{ccc} (g_{|Y^{00}})_*\Upsilon_{f\circ g}(\mathscr{F}) & \xrightarrow{\sim} & (g_{|Y^{00}})_*\Psi^{\mathrm{un}}_{(f\circ g)_\Delta}(k_Y^*\mathscr{F}[-1]) \\ \big\downarrow\wr & & \big\downarrow\wr \\ \Upsilon_f((g_{|Y^{\times\times}})_*\mathscr{F}) & \longrightarrow & \Psi^{\mathrm{un}}_{f_\Delta}(k_X^*(g_{|Y^{\times\times}})_*\mathscr{F}). \end{array}$$

Here, the left-hand vertical arrow comes from Corollary 9.4.9, and the right-hand vertical arrow from Proposition 9.2.3(1) and the proper base change theorem. The top horizontal arrow is an isomorphism by assumption, so the bottom one is as well.

(2) Note that in this setting the complex $\Upsilon_{f\circ g}((g_{|Y^{\times\times}})^*\mathscr{F}[r])$ is well defined thanks to Proposition 9.4.3(2). The first part of our assertion follows from a similar commutative diagram to the one above, using Proposition 9.4.3(2) and Proposition 9.2.3(2). The last part follows from the fact that if g is surjective, and if ρ is a morphism of complexes on X^{00}, then $(g_{|Y^{00}})^*\rho$ is an isomorphism if and only if ρ is. $\square$

Remark 9.4.14. — In the setting of Proposition 9.4.13(1), if $\Psi_{(f\circ g)_1}(\mathscr{F}) = \Psi^{\mathrm{un}}_{(f\circ g)_1}(\mathscr{F})$ and $\Psi_{(f\circ g)_2}\circ\Psi_{(f\circ g)_1}(\mathscr{F}) = \Psi^{\mathrm{un}}_{(f\circ g)_2}\circ\Psi_{(f\circ g)_1}(\mathscr{F})$ then we also have

$$\Psi^{\mathrm{un}}_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}) = \Psi_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F})$$

and

$$\Psi_{f_2}\circ\Psi_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}) = \Psi^{\mathrm{un}}_{f_2}\circ\Psi_{f_1}((g_{|Y^{\times\times}})_*\mathscr{F}).$$

Similarly, in the setting of Proposition 9.4.13(2), if $\Psi_{f_1}(\mathscr{F}) = \Psi^{\mathrm{un}}_{f_1}(\mathscr{F})$ and $\Psi_{f_2}\circ\Psi_{f_1}(\mathscr{F}) = \Psi^{\mathrm{un}}_{f_2}\circ\Psi_{f_1}(\mathscr{F})$, then we also have

$$\Psi^{\mathrm{un}}_{(f\circ g)_1}((g_{|Y^{\times\times}})^*\mathscr{F}) = \Psi_{(f\circ g)_1}((g_{|Y^{\times\times}})^*\mathscr{F})$$

and

$$\Psi_{(f\circ g)_2}\circ\Psi_{(f\circ g)_1}((g_{|Y^{\times\times}})^*\mathscr{F}) = \Psi^{\mathrm{un}}_{(f\circ g)_2}\circ\Psi_{(f\circ g)_1}((g_{|Y^{\times\times}})^*\mathscr{F}).$$

If g is surjective, then the converse implication holds.

9.4.6. Products. — In this subsection, we assume that

$$X = Y_{(1)}\times Y_{(2)},$$

where $Y_{(1)}$ and $Y_{(2)}$ are separated $\mathbb{C}$-schemes of finite type. We also suppose that we are given algebraic maps $f_{(1)}: Y_{(1)}\to\mathbb{C}$ and $f_{(2)}: Y_{(2)}\to\mathbb{C}$, and we set

$$f := f_{(1)}\times f_{(2)}: X\to\mathbb{C}^2.$$

For $i=1,2$, we let

$$u_{(i)}: Y^\times_{(i)}\hookrightarrow Y_{(i)},\qquad s_{(i)}: Y_{(i),0}\hookrightarrow Y_{(i)}$$

be the inclusion maps. Here, of course, we set $Y^\times_{(i)} = f^{-1}_{(i)}(\mathbb{C}^\times)$ and $Y_{(i),0} = f^{-1}_{(i)}(0)$.

Lemma 9.4.15. — *Let $\mathscr{G}_{(1)},\mathscr{G}_{(2)}$ be perverse sheaves on $Y_{(1)},Y_{(2)}$, respectively. If $\mathscr{G}_{(1)}\boxtimes^L\mathscr{G}_{(2)}$ is perverse, then it is iterated-clean.*

Proof. — By [**Ac3**, Propositions 1.4.21 and 2.9.1], both $j_{1!}^{\times} j_{2*}(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})$ and $j_{2*}^{\times} j_{1!}(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})$ can be identified with $u_{(1)!}\mathscr{G}_{(1)} \boxtimes^L u_{(2)*}\mathscr{G}_{(2)}$. We then have

$$\mathbf{i}^* j_{2*}^{\times} j_{1!}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}) \cong (u_{(1)!}\mathscr{G}_{(1)})_{|Y_{(1),0}} \overset{L}{\boxtimes} (u_{(2)*}\mathscr{G}_{(2)})_{|Y_{(2),0}} = 0$$

since the left-hand term in the tensor product vanishes. □

In this setting iterated nearby cycles can be easily computed, as shown in the following lemma, where we denote by $p_{(1)} : Y_{(1)} \times Y_{(2)} \to Y_{(1)}$ and $p_{(2)}^0 : Y_{(1),0} \times Y_{(2)} \to Y_{(2)}$ the projections.

Lemma 9.4.16. — *Let $\mathscr{G}_{(1)} \in D_c^{\mathrm{b}}(Y_{(1)}^{\times}, \Bbbk)$ and $\mathscr{G}_{(2)} \in D_c^{\mathrm{b}}(Y_{(2)}^{\times}, \Bbbk)$. Then there exists a canonical isomorphism*

$$\Psi_{f_{(1)}}(\mathscr{G}_{(1)}) \overset{L}{\boxtimes} \Psi_{f_{(2)}}(\mathscr{G}_{(2)}) \xrightarrow{\sim} \Psi_{f_{(2)} \circ p_{(2)}^0} \circ \Psi_{f_{(1)} \circ p_{(1)}}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}).$$

If $\mathscr{G}_{(1)}$, $\mathscr{G}_{(2)}$, and $\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)}$ are perverse, this isomorphism induces an isomorphism

$$\Psi_{f_{(1)}}^{\mathrm{un}}(\mathscr{G}_{(1)}) \overset{L}{\boxtimes} \Psi_{f_{(2)}}^{\mathrm{un}}(\mathscr{G}_{(2)}) \xrightarrow{\sim} \Psi_{f_{(2)} \circ p_{(2)}^0}^{\mathrm{un}} \circ \Psi_{f_{(1)} \circ p_{(1)}}^{\mathrm{un}}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}).$$

If moreover

$$\Psi_{f_{(1)}}(\mathscr{G}_{(1)}) = \Psi_{f_{(1)}}^{\mathrm{un}}(\mathscr{G}_{(1)}), \quad \Psi_{f_{(2)}}(\mathscr{G}_{(2)}) = \Psi_{f_{(2)}}^{\mathrm{un}}(\mathscr{G}_{(2)}),$$

then in the notation of §9.4.3 we have

$$\Psi_{f_1}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}) = \Psi_{f_1}^{\mathrm{un}}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}),$$
$$\Psi_{f_2} \circ \Psi_{f_1}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}) = \Psi_{f_2}^{\mathrm{un}} \circ \Psi_{f_1}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}).$$

Proof. — This follows from Lemma 9.1.8. □

In the notation of §9.4.3, this lemma shows that if $\mathscr{G}_{(1)}$, $\mathscr{G}_{(2)}$ are perverse sheaves on $Y_{(1)}$, $Y_{(2)}$, respectively, such that $\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)}$ is perverse, then we have

$$\Psi_f^{(2)}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}) \cong \Psi_{f_{(1)}}^{\mathrm{un}}(\mathscr{G}_{(1)}) \overset{L}{\boxtimes} \Psi_{f_{(2)}}^{\mathrm{un}}(\mathscr{G}_{(2)}). \tag{9.4.6}$$

Remark 9.4.17. — Combining (9.4.6) with Proposition 9.4.8 and Lemma 9.4.15, we see that if $\mathscr{G}_{(1)}$, $\mathscr{G}_{(2)}$ are perverse sheaves on $Y_{(1)}$, $Y_{(2)}$, respectively, such that $\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)}$ is perverse, and if either $\Bbbk$ is artinian or

$$\Psi_{f_{(1)}}(\mathscr{G}_{(1)}) = \Psi_{f_{(1)}}^{\mathrm{un}}(\mathscr{G}_{(1)}), \quad \Psi_{f_{(2)}}(\mathscr{G}_{(2)}) = \Psi_{f_{(2)}}^{\mathrm{un}}(\mathscr{G}_{(2)}),$$

then $\Upsilon_f(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})$ is well defined, and moreover there is a natural isomorphism

$$\Upsilon_f(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}) \xrightarrow{\sim} \Psi_{f_{(1)}}^{\mathrm{un}}(\mathscr{G}_{(1)}) \overset{L}{\boxtimes} \Psi_{f_{(2)}}^{\mathrm{un}}(\mathscr{G}_{(2)}).$$

Here is an alternative description of this map, in the spirit of Remark 9.2.2. Let $p_{a,b} : L_a \otimes L_b \to \Bbbk$ be the linear map given by

$$p_{a,b}(x^i \otimes x^j) = \begin{cases} 0 & \text{if } i < a-1 \text{ or } j < b-1, \\ 1 & \text{if } i = a-1 \text{ and } j = b-1. \end{cases}$$

This corresponds to a map of local systems $\exp^* \mathscr{L}_a \boxtimes^L \exp^* \mathscr{L}_b \to \underline{\Bbbk}_{\mathbb{C}^2}$ on $\mathbb{C}^2$. Next, let $\exp^{(2)} = \exp_{Y_{(1)}} \times \exp_{Y_{(2)}} : \tilde{Y}^{\times}_{(1)} \times \tilde{Y}^{\times}_{(2)} \to X^{\times\times}$. We obtain an induced map $\exp^{(2)*}((\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)}) \otimes^L (f^{\times\times})^*(\mathscr{L}_a \boxtimes^L \mathscr{L}_b)) \to \exp^{(2)*}(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})$, and then, by adjunction, we have a natural map

$$(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}) \overset{L}{\otimes} (f^{\times\times})^*(\mathscr{L}_a \overset{L}{\boxtimes} \mathscr{L}_b) \to \exp^{(2)}_* \exp^{(2)*}(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}).$$

Now apply ${}^{\mathrm{p}}\mathscr{H}^{-2}(\mathbf{i}^*\mathbf{j}_*(-))$ to this map. When a and b are large enough, the left-hand side is identified with $\Upsilon_f(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})$. On the other hand, as explained in Remark 9.1.7, the right-hand side is identified with $\Psi_{f_{(1)}}(\mathscr{G}_{(1)}) \boxtimes^L \Psi_{f_{(2)}}(\mathscr{G}_{(2)})$.

Proposition 9.4.18. — *Let $\mathscr{G}_{(1)}, \mathscr{G}_{(2)}$ be perverse sheaves on $Y_{(1)}, Y_{(2)}$, respectively. Assume that the complexes*

$$\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}, \quad \textit{and} \quad k^*(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)})[-1]$$

are perverse, and that

$$\Psi^{\mathrm{un}}_{f_{(1)}}(\mathscr{G}_{(1)}) = \Psi_{f_{(1)}}(\mathscr{G}_{(1)}), \quad \Psi^{\mathrm{un}}_{f_{(2)}}(\mathscr{G}_{(2)}) = \Psi_{f_{(2)}}(\mathscr{G}_{(2)}).$$

Then $\Upsilon_f(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})$ is well defined, we have

$$\Psi_{f_\Delta}(k^*(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)})[-1]) = \Psi^{\mathrm{un}}_{f_\Delta}(k^*(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)})[-1]),$$

and the natural map

$$\Upsilon_f(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}) \to \Psi^{\mathrm{un}}_{f_\Delta}(k^*(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)})[-1])$$

from Lemma 9.4.12 is an isomorphism.

Proof. — As explained in Remark 9.4.17, the complex $\Upsilon_f(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})$ is well defined, and moreover we have an identification

$$\Upsilon_f(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)}) \xrightarrow{\sim} \Psi^{\mathrm{un}}_{f_{(1)}}(\mathscr{G}_{(1)}) \overset{L}{\boxtimes} \Psi^{\mathrm{un}}_{f_{(2)}}(\mathscr{G}_{(2)}).$$

Under our assumptions, the right-hand side is $\Psi_{f_{(1)}}(\mathscr{G}_{(1)}) \boxtimes^L \Psi_{f_{(2)}}(\mathscr{G}_{(2)})$, and by Proposition 9.1.6(3) this complex identifies with $\Psi_{f_\Delta}(k^*(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})[-1])$. Since all these identifications are compatible with the monodromy automorphisms in the natural way, we deduce that

$$\Psi_{f_\Delta}(k^*(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)})[-1]) = \Psi^{\mathrm{un}}_{f_\Delta}(k^*(\mathscr{G}_{(1)} \overset{L}{\boxtimes} \mathscr{G}_{(2)})[-1]).$$

One checks that this identification of $\Upsilon_f(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})$ with $\Psi^{\mathrm{un}}_{f_\Delta}(k^*(\mathscr{G}_{(1)} \boxtimes^L \mathscr{G}_{(2)})[-1])$ coincides with the morphism of Lemma 9.4.12, which is therefore an isomorphism. □

9.5. Nearby cycles for étale sheaves

In this section we briefly recall the analogues of the constructions above in the setting of étale sheaves, see [**De1, I1**].

9.5.1. Definition. — Let A be a henselian discrete valuation ring, with field of fractions K and residue field k, whose characteristic exponent is denoted p. Set $S := \mathrm{Spec}(A)$, so that S is a henselian trait with closed point $s := \mathrm{Spec}(k)$ and generic point $\eta := \mathrm{Spec}(K)$. Denote by $\overline{K}$ a separable closure of K, and set $\overline{\eta} := \mathrm{Spec}(\overline{K})$. Let also $\overline{A}$ be the normalization of A in $\overline{K}$; then $\overline{A}$ is an absolutely integrally closed henselian valuation ring, with fraction field $\overline{K}$ and residue field a separable closure $\overline{k}$ of k. We set $\overline{S} := \mathrm{Spec}(\overline{A})$, $\overline{s} := \mathrm{Spec}(\overline{k})$.

Let X be a separated S-scheme of finite type. We then consider the diagram

$$\begin{tikzcd} X_\eta \arrow[r, "j"] \arrow[d] & X \arrow[d, "f"] & X_s \arrow[l, "i"'] \arrow[d] \\ \eta \arrow[r] & S & s \arrow[l] \end{tikzcd}$$

in which both squares are cartesian. We also set $X_{\overline{S}} := X \times_S \overline{S}$, and consider the following diagram, where again all squares are cartesian:

$$\begin{tikzcd} X_{\overline{\eta}} \arrow[r, "\overline{\jmath}"] \arrow[d] & X_{\overline{S}} \arrow[d] & X_{\overline{s}} \arrow[l, "\overline{\imath}"'] \arrow[d] \\ \overline{\eta} \arrow[r] & \overline{S} & \overline{s}. \arrow[l] \end{tikzcd}$$

The morphism $\overline{\eta} \to \eta$ induces a morphism $\rho : X_{\overline{\eta}} \to X_\eta$.

Now, let ℓ be a prime number invertible in k (i.e. $\ell \neq p$), and consider a finite commutative ring $\Bbbk$ annihilated by ℓ^n for some $n \in \mathbb{Z}_{\geq 1}$. Then we can consider the bounded étale derived category of $\Bbbk$-sheaves on X, X_η, $X_{\overline{S}}$, $X_{\overline{\eta}}$, or $X_{\overline{s}}$. Of particular interest are the constructible derived categories of sheaves on X_η and $X_{\overline{s}}$, which will be denoted $D^{\mathrm{b}}_c(X_\eta, \Bbbk)$ and $D^{\mathrm{b}}_c(X_{\overline{s}}, \Bbbk)$ respectively. The nearby cycles functor associated with f is then the functor

$$\Psi_f : D^{\mathrm{b}}_c(X_\eta, \Bbbk) \to D^{\mathrm{b}}_c(X_{\overline{s}}, \Bbbk) \quad \text{defined by } \Psi_f(\mathscr{F}) := \overline{\imath}^* \overline{\jmath}_* \rho^* \mathscr{F}[-1].$$

(In [**De1**] this functor is denoted $R\Psi_\eta[-1]$.) Here the fact that the result is a constructible complex follows from [**HS**, Corollary 4.2]; more specifically, this reference proves that the functor $\overline{\imath}^* \overline{\jmath}_*$ sends constructible complexes to constructible complexes, and it is clear from definitions that ρ^* also has this property. (See also [**De2**, Théorème 3.2] for an earlier result of this sort.) It is known also that Ψ_f is exact with respect to the perverse t-structures, see e.g. [**BBDG**, Appendix A] or [**HS**, Lemma 6.3].

In practice we will consider this construction in the following setting. We start with a field k which is either finite or algebraically closed, a smooth k-curve C (in fact, equal to $\mathbb{A}^1_k$), a k-point c of C (usually 0), and a separated k-scheme X of finite type endowed with a map $f : X \to C$. We will then chose for A the henselization of the local ring of C at c (which has residue field k), and apply the above construction for the fiber product $\tilde{X} = X \times_C \mathrm{Spec}(A)$ and the map $\tilde{f} : \tilde{X} \to \mathrm{Spec}(A)$ induced by f. The projection $\tilde{X} \to X$ induces a morphism $\tilde{X}_\eta \to X^\times := f^{-1}(C \smallsetminus \{c\})$, and we will denote by

$$\Psi_f : D^{\mathrm{b}}_c(X^\times, \Bbbk) \to D^{\mathrm{b}}_c(X \times_C \mathrm{Spec}(\overline{k}), \Bbbk) \tag{9.5.1}$$

the composition of the pullback under this map with $\Psi_{\tilde{f}}$. (Here, the fiber product on the right-hand side is defined with respect to the composition $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k) \xrightarrow{c} C$.)

In this setting, using the usual constructions (see e.g. [**BBDG**, §2.2]) one can allow more general coefficients; in addition to the case considered above, one can allow $\Bbbk$ to be one of the following:

- a finite extension of $\mathbb{Q}_\ell$;
- the ring of integers in a finite extension of $\mathbb{Q}_\ell$;
- an algebraic closure of $\mathbb{Q}_\ell$;
- or an algebraic closure of $\mathbb{F}_\ell$.(3)

In each of these cases one obtains a functor (9.5.1), which is t-exact with respect to the perverse t-structures.

9.5.2. Monodromy action. — As in the complex setting, the objects $\Psi_f(\mathscr{F})$ for $\mathscr{F}$ in $D^{\mathrm{b}}_c(X_\eta, \Bbbk)$ or $D^{\mathrm{b}}_c(X^\times, \Bbbk)$ come with some extra structure gathered under the name "monodromy."

We have a surjective morphism

$$\mathrm{Gal}(\overline{K}/K) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k), \tag{9.5.2}$$

whose kernel is denoted I, and a canonical surjective morphism

$$t : I \to \mathbb{Z}_{(p')}(1),$$

where $\mathbb{Z}_{(p')}(1)$ is the Tate module, i.e. the inverse limit over integers n prime to p of the groups of n-th roots of unity in $\overline{k}$. The latter group identifies with the product over all primes $r \neq p$ of the r-adic Tate groups $\mathbb{Z}_r(1)$, i.e. the inverse limits of the groups of r^n-th roots of unity in $\overline{k}$ (a free $\mathbb{Z}_r$-module of rank 1). In particular, we will denote by

$$t_\ell : I \to \mathbb{Z}_\ell(1)$$

the composition of t with projection on $\mathbb{Z}_\ell(1)$.

The group $\mathbb{Z}_\ell(1)$, with the natural action of $\mathrm{Gal}(\overline{k}/k)$, defines a $\mathbb{Z}_\ell$-sheaf on $\mathrm{Spec}(k)$; if $\Bbbk$ is one of the rings considered in §9.5.1 we will denote by $\Bbbk(1)$ the $\Bbbk$-sheaf obtained from this by extension of scalars. For $n \in \mathbb{Z}$, taking the n-th tensor power of this sheaf we obtain an invertible sheaf $\Bbbk(n)$ on $\mathrm{Spec}(k)$. Then, for any k-scheme X we can consider the operation of tensoring with the pullback to X of the latter sheaf; it is called the n-th Tate twist, and denoted $\mathscr{F} \mapsto \mathscr{F}(n)$. In practice, we will consider Tate sheaves only when k is finite and $\Bbbk = \overline{\mathbb{Q}}_\ell$; in this case one can choose a square root of $\overline{\mathbb{Q}}_\ell(1)$, which will be denoted $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$.

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(3) This case is perhaps less familiar than the other ones. Here one can simply define the constructible derived category as the category whose objects are pairs $(\mathscr{F}, \Bbbk_0)$ where $\Bbbk_0$ is a finite subfield of $\Bbbk$ and $\mathscr{F}$ a complex in $D^{\mathrm{b}}_c(X, \Bbbk_0)$, and where the space of morphisms from $(\mathscr{F}, \Bbbk_0)$ to $(\mathscr{F}', \Bbbk'_0)$ is $\Bbbk \otimes_{\Bbbk''_0} \mathrm{Hom}_{D^{\mathrm{b}}_c(X, \Bbbk''_0)}(\Bbbk''_0 \otimes_{\Bbbk_0} \mathscr{F}, \Bbbk''_0 \otimes_{\Bbbk'_0} \mathscr{F}')$ where $\Bbbk''_0$ is a finite subfield of $\Bbbk$ containing $\Bbbk_0$ and $\Bbbk'_0$.

As explained in [**De1**], the functor Ψ_f factors through an appropriate derived category of $\Bbbk$-sheaves on the 2-fibered product of topoi $X \times_s \eta$. (See in particular [**De1**, Construction 1.2.4] for a "concrete" description of sheaves on this topos, in terms of a continuous action of $\mathrm{Gal}(\overline{K}/K)$ compatible with the action of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{s}}$.) For us, this fact will have two consequences:

1. For any $\mathscr{F}$ in $D^{\mathrm{b}}_c(X_\eta, \Bbbk)$ or $D^{\mathrm{b}}_c(X^\times, \Bbbk)$, the complex $\Psi_f(\mathscr{F})$ is endowed with a canonical action of I, which will be called the *monodromy* action.
2. Assume that k is a finite field. After chosing (once and for all) a preimage of the Frobenius morphism under (9.5.2), we obtain that Ψ_f factors through the category of pairs consisting of a complex and a "Weil structure" as in [**BBDG**, §5.1]. If $\mathscr{F}$ is perverse, this structure defines a "lift" of the perverse sheaf $\Psi_f(\mathscr{F})$ to X_s by [**BBDG**, Proposition 5.1.2 and Remarque following it]. (Note that, even in case $\Bbbk$ is an extension of $\mathbb{Q}_\ell$, $\Psi_f(\mathscr{F})$ comes with an integral structure, so that the lift indeed exists.) In this way one obtains an exact functor

$$\Psi^\circ_f : \mathsf{Perv}(X^\times, \Bbbk) \to \mathsf{Perv}(X_s, \Bbbk). \tag{9.5.3}$$

In fact, we will only consider this construction in the case $\Bbbk = \overline{\mathbb{Q}}_\ell$; then we will twist the "natural" lift of Ψ_f by $\overline{\mathbb{Q}}_\ell(-\frac{1}{2})$ so that the resulting functor Ψ°_f commutes with Verdier duality.

With these constructions at hand, the results and constructions considered in Sections 9.1, 9.2 and 9.4 have natural analogues in the étale setting. See e.g. [**De1**] for the analogue of Proposition 9.1.4, [**I1**, §4] for the analogue of Proposition 9.1.6 (see also [**BB**, Lemma 5.1.1]), [**Beĭ2**] for the analogue of Proposition 9.2.1.

The counterpart of Section 9.3 is treated in the next subsection.

Remark 9.5.1. — Our construction of the functor (9.5.3) is non canonical, since it requires a choice of lift of the Frobenius morphism. Some of the results stated below and involving this functor can be formulated in a way that does not require such a choice, but we believe this presentation (suggested by the constructions in [**GH**] or [**G1**, Footnote 4 on p. 263]) might be easier to understand for nonspecialists.

9.5.3. Monodromic étale sheaves. — In this subsection, we briefly discuss the theory of monodromic sheaves in the étale setting, following [**Ve**]. This theory necessarily looks rather different from its counterpart in Section 9.3, as there is no "exponential map" in the étale setting.

Let k be an algebraically closed field, of characteristic exponent p, and consider a principal $\mathbb{G}_{\mathrm{m}}$-bundle Y. We also fix a prime number ℓ invertible in k, and a coefficient ring $\Bbbk$ as in §9.5.1. Recall that the étale maps $z \mapsto z^n$ (for n prime to p) define a continuous surjective morphism of pro-finite groups

$$\pi_1^{\text{ét}}(\mathbb{G}_{\mathrm{m}}, 1) \to \mathbb{Z}_{(p')}(1)$$

(where $\mathbb{Z}_{(p')}(1)$ is as in §9.5.2), and that a local system on $\mathbb{G}_{\mathrm{m}}$ is called *tame* if the corresponding representation of $\pi_1^{\text{ét}}(\mathbb{G}_{\mathrm{m}}, 1)$ factors through this quotient. Following Verdier [**Ve**], a constructible étale $\Bbbk$-sheaf on Y is called *monodromic* if its pullback along each morphism

$$\mathbb{G}_{\mathrm{m}} \to Y, \quad t \mapsto y \cdot t$$

(with $y \in Y$) is a tame local system. One then says that a complex $\mathscr{F} \in D^{\mathrm{b}}_c(Y, \Bbbk)$ is monodromic if $\mathscr{H}^i(\mathscr{F})$ is a monodromic sheaf for each $i \in \mathbb{Z}$. Given a monodromic complex $\mathscr{F}$ on Y, Verdier constructs in [**Ve**, §5] a canonical group homomorphism

$$\mu_{\mathscr{F}} : \mathbb{Z}_{(p')}(1) \to \mathrm{Aut}(\mathscr{F}).$$

Consider now a k-scheme Z of finite type endowed with a morphism $f : Z \to \mathbb{A}^1_k$. One can then set $Y := Z \times \mathbb{G}_{\mathrm{m}}$, considered as a principal $\mathbb{G}_{\mathrm{m}}$-bundle for the action on the second factor, and consider the function $g : Y \to \mathbb{A}^1_k$ defined by $g(x,t) = t \cdot f(x)$. We have $g^{-1}(0) = Z_0 \times \mathbb{G}_{\mathrm{m}}$, where $Z_0 = f^{-1}(0)$. The following result is [**Ve**, Proposition 7.1(a)]. (Here the nearby cycles functor is as in (9.5.1)).

Proposition 9.5.2. — *For any $\mathscr{F}$ in $D^{\mathrm{b}}_c(f^{-1}(\mathbb{A}^1_k \smallsetminus \{0\}), \Bbbk)$, the complex $\Psi_g(\mathscr{F} \boxtimes^L \underline{\Bbbk}_{\mathbb{G}_{\mathrm{m}}})$ is monodromic. Moreover, the monodromy action of $I = \mathrm{Gal}(\overline{K}/K)$ on this complex coming from the nearby cycles construction factors through an action of $\mathbb{Z}_{(p')}(1)$, which is given by $\alpha \mapsto \mu_{\Psi_g(\mathscr{F} \boxtimes^L \underline{\Bbbk}_{\mathbb{G}_{\mathrm{m}}})}(\alpha^{-1})$.*

We will apply this statement in the following context, following [**AB**, §5.2]. Given a k-scheme X' of finite type endowed with a $\mathbb{G}_{\mathrm{m}}$-action, the action map $a_{X'} : \mathbb{G}_{\mathrm{m}} \times X' \to X'$ is $\mathbb{G}_{\mathrm{m}}$-equivariant if $\mathbb{G}_{\mathrm{m}}$ acts on $\mathbb{G}_{\mathrm{m}} \times X'$ via multiplication on the first factor, and we will say that a perverse sheaf $\mathscr{F}$ on X' is *monodromic* if its pullback $a^*_{X'}(\mathscr{F})$ is monodromic in the sense above. In this case we have the monodromy morphism $\mu_{a^*_{X'}(\mathscr{F})}$. Since the pullback functor $a^*_{X'}$ is fully faithful on perverse sheaves by [**BBDG**, §§4.2.5–4.2.6] (because $a_{X'}$ is smooth with connected fibers), from this morphism we obtain a canonical morphism

$$\mu_{\mathscr{F}} : \mathbb{Z}_{(p')}(1) \to \mathrm{Aut}(\mathscr{F}). \tag{9.5.4}$$

We now assume given a k-scheme X of finite type endowed with a $\mathbb{G}_{\mathrm{m}}$-action, and a $\mathbb{G}_{\mathrm{m}}$-equivariant morphism $f : X \to \mathbb{A}^1_k$ (for the obvious action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^1_k$). As usual we set $X^\times = f^{-1}(\mathbb{A}^1_k \smallsetminus \{0\})$, and we consider the nearby cycles functor Ψ_f as in (9.5.1). Note that the $\mathbb{G}_{\mathrm{m}}$-action preserves $X_0 = f^{-1}(0)$.

Corollary 9.5.3. — *If $\mathscr{F}$ belongs to $\mathsf{Perv}_{\mathbb{G}_{\mathrm{m}}}(X^\times, \Bbbk)$, then the perverse sheaf $\Psi_f(\mathscr{F})$ on X_0 is monodromic. Moreover, the monodromy action of $I = \mathrm{Gal}(\overline{K}/K)$ on $\Psi_f(\mathscr{F})$ coming from the nearby cycles construction factors through an action of $\mathbb{Z}_{(p')}(1)$, which is given by $\alpha \mapsto \mu_{\Psi_f(\mathscr{F})}(\alpha^{-1})$.*

Proof. — Since $\mathscr{F}$ is $\mathbb{G}_{\mathrm{m}}$-equivariant we have $a^*_X\mathscr{F} \cong \underline{\Bbbk}_{\mathbb{G}_{\mathrm{m}}} \boxtimes^L \mathscr{F}$. Using Proposition 9.5.2 we deduce that $\Psi_g(a^*_X\mathscr{F})$ is monodromic with the expected property of monodromy, where $g = f \circ a_X$. Now by compatibility of nearby cycles with smooth pullback (see Proposition 9.1.4(2)) we have $\Psi_g(a^*_X\mathscr{F}) = a^*_{X_0}\Psi_f(\mathscr{F})$, so that this claim exactly translates into the desired statement. □

Corollary 9.5.3 can be used as a substitute for Proposition 9.3.6 in the étale setting. (Here we have to assume that $\mathscr{F}$ is perverse, but this will be the case in the applications considered in this book.)

9.5.4. The case $\Bbbk = \overline{\mathbb{Q}}_\ell$. — The monodromy action on complexes $\Psi_f(\mathscr{F})$ has been studied extensively in the case $\Bbbk = \overline{\mathbb{Q}}_\ell$, see e.g. [**GH**, §5] for a detailed overview. In particular it is known in this case (due to results of Grothendieck, see e.g. [**I3**]) that the action of I on each $\Psi_f(\mathscr{F})$ is quasi-unipotent, i.e. that there exists a subgroup $I' \subset I$ of finite index such that each element of I' acts by a unipotent automorphism. One can deduce that there exists a unique nilpotent[(4)] morphism

$$\mathsf{n}_{\mathscr{F}} : \Psi_f(\mathscr{F}) \to \Psi_f(\mathscr{F})(-1)$$

such that each $\gamma' \in I'$ acts on $\Psi_f(\mathscr{F})$ as $\exp(t_\ell(\gamma') \cdot \mathsf{n}_{\mathscr{F}})$. This morphism is called the "logarithm of (the unipotent part of) the monodromy." (In the cases that occur for central sheaves one in fact has $I' = I$; see in particular Remark 5.3.1.)

Now we specialize our setting to that of (9.5.1). More precisely we consider a finite field k, and assume that our henselian ring $A_\circ$ is the henselization of the local ring at a k-point $s_\circ$ of a smooth k-curve $C_\circ$. Then, setting $S_\circ := \mathrm{Spec}(A_\circ)$, we have a natural map $S_\circ \to C_\circ$ sending the closed point of $S_\circ$ to $s_\circ$, and the generic point $\eta_\circ$ of $S_\circ$ in $U_\circ := C_\circ \smallsetminus s_\circ$. Given a $C_\circ$-scheme $X_\circ$ of finite type, with structure morphism $f : X_\circ \to C_\circ$, following the discussion in Item (2) in §9.5.2, we have a functor

$$\Psi_f^\circ : \mathsf{Perv}(X_\circ \times_{C_\circ} U_\circ, \overline{\mathbb{Q}}_\ell) \to \mathsf{Perv}(X_\circ \times_{C_\circ} s_\circ, \overline{\mathbb{Q}}_\ell).$$

In this setting, we have a notion of *mixed* perverse sheaves on $X_\circ \times_{C_\circ} U_\circ$ and $X_\circ \times_{C_\circ} s_\circ$, see [**BBDG**, §5.1.5]. The following result is due to Deligne, see [**De3**, Théorème 6.1.13]. (See also [**GH**, §10] for closely related results.)

Theorem 9.5.4. — *The functor Ψ_f° sends mixed perverse sheaves to mixed perverse sheaves.*

Consider now the scheme $X := X_\circ \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ over $C := C_\circ \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$, where as usual $\overline{k}$ is a separable closure of k. The morphism $X \to C$ induced by f will still be denoted f. Then one can play the same game as above by considering $S = \mathrm{Spec}(A)$ with A the henselization of the local ring at the $\overline{k}$-point s of C obtained from $s_\circ$, the associated nearby cycles functor, and the induced functor

$$\Psi_f : D^{\mathrm{b}}_c(X \times_C U, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_c(X \times_C s, \overline{\mathbb{Q}}_\ell)$$

where $U = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} U_\circ = C \smallsetminus s$.

Lemma 9.5.5. — *The following diagram commutes up to an isomorphism of functors, where the vertical arrows are the pullback functors associated with the morphisms $X \times_C U \to X_\circ \times_{C_\circ} U_\circ$ and $X \times_C s \to X_\circ \times_{C_\circ} s_\circ$:*

$$\begin{tikzcd}
\mathsf{Perv}(X_\circ \times_{C_\circ} U_\circ, \overline{\mathbb{Q}}_\ell) \arrow[r, "\Psi_f^\circ"] \arrow[d, "\varkappa"'] & \mathsf{Perv}(X_\circ \times_{C_\circ} s_\circ, \overline{\mathbb{Q}}_\ell) \arrow[d, "\varkappa"] \\
D^{\mathrm{b}}_c(X \times_C U, \overline{\mathbb{Q}}_\ell) \arrow[r, "\Psi_f"] & D^{\mathrm{b}}_c(X \times_C s, \overline{\mathbb{Q}}_\ell).
\end{tikzcd}$$

(4) Here, when writing "nilpotent" we mean that $\mathsf{n}_{\mathscr{F}}(-n) \circ \cdots \circ \mathsf{n}_{\mathscr{F}}(-1) \circ \mathsf{n}_{\mathscr{F}} = 0$ for some $n \in \mathbb{Z}_{\geq 0}$.

Proof. — To define Ψ°_f, resp. Ψ_f, we need to choose a separable closure $\overline{K}_\circ$ of the fraction field $K_\circ$ of $A_\circ$, resp. a separable closure $\overline{K}$ of the fraction field K of A. By the universal property of the henselization we have a canonical morphism $A_\circ \to A$, and an associated "base change" morphism for nearby cycles functors, see [**De1**, (2.1.7.5)]. More specifically, the morphism $A_\circ \to A$ induces an embedding $K_\circ \hookrightarrow K$. We can then assume that $\overline{K}_\circ$ embeds in $\overline{K}$, so that the normalization $\overline{A}_\circ$ of $A_\circ$ in $\overline{K}_\circ$ is a subring in the normalization $\overline{A}$ of A in $\overline{K}$. The fact that our base change morphism is an isomorphism then follows from [**De2**, Proposition 3.7]. □

We conclude this chapter with a deep result of Gabber, see [**BB**, §5.1]. This statement compares, in the setting considered above, and for $\mathscr{F}_\circ$ a pure perverse sheaf of weight 0 on $X_\circ \times_{C_\circ} U_\circ$, the *weight filtration* of the mixed perverse sheaf $\Psi^\circ_f(\mathscr{F}_\circ)$ (for which we refer to [**BBDG**, Théorème 5.3.5]), and the *Jacobson–Morozov–Deligne filtration* associated with the logarithm of monodromy, which we now review. Given a $\overline{\mathbb{Q}}_\ell$-perverse sheaf $\mathscr{F}$ on $X \times_C s$ and a morphism $N : \mathscr{F} \to \mathscr{F}(-1)$ that is "nilpotent" in the sense considered above, the Jacobson–Morozov–Deligne filtration associated to N is the unique increasing filtration $(F_i(\mathscr{F}))_{i\in\mathbb{Z}}$ of $\mathscr{F}$ such that $F_i(\mathscr{F}) = 0$ for $i \ll 0$, $F_i(\mathscr{F}) = \mathscr{F}$ for $i \gg 0$, $N(F_i(\mathscr{F})) \subset F_{i-2}(\mathscr{F})(-1)$ for any i, and such that for $i \geq 0$ the i-th power N^i induces an isomorphism

$$\mathrm{gr}^F_i(\mathscr{F}) \xrightarrow{\sim} \mathrm{gr}^F_{-i}(\mathscr{F})(-i).$$

The existence and uniqueness of this filtration are proved in [**De3**, Proposition 1.6.1] (which treats more generally an object of an abelian category equipped with a nilpotent endomorphism).

Theorem 9.5.6. — *If $\mathscr{F}_\circ$ is a pure perverse sheaf of weight* 0*, then the Jacobson–Morozov–Deligne filtration of $\Psi_f(\varkappa(\mathscr{F}_\circ))$ associated with $\mathsf{n}_{\varkappa(\mathscr{F}_\circ)}$ is the image under $\varkappa$ of the weight filtration on the mixed perverse sheaf $\Psi^\circ_f(\mathscr{F}_\circ)$.*

Remark 9.5.7. — In the setting above, $\mathsf{n}_{\varkappa(\mathscr{F}_\circ)}$ is in fact the image under $\varkappa$ of a "nilpotent" endomorphism $\mathsf{n}_{\mathscr{F}_\circ}$ of $\mathscr{F}_\circ$, so that the associated Jacobson–Morozov–Deligne filtration of $\Psi_f(\varkappa(\mathscr{F}_\circ))$ is the image under $\varkappa$ of a filtration of $\Psi^\circ_f(\mathscr{F}_\circ)$. What is proved in [**BB**] is that, in the setting of Theorem 9.5.6, the latter filtration coincides with the weight filtration of this perverse sheaf. In practice we will only need the consequence of this fact that is stated in the theorem.

CHAPTER 10

EQUIVARIANT DERIVED CATEGORIES IN FAMILIES

If G is a complex algebraic group and X is a G-variety, there is a well known theory of G-equivariant sheaves on X and of the G-equivariant derived category $D^{\mathrm{b}}_G(X, \Bbbk)$, following Bernstein–Lunts [**BL**]. In this chapter, we develop an analogous theory "relative to a curve." That is: given a smooth connected complex affine curve C, a smooth group scheme $\mathcal{G}$ over C, and a scheme X of finite type over C equipped with a $\mathcal{G}$-action (over C), we explain how to define the "$\mathcal{G}$-equivariant derived category of X," denoted by $D^{\mathrm{b}}_{\mathcal{G}}(X, \Bbbk)$. We will also treat a minor generalization, in which C, $\mathcal{G}$, and X all carry compatible actions of a complex algebraic group K.

A key technical tool in this chapter, as in [**BL**], is the notion of "acyclic $\mathcal{G}$-resolutions of X," to be defined in Section 10.1. As a prerequisite to defining $D^{\mathrm{b}}_{\mathcal{G}}(X, \Bbbk)$, one must show that every scheme X as above admits a rich enough family of acyclic resolutions. In [**BL**], this problem is solved in two steps: (i) if $G = \mathrm{GL}_n$, then one can use Stiefel manifolds to construct enough acyclic resolutions; and (ii) for general G, one can reduce to the previous case by choosing a closed embedding $G \hookrightarrow \mathrm{GL}_n$. It turns out that both steps go through over a curve C as well: step (i) is essentially covered by [**LMB**, Lemma 18.7.5], and step (ii) follows from a comment in unpublished notes of Milne [**Mi2**]. In Section 10.1, we will review the steps of this construction carefully, and we will include full details of proofs whenever they differ from the corresponding proofs in [**BL**].

In Section 10.2, we give the actual definition of $D^{\mathrm{b}}_{\mathcal{G}}(X, \Bbbk)$ (as well as its K-equivariant variant). This section also discusses how to define equivariant sheaf functors, and it contains fundamental results such as the quotient and induction equivalences. Lastly, Section 10.3 contains a discussion of the nearby cycles functor and monodromy automorphisms in a $\mathcal{G}$-equivariant context.

10.1. Acyclic bundles

In this section we fix a coefficient ring $\Bbbk$ which is (as usual) noetherian, commutative, of finite global dimension.

10.1.1. Acyclic morphisms. — The following definition is the starting point for this section.

Definition 10.1.1. — Let $n \in \mathbb{Z}_{\geq 0}$, and X, Z be separated $\mathbb{C}$-schemes of finite type. A morphism of $\mathbb{C}$-schemes $f : X \to Z$ is said to be *n-acyclic* if the following condition holds: for any morphism of $\mathbb{C}$-schemes $g : Y \to Z$ (with Y separated of finite type), and for any sheaf $\mathscr{F}$ of $\Bbbk$-modules on Y, if $f' : Y \times_Z X \to Y$ is the morphism induced by f then the natural map

$$\mathscr{F} \to \mathscr{H}^0(f'_*(f')^*\mathscr{F})$$

is an isomorphism, and moreover

$$\mathscr{H}^n(f'_*(f')^*\mathscr{F}) = 0 \qquad \text{for } i = 1, \ldots, n.$$

The morphism $f : X \to Z$ is said to be *∞-acyclic* if it is n-acyclic for all n.

Remark 10.1.2. —

1. For smooth maps, n-acyclicity can be checked "fiber-wise": according to [**BL**, Criterion 1.9.4],(1) a sufficient condition for a smooth map $f : X \to Z$ to be n-acyclic is as follows: for any point $z \in Z$, we have
$$\begin{aligned} \mathsf{H}^0(p^{-1}(z); \mathbb{Z}) &\cong \mathbb{Z}, \\ \mathsf{H}^i(p^{-1}(z); \mathbb{Z}) &= 0 \qquad \text{for } i = 1, \ldots, n, \\ \mathsf{H}^{n+1}(p^{-1}(z); \mathbb{Z}) &\text{ is torsion-free.} \end{aligned}$$
2. It is easily seen that a composition of n-acyclic morphisms is n-acyclic, and that the base change of an n-acyclic morphism is n-acyclic. We also have a partial converse to this statement: given a cartesian square
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f'} & Y' \end{array}$$
where π is smooth and surjective, if f is n-acyclic then f' is n-acyclic. (This can be checked by noticing that a morphism in the derived category is invertible iff the induced morphism on each stalk is invertible, and then using the smooth base change theorem; see e.g. [**We**, Proposition 2.12].)

10.1.2. Stiefel and Grassmannian schemes. — In this subsection, C can be any affine scheme. Let $\mathcal{V}$ be a locally free $\mathscr{O}(C)$-module of finite rank. Let $\mathrm{GL}(\mathcal{V})$ be the smooth group scheme over C defined as follows: for any $\mathscr{O}(C)$-algebra R, we set

$$\mathrm{GL}(\mathcal{V})(R) = \{R\text{-module automorphisms of } \mathcal{V} \otimes_{\mathscr{O}(C)} R\}.$$

For any integer $k \geq 0$, the *Grassmannian scheme of* $\mathcal{V}$, denoted by $G(k, \mathcal{V})$, is the functor sending an $\mathscr{O}(C)$-algebra R to

$$G(k, \mathcal{V})(R) = \left\{ \begin{array}{c} \text{isomorphism classes of surjective maps } \mathcal{V} \otimes_{\mathscr{O}(C)} R \twoheadrightarrow \mathcal{Q}, \\ \text{where } \mathcal{Q} \text{ is a locally free } R\text{-module of rank } k \end{array} \right\}.$$

(1) Note that a smooth morphism of algebraic varieties is "fibered" in the sense of [**BL**, §1.4.7], e.g. by [**Ac3**, §1.4.7].

It is well known that $G(k, \mathcal{V})$ is represented by a projective (and hence separated) scheme over C. (In fact, it suffices to treat the case when $\mathcal{V}$ is free; in this setting, the representability in the case $C = \mathrm{Spec}(\mathbb{Z})$ is treated in [**SP**, Tag 089T], and the general case follows by base change. See also [**GW**, §8.4].)

Next, let $\mathcal{V}'$ be another locally free $\mathscr{O}(C)$-module of finite rank. The *Stiefel scheme* $\mathrm{St}(\mathcal{V}, \mathcal{V}')$ is the functor sending an $\mathscr{O}(C)$-algebra R to

$$\mathrm{St}(\mathcal{V}, \mathcal{V}')(R) = \left\{ \begin{array}{c} \text{injective } R\text{-module homomorphisms } \mathcal{V} \otimes_{\mathscr{O}(C)} R \to \mathcal{V}' \otimes_{\mathscr{O}(C)} R \\ \text{whose cokernel is locally free} \end{array} \right\}.$$

Once again, this functor is represented by a scheme, which is moreover smooth and quasi-affine (in particular, of finite type) over C. (Here also one can assume that $\mathcal{V}$ and $\mathcal{V}'$ are free, and then that $C = \mathrm{Spec}(\mathbb{Z})$; then the Stiefel scheme is the open subscheme in the space of $(\mathrm{rk}(\mathcal{V}') \times \mathrm{rk}(\mathcal{V}))$-matrices defined by the invertibility of some minor of size $\mathrm{rk}(\mathcal{V})$.) There is an obvious action of $\mathrm{GL}(\mathcal{V})$ on $\mathrm{St}(\mathcal{V}, \mathcal{V}')$ by precomposition with automorphisms of $\mathcal{V} \otimes_{\mathscr{O}(C)} R$.

Suppose now that $\mathcal{V}$ has rank k, and $\mathcal{V}'$ has rank $k + n$. There is an obvious map

$$\mathrm{St}(\mathcal{V}, \mathcal{V}') \to G(n, \mathcal{V}')$$

given by $(\mathcal{V} \xrightarrow{\phi} \mathcal{V}') \mapsto (\mathcal{V}' \twoheadrightarrow \operatorname{cok} \phi)$. The following lemma can be proved by a minor variation on the proof that $G(n, \mathcal{V}')$ is representable.

Lemma 10.1.3. — *The map* $\mathrm{St}(\mathcal{V}, \mathcal{V}') \to G(n, \mathcal{V}')$ *makes* $\mathrm{St}(\mathcal{V}, \mathcal{V}')$ *into a Zariski locally trivial principal* $\mathrm{GL}(\mathcal{V})$*-bundle over* $G(n, \mathcal{V}')$.

Remark 10.1.4. — If K is an affine group scheme acting on C, and if $\mathcal{V}$ and $\mathcal{V}'$ have the structure of K-equivariant locally free $\mathscr{O}(C)$-modules, then there is an obvious action of K on $\mathrm{St}(\mathcal{V}, \mathcal{V}')$ and on $G(n, \mathcal{V}')$, and the map $\mathrm{St}(\mathcal{V}, \mathcal{V}') \to G(n, \mathcal{V}')$ considered above is K-equivariant.

In the following lemma we assume that C is an affine $\mathbb{C}$-scheme of finite type.

Lemma 10.1.5. — *Let* $\mathcal{V}$ *be a locally free* $\mathscr{O}(C)$*-module of rank* k*. For any* $n \in \mathbb{Z}_{\geq 0}$*, the structure map* $\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n}) \to C$ *is* $2n$*-acyclic.*

Proof. — Let M be the space of injective linear maps $\mathbb{C}^k \to \mathbb{C}^{k+n}$. This is a smooth complex variety, and as a topological space, it is homotopy-equivalent to the traditional complex Stiefel manifold of k-frames in a Hermitian $(k+n)$-space, see e.g. [**Hat**, Example 4.53]. Now it is well known that the integral cohomology of the latter manifold is an exterior algebra, with generators in degrees $2n + 1, \ldots, 2(k + n) - 1$. In particular, this cohomology satisfies the conditions in Remark 10.1.2(1), and so the same holds for M.

To deduce the lemma, we can replace C by an open subset over which $\mathcal{V}$ is trivial, and fix a trivialization $\mathcal{V} \cong \mathscr{O}(C)^{\oplus k}$. Then $\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$ is isomorphic to $C \times M$, and the desired claim follows from Remark 10.1.2(1). $\square$

10.1.3. Embedding a group scheme in a general linear group. — In this subsection, let C be an smooth connected complex affine curve, and let $\mathcal{G}$ be a flat affine group scheme of finite type over C. Our goal is to prove that $\mathcal{G}$ is isomorphic to a closed subgroup scheme of $\mathrm{GL}(\mathcal{V})$ for some vector bundle $\mathcal{V}$ over C. The argument is based on a sketch of a proof in [**Mi2**, Chap. VIII, Aside 9.4] (see also [**BT**, §1.4.5]).

In fact, our argument will also apply in the presence of supplemental action of an affine group scheme K of finite type over $\mathbb{C}$. In more detail: suppose K acts on C, and suppose it acts on $\mathcal{G}$ in such a way that the structure morphism $\mathcal{G} \to C$ and the multiplication map $\mathcal{G} \times_C \mathcal{G} \to \mathcal{G}$ are K-equivariant. In the following statements, "$K \ltimes \mathcal{G}$-representation" should be understood to mean a $\mathcal{G}$-representation (i.e., an $\mathscr{O}(C)$-module that is also an $\mathscr{O}(\mathcal{G})$-comodule in a compatible way) M that is equipped with an action of the group scheme K such that the multiplication and comultiplication maps

$$\mathscr{O}(C) \otimes_{\mathbb{C}} M \to M \qquad \text{and} \qquad M \to M \otimes_{\mathscr{O}(C)} \mathscr{O}(\mathcal{G})$$

are K-equivariant. (The notation "$K \ltimes \mathcal{G}$" is a convenient shorthand, but the reader should beware that there is no literal algebro-geometric object called $K \ltimes \mathcal{G}$.)

On a first reading, the reader may wish to assume that K is trivial; however, some applications of this theory in the main part of the book require nontrivial K.

Lemma 10.1.6. — *Let M be a $K \ltimes \mathcal{G}$-representation, and let $V_0 \subset M$ be a finite-dimensional K-subrepresentation. Then there exists a $K \ltimes \mathcal{G}$-subrepresentation $V \subset M$ that contains V_0 and is finitely generated over $\mathscr{O}(C)$.*

(The following proof actually goes through when C is any affine $\mathbb{C}$-scheme, not just for smooth affine curves, but we will not need this generality.)

Proof. — Choose a basis $v_1, \dots, v_k$ for V_0, and write their coproducts as

$$\Delta(v_i) = \sum_j v_{ij} \otimes a_{ij} \in M \otimes_{\mathscr{O}(C)} \mathscr{O}(\mathcal{G}).$$

Let V_1 be the $\mathbb{C}$-span of the elements $v_1, \dots, v_k$ together with all the v_{ij}'s; then there exists a finite-dimensional K-subrepresentation V_1' of M containing V_1, see [**J1**, I.2.13(3)]. Let $V^+ \subset M$ be the $\mathscr{O}(C)$-submodule generated by V_1'. Note that V^+ is a K-submodule of M, and that it is finitely generated over $\mathscr{O}(C)$.

Since $\mathscr{O}(\mathcal{G})$ is flat over $\mathscr{O}(C)$, we can identify $V^+ \otimes_{\mathscr{O}(C)} \mathscr{O}(\mathcal{G})$ with a subset of $M \otimes_{\mathscr{O}(C)} \mathscr{O}(\mathcal{G})$. Let

$$V = \{v \in M \mid \Delta(v) \in V^+ \otimes_{\mathscr{O}(C)} \mathscr{O}(\mathcal{G})\}.$$

According to [**Se**, §1.4, Proposition 3], V is a $\mathcal{G}$-subrepresentation of M contained in V^+. In particular, it is finitely generated over $\mathscr{O}(C)$. It also clearly contains V_0.

It remains to check that V is a K-subrepresentation. However V is the kernel of the composition of the maps

$$M \xrightarrow{\Delta} M \otimes_{\mathscr{O}(C)} \mathscr{O}(\mathcal{G}) \twoheadrightarrow (M/V^+) \otimes_{\mathscr{O}(C)} \mathscr{O}(\mathcal{G}).$$

Both maps here are K-equivariant, so V is indeed a K-subrepresentation. $\square$

Lemma 10.1.7. — *There exist a K-equivariant locally free $\mathscr{O}(C)$-module $\mathcal{V}$ and a K-equivariant closed embedding of group schemes $\mathcal{G} \hookrightarrow \mathrm{GL}(\mathcal{V})$ over C.*

Proof. — Since $\mathcal{G}$ is of finite type, again by [**J1**, I.2.13(3)] there exists a finite-dimensional subspace $V_0 \subset \mathscr{O}(\mathcal{G})$ that is K-stable and that generates $\mathscr{O}(\mathcal{G})$ as an $\mathscr{O}(C)$-algebra. Applying Lemma 10.1.6 to $M = \mathscr{O}(\mathcal{G})$ (i.e., to the regular representation) we find a $K \ltimes \mathcal{G}$-subrepresentation $\mathcal{V} \subset \mathscr{O}(\mathcal{G})$ which is finitely generated over $\mathscr{O}(C)$ and contains V_0.

Since $\mathscr{O}(\mathcal{G})$ is a flat $\mathscr{O}(C)$-module, it is torsion-free, and hence its submodule $\mathcal{V}$ is also torsion-free. Since C is a connected smooth affine curve, its coordinate ring $\mathscr{O}(C)$ is a Dedekind domain: see [**SP**, Tag 09IG]. By [**SP**, Tag 0AUW], a finitely generated torsion-free module over a Dedekind domain is automatically locally free. We conclude that $\mathcal{V}$ is locally free (and of finite rank).

We can then consider the C-group scheme $\mathrm{GL}(\mathcal{V})$ as in §10.1.2. The K-action on $\mathcal{V}$ gives rise to a K-action on the group scheme $\mathrm{GL}(\mathcal{V}) \to C$, and of course the $\mathcal{G}$-action on $\mathcal{V}$ gives rise to a K-equivariant map $\mathcal{G} \to \mathrm{GL}(\mathcal{V})$.

It remains to show that our map $\mathcal{G} \to \mathrm{GL}(\mathcal{V})$ is a closed embedding, i.e. that the associated morphism of $\mathscr{O}(C)$-algebras $\mathscr{O}(\mathrm{GL}(\mathcal{V})) \to \mathscr{O}(\mathcal{G})$ is surjective. (The K-action plays no role in this assertion, and we will ignore it for the rest of the proof.) By [**Mi2**, Chap. VII, Proposition 2.3], it is enough to check this Zariski-locally over C; so, we can assume that $\mathcal{V}$ is free over $\mathscr{O}(C)$. Let $e_1, \dots, e_m$ be an $\mathscr{O}(C)$-basis for $\mathcal{V}$, and write their coproducts as

$$\Delta(e_i) = \sum_j e_j \otimes a_{ij} \in \mathcal{V} \otimes \mathscr{O}(\mathcal{G}).$$

According to [**Mi2**, Chap. VIII, Corollary 6.9], the image of the map $\mathscr{O}(\mathrm{GL}(\mathcal{V})) \to \mathscr{O}(\mathcal{G})$ contains the elements a_{ij}. Next, using the counit $\epsilon : \mathscr{O}(\mathcal{G}) \to \mathscr{O}(C)$, we have

$$e_i = ((\epsilon \otimes \mathrm{id}) \circ \Delta)(e_i) = \sum \epsilon(e_j) a_{ij}.$$

This calculation shows that the image of the $\mathscr{O}(C)$-algebra morphism $\mathscr{O}(\mathrm{GL}(\mathcal{V})) \to \mathscr{O}(\mathcal{G})$ contains $\mathcal{V}$. Since by construction $\mathcal{V}$ generates $\mathscr{O}(\mathcal{G})$ as an $\mathscr{O}(C)$-algebra, this morphism is surjective, as desired. □

The rest of this subsection is devoted to the proof of a refined version of Lemma 10.1.7 (see Proposition 10.1.9). This version is not used in this chapter, but is required to solve some representability questions in Chapter 2. The proofs are adapted from those in [**PR**, §1.b] and [**PZ**, §11].

Lemma 10.1.8. — *Let $\mathcal{H} \subset \mathcal{G}$ be a K-stable closed subgroup scheme that is itself flat over C. There exists a $K \ltimes \mathcal{G}$-representation $\mathcal{V}$ that is locally free of finite rank over $\mathscr{O}(C)$, along with a direct summand $\mathcal{V}' \subset \mathcal{V}$ that is locally free of rank* 1, *and such that the normalizer of $\mathcal{V}'$ in $\mathcal{G}$ is $\mathcal{H}$.*

In this statement, the *normalizer* of $\mathcal{V}'$ is the subgroup functor $N_{\mathcal{G}}(\mathcal{V}') \subset \mathcal{G}$ defined as follows: for any $\mathscr{O}(C)$-algebra R, let $\mathcal{V}'_R = R \otimes_{\mathscr{O}(C)} \mathcal{V}'$, and set

$$N_{\mathcal{G}}(\mathcal{V}')(R) = \{g \in \mathcal{G}(R) \mid g \cdot \mathcal{V}'_R \subset \mathcal{V}'_R\}.$$

It follows from [**Mi2**, Chap. V, Theorem 6.9] that $N_{\mathcal{G}}(\mathcal{V}')$ is represented by a closed subgroup scheme of $\mathcal{G}$.

Proof. — Note that $\mathcal{H}$ is automatically affine and of finite type over C. Let $\mathcal{J} \subset \mathscr{O}(\mathcal{G})$ be the defining ideal of $\mathcal{H}$. Then $\mathcal{J}$ is K-stable and finitely generated as an ideal in $\mathscr{O}(\mathcal{G})$, so it contains a finite-dimensional K-stable subspace J that generates $\mathcal{J}$ as an ideal. By Lemma 10.1.6, we can find a $K \ltimes \mathcal{G}$-subrepresentation $\mathcal{V}_1 \subset \mathscr{O}(\mathcal{G})$ that contains J and is finitely generated over $\mathscr{O}(C)$. As in the proof of Lemma 10.1.7, $\mathcal{V}_1$ is locally free of finite rank over $\mathscr{O}(C)$. Now consider the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{V}_1 \cap \mathcal{J} & \longrightarrow & \mathcal{V}_1 & \longrightarrow & \mathcal{V}_1/(\mathcal{V}_1 \cap \mathcal{J}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathscr{O}(\mathcal{G}) & \longrightarrow & \mathscr{O}(\mathcal{H}) & \longrightarrow & 0
\end{array}$$

in which the vertical maps are all injective. Since $\mathcal{H}$ is flat over C, $\mathscr{O}(\mathcal{H})$ is torsion-free, and (again as in the proof of Lemma 10.1.7), $\mathcal{V}_1/(\mathcal{V}_1 \cap \mathcal{J})$ is locally free, and hence projective, so the top short exact sequence splits. We conclude that $\mathcal{V}_1 \cap \mathcal{J}$ is a direct summand of $\mathcal{V}_1$, and hence also locally free.

Let d be the rank of $\mathcal{V}_1 \cap \mathcal{J}$. Set $\mathcal{V} := \wedge^d \mathcal{V}_1$, and set $\mathcal{V}' := \wedge^d(\mathcal{V}_1 \cap \mathcal{J})$. Then both $\mathcal{V}$ and $\mathcal{V}'$ are locally free; $\mathcal{V}'$ is a direct summand of $\mathcal{V}$; and the rank of $\mathcal{V}'$ is 1. It is easy to see that the normalizer $N_{\mathcal{G}}(\mathcal{V}')$ of $\mathcal{V}'$ in $\mathcal{V}$ coincides with the normalizer $N_{\mathcal{G}}(\mathcal{V}_1 \cap \mathcal{J})$ of $\mathcal{V}_1 \cap \mathcal{J}$ in $\mathcal{V}_1$.

To finish the proof, we must show that $N_{\mathcal{G}}(\mathcal{V}_1 \cap \mathcal{J}) = \mathcal{H}$. It is clear that $\mathcal{H} \subset N_{\mathcal{G}}(\mathcal{V}_1 \cap \mathcal{J})$. Conversely, since $\mathcal{V}_1 \cap \mathcal{J}$ generates $\mathcal{J}$ as an ideal (because it contains J), the normalizer of $\mathcal{V}_1 \cap \mathcal{J}$ must normalize all of $\mathcal{J}$ (because $\mathcal{G}$ acts on $\mathscr{O}(\mathcal{G})$ by ring automorphisms). That is, $N_{\mathcal{G}}(\mathcal{V}_1 \cap \mathcal{J}) \subset N_{\mathcal{G}}(\mathcal{J})$, and the latter normalizer is easily seen to coincide with $\mathcal{H}$. $\square$

Proposition 10.1.9. — *There exist a K-equivariant locally free $\mathscr{O}(C)$-module $\mathcal{V}$ and a K-equivariant closed immersion of group schemes $\mathcal{G} \hookrightarrow \mathrm{GL}(\mathcal{V})$ over C such that the quotient fppf sheaf $\mathrm{GL}(\mathcal{V})/\mathcal{G}$ is represented by a quasi-affine scheme over C.*

Proof. — Apply Lemma 10.1.7 to obtain a K-equivariant closed embedding $\iota : \mathcal{G} \hookrightarrow \mathrm{GL}(\mathcal{V}_1)$. Then apply Lemma 10.1.8 to the pair $\mathcal{G} \subset \mathrm{GL}(\mathcal{V}_1)$ to obtain a representation $\rho : \mathrm{GL}(\mathcal{V}_1) \to \mathrm{GL}(\mathcal{V}_2)$ where $\mathcal{V}_2$ is locally free of finite rank, along with a direct summand $\mathcal{V}_2' \subset \mathcal{V}_2$ of rank 1 whose normalizer is $\mathcal{G}$. Restricting ρ, we obtain a morphism $\phi : \mathcal{G} \to \mathrm{GL}(\mathcal{V}_2')$. Combining these, we obtain a morphism of group schemes

$$(\iota, \phi) : \mathcal{G} \hookrightarrow \mathrm{GL}(\mathcal{V}_1) \times_C \mathrm{GL}(\mathcal{V}_2'),$$

which is a closed immersion by [**SP**, Tag 07RK]. By [**An**, Théorème 4.C], the fppf quotient $(\mathrm{GL}(\mathcal{V}_1) \times_C \mathrm{GL}(\mathcal{V}_2'))/\mathcal{G}$ is represented by a scheme over C.

We will now show that $(\mathrm{GL}(\mathcal{V}_1) \times_C \mathrm{GL}(\mathcal{V}_2'))/\mathcal{G}$ is quasi-affine over C. Since this property is Zariski-local (see e.g. [**SP**, Tag 01SM]), we may replace C by a Zariski-open subscheme and assume that $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_2'$ are free. Let $\mathrm{GL}(\mathcal{V}_1) \times_C \mathrm{GL}(\mathcal{V}_2')$ act on $\mathcal{V}_2 \otimes_{\mathscr{O}(C)} \mathcal{V}_2'$ by

$$(g,h)\cdot(x\otimes y) = \rho(g)(x)\otimes h^{-1}y.$$

Next, choose an $\mathscr{O}(C)$-basis vector v for $\mathcal{V}_2'$, and consider the vector $v\otimes v \in \mathcal{V}_2\otimes_{\mathscr{O}(C)}\mathcal{V}_2'$. We can see $\mathcal{V}_2\otimes_{\mathscr{O}(C)}\mathcal{V}_2'$ as a scheme over C (an affine space), and $v\otimes v$ as a C-point of this scheme. The stabilizer of this point is $\mathcal{G}$, so the quotient $(\mathrm{GL}(\mathcal{V}_1)\times_C \mathrm{GL}(\mathcal{V}_2'))/\mathcal{G}$ is identified with the image of the action map

$$f : \mathrm{GL}(\mathcal{V}_1)\times_C \mathrm{GL}(\mathcal{V}_2') \to \mathcal{V}_2\otimes_{\mathscr{O}(C)}\mathcal{V}_2' \qquad \text{given by} \qquad (g,h)\mapsto \rho(g)(v)\otimes_{\mathscr{O}(C)} h^{-1}v.$$

A routine argument (cf. [**Bri2**, Proposition 2.1.10]) shows that the image of f is open in its closure, and hence quasi-affine.

Finally we set $\mathcal{V} := \mathcal{V}_1\oplus\mathcal{V}_2'$, and consider the obvious closed immersion

$$\mathrm{GL}(\mathcal{V}_1)\times_C \mathrm{GL}(\mathcal{V}_2') \hookrightarrow \mathrm{GL}(\mathcal{V}).$$

As above the fppf quotient $\mathrm{GL}(\mathcal{V})/\mathcal{G}$ is a scheme, and we have a natural morphism

$$\mathrm{GL}(\mathcal{V})/\mathcal{G} \to \mathrm{GL}(\mathcal{V})/\mathrm{GL}(\mathcal{V}_1)\times_C \mathrm{GL}(\mathcal{V}_2'),$$

which is quasi-affine since it is an fppf locally trivial fibration with fiber the quasi-affine scheme $(\mathrm{GL}(\mathcal{V}_1)\times_C \mathrm{GL}(\mathcal{V}_2'))/\mathcal{G}$, and the property of being quasi-affine is fpqc local on the base (see [**SP**, Tag 02L7]). On the other hand, $\mathrm{GL}(\mathcal{V})/\mathrm{GL}(\mathcal{V}_1)\times_C \mathrm{GL}(\mathcal{V}_2')$ is affine over C since it is Zariski locally a base change of an affine scheme $\mathrm{GL}(n+m,\mathbb{C})/(\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(m,\mathbb{C}))$. Hence $\mathrm{GL}(\mathcal{V})/\mathcal{G}$ is quasi-affine over C, which concludes the proof. □

10.1.4. Equivariant acyclic resolutions. — We continue with the assumptions of §10.1.3: C is a connected smooth complex affine curve, $\mathcal{G}$ is a flat affine C-group scheme of finite type, and K is a complex affine group scheme acting compatibly on C and $\mathcal{G}$. In the following definition, a "K-equivariant principal $\mathcal{G}$-bundle" is a principal $\mathcal{G}$-bundle in which the source and target are both equipped with actions of K, and the associated map is K-equivariant. A "$K\ltimes\mathcal{G}$-equivariant map" is just a map that is both K- and $\mathcal{G}$-equivariant. (Recall that there is no scheme called $K\ltimes\mathcal{G}$.)

Definition 10.1.10. — Let X be a separated $\mathbb{C}$-scheme of finite type endowed with a morphism of $\mathbb{C}$-schemes $X\to C$ and of compatible actions of $\mathcal{G}$ and K. A *K-equivariant $\mathcal{G}$-resolution* of X is a triple

$$(\bar{P}, q : P\to\bar{P}, p : P\to X)$$

where

– $\bar{P}$ is a separated $\mathbb{C}$-scheme of finite type endowed with an action of K and a K-equivariant morphism $\bar{P}\to C$;
– $q : P\to\bar{P}$ is a K-equivariant principal $\mathcal{G}$-bundle over $\bar{P}$;
– $p : P\to X$ is a $K\ltimes\mathcal{G}$-equivariant map.

Note that in this setting, the scheme P is automatically separated and of finite type by Remark 2.1.2(2) and the fact that an affine morphism is separated (see [**SP**, Tag 01S7]), so that it will make sense to consider sheaves on X, P and $\bar{P}$. Note also that since C is separated of finite type, given a morphism of $\mathbb{C}$-schemes $f : Y\to C$, Y is separated and of finite type over $\mathbb{C}$ iff f is separated and of finite type (see [**SP**, Tag 01KV] and [**GW**, Proposition 10.7]). For brevity, we often simply write the last component $p : P \to X$ of a resolution.

Proposition 10.1.11. — *Let $X \to C$ be as above, and assume that X admits a $\mathcal{G}$-equivariant line bundle which is relatively ample for the map $X \to C$. Then for any $n \geq 0$, X admits an n-acyclic K-equivariant $\mathcal{G}$-resolution $(\bar{P}, q : P \to \bar{P}, p : P \to X)$ in which the map p is smooth and quasi-affine.*

Proof. — By Lemma 10.1.7, there exists a K-equivariant locally free $\mathscr{O}(C)$-module $\mathcal{V}$ together with a K-equivariant closed embedding $\mathcal{G} \hookrightarrow \mathrm{GL}(\mathcal{V})$. We then set

$$P = \mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n}) \times_C X.$$

The K-actions on $\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$ and on X induce a K-action on P, such that the projection $P \to X$ is K-equivariant. This projection is smooth and quasi-affine as a base change of a smooth and quasi-affine morphism (see §10.1.2), and it follows from (the proof of) Lemma 10.1.5 that it is $2n$-acyclic.

It remains to construct the scheme $\bar{P}$ and the morphism $q : P \to \bar{P}$. Before doing this, we will first construct a separated, finite-type C-scheme T equipped with a K-action and a K-equivariant morphism

$$\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n}) \to T \tag{10.1.1}$$

of schemes over C that makes $\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$ into a principal $\mathcal{G}$-module over T. (The scheme T plays the role of $\bar{P}$ in the special case where $X = C$.) We define T as the fppf sheafification of the functor sending a $\mathbb{C}$-algebra R to the quotient $\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})(R)/\mathcal{G}(R)$. Then T is an fppf sheaf, with a natural K-action (obtained by functoriality of the sheafification process). To establish the desired properties of T, we will make use of the following facts:

1. By [**An**, Théorème 4.C], the fppf sheafification of the functor sending R to $\mathrm{GL}(\mathcal{V})(R)/\mathcal{G}(R)$ is a scheme, which will be denoted $\mathrm{GL}(\mathcal{V})/\mathcal{G}$.
2. This property being known, the morphism $\mathrm{GL}(\mathcal{V}) \to \mathrm{GL}(\mathcal{V})/\mathcal{G}$ is faithfully flat and of finite presentation by [**DG**, III, §3, Proposition 2.5]. Since moreover we have an isomorphism
$$\mathrm{GL}(\mathcal{V}) \times_{\mathrm{GL}(\mathcal{V})/\mathcal{G}} \mathrm{GL}(\mathcal{V}) \cong \mathrm{GL}(\mathcal{V}) \times_C \mathcal{G}$$
(see the proof of [**DG**, III, §3, Proposition 2.5]), by Remark 2.1.2(1) we deduce that $\mathrm{GL}(\mathcal{V}) \to \mathrm{GL}(\mathcal{V})/\mathcal{G}$ is a principal $\mathcal{G}$-bundle over $\mathrm{GL}(\mathcal{V})/\mathcal{G}$.
3. The quotient $\mathrm{GL}(\mathcal{V})/\mathcal{G}$ is separated over C; in fact we have a cartesian diagram
$$\begin{array}{ccc} \mathrm{GL}(\mathcal{V}) \times_C \mathcal{G} & \longrightarrow & \mathrm{GL}(\mathcal{V}) \times_C \mathrm{GL}(\mathcal{V}) \\ \downarrow & & \downarrow \\ \mathrm{GL}(\mathcal{V})/\mathcal{G} & \longrightarrow & \mathrm{GL}(\mathcal{V})/\mathcal{G} \times_C \mathrm{GL}(\mathcal{V})/\mathcal{G} \end{array}$$
where the upper arrow is defined by $(g, h) \mapsto (g, gh)$, the bottom arrow is the diagonal embedding, and the other arrows are the obvious maps. Here the upper arrow is a closed immersion since $\mathcal{G}$ is a closed subgroup of $\mathrm{GL}(\mathcal{V})$. Since the property of being a closed immersion is fpqc local on the base, see [**SP**, Tag

02L6], we deduce that the diagonal embedding of $\mathrm{GL}(\mathcal{V})/\mathcal{G}$ (over C) is a closed immersion, i.e. that this scheme is separated.

4. The quotient $\mathrm{GL}(\mathcal{V})/\mathcal{G}$ is of finite type; in fact, this follows from [**EGA4.4**, Lemme 17.7.5(iii)] applied to the diagram

$$\begin{tikzcd} \mathrm{GL}(\mathcal{V}) \arrow[rr] \arrow[dr] & & \mathrm{GL}(\mathcal{V})/\mathcal{G} \arrow[dl] \\ & C & \end{tikzcd}$$

where the upper arrow is faithfully flat and of finite presentation by (2).

Consider the morphism of functors $T \to G(n, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$ induced by the map $\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n}) \to G(n, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$. Since the latter map is a Zariski locally trivial principal $\mathrm{GL}(\mathcal{V})$-bundle (see Lemma 10.1.3), there exists a (Zariski) covering of $G(n, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$ by open subsets U over which $\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$ identifies with $U \times_C \mathrm{GL}(\mathcal{V})$. Over such an open subset, T identifies with the product $U \times_C \mathrm{GL}(\mathcal{V})/\mathcal{G}$, and hence is a scheme by (1) above. By [**GW**, Theorem 8.9], we conclude that T is a scheme. (The K-action on T as a functor now defines a K-action on T as a scheme by full faithfulness of the assignment $X \mapsto h_X$, see §2.1.1.)

The map $T \to G(n, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$ is separated by the "local" description of T and property (3) above. Since $G(n, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n})$ is separated (see §10.1.2), it follows that T is separated. Similar reasoning using property (4) shows that T is of finite type.

Finally, the claim that the morphism $\mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n}) \to T$ is a principal $\mathcal{G}$-bundle over T can be checked (Zariski) locally over T, and hence follows from property (2) above. This finishes the proof of the desired properties of T.

We return to the problem of constructing $q : P \to \bar{P}$. Using the principal bundle (10.1.1) and our assumption on the existence of a suitable line bundle on X, we can invoke Proposition 2.1.7 and set $\bar{P} := \mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n}) \times_C^{\mathcal{G}} X$. (Note that the K-action on P automatically descends to a K-action on $\bar{P}$ by fpqc descent; see e.g. [**GW**, §14.17].) By construction, this scheme fits into a cartesian square

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d] & \bar{P} \arrow[d] \\ \mathrm{St}(\mathcal{V}, \mathcal{V} \oplus \mathscr{O}(C)^{\oplus n}) \arrow[r] & T. \end{tikzcd}$$

Since the properties of being separated or of finite type are local in the fpqc topology (see [**SP**, Tag 02KU, Tag 02KZ]) and since the left vertical arrow is separated and of finite type (because $X \to C$ has those properties), we deduce that $\bar{P} \to T$ is separated and of finite type, and that hence that $\bar{P}$ is separated and of finite type. $\square$

10.2. Definition of the equivariant derived category

In this section, we let C be a connected smooth complex affine curve, and we let $\mathcal{G}$ be a smooth affine C-group scheme. We also let K be a smooth (and in particular, finite type) affine group scheme over $\mathbb{C}$, and we assume that C and $\mathcal{G}$ are equipped with actions of K that make the structure map $\mathcal{G} \to C$ and the multiplication map

$\mathcal{G} \times_C \mathcal{G} \to \mathcal{G}$ both K-equivariant. (Compared to the settings of §§10.1.3–10.1.4, we are now imposing the additional requirement that $\mathcal{G}$ and K be smooth.)

A C-scheme X will be called a *$K \ltimes \mathcal{G}$-variety* if it satisfies the following conditions:

- it is separated and of finite type (over $\mathbb{C}$, or equivalently over C);
- it is equipped with an action of $\mathcal{G}$ over C;
- it is equipped with an action of K such that the structure map $X \to C$ and the action map $\mathcal{G} \times_C X \to X$ are both K-equivariant;
- it admits a $\mathcal{G}$-equivariant line bundle that is relatively ample for the structure map $X \to C$ (so that Proposition 10.1.11 applies).

Let X be a $K \ltimes \mathcal{G}$-variety. The goal of this section is to explain (following the strategy of [**BL**, §2]) the definition and basic properties of the "$K \ltimes \mathcal{G}$-equivariant derived category of X," to be denoted $D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, \Bbbk)$. When K is trivial, this notation can be simplified to just $D^{\mathrm{b}}_{\mathcal{G}}(X, \Bbbk)$. We reiterate that the notation $K \ltimes \mathcal{G}$ is a convenient shorthand that does not refer to any algebro-geometric object.

Note that if Y is any separated, finite-type $\mathbb{C}$-scheme equipped with a K-action, the equivariant derived category $D^{\mathrm{b}}_K(Y, \Bbbk)$ is already available thanks to [**BL**].

10.2.1. Categories associated with resolutions. — Let X be a $K \ltimes \mathcal{G}$-variety. Following [**BL**, §2.1.3], given a K-equivariant $\mathcal{G}$-resolution $p : P \to X$, we define the additive category

$$D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, P, \Bbbk)$$

as follows. Its objects are triples $(\mathscr{F}_X, \bar{\mathscr{F}}, \beta)$ where $\mathscr{F}_X \in D^{\mathrm{b}}_K(X, \Bbbk)$ and $\bar{\mathscr{F}} \in D^{\mathrm{b}}_K(\bar{P}, \Bbbk)$, and where $\beta : p^*\mathscr{F}_X \xrightarrow{\sim} q^*\bar{\mathscr{F}}$ is an isomorphism in $D^{\mathrm{b}}_K(P, \Bbbk)$. A morphism $\alpha : \mathscr{F} \to \mathscr{G}$ in $D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, P, \Bbbk)$ is a pair $\alpha = (\alpha_X : \mathscr{F}_X \to \mathscr{G}_X, \bar{\alpha} : \bar{\mathscr{F}} \to \bar{\mathscr{G}})$ such that $\beta_{\mathscr{G}} \circ p^*\alpha_X = q^*\bar{\alpha} \circ \beta_{\mathscr{F}}$. For integers $a \leq b$, let $D^{[a,b]}_{K \ltimes \mathcal{G}}(X, P, \Bbbk)$ be the full additive subcategory of $D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, P, \Bbbk)$ consisting of objects $(\mathscr{F}_X, \bar{\mathscr{F}}, \beta)$ with $\mathscr{H}^i(\mathscr{F}_X) = 0$ if $i < a$ or $i > b$.

Remark 10.2.1. — We have defined $D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, P, \Bbbk)$ in terms of the K-equivariant derived categories of X, P, and $\bar{P}$. Those latter categories are, of course, defined in [**BL**] in terms of K-resolutions of various spaces. One could unwrap that construction as well, and give an alternative definition of $D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, P, \Bbbk)$ directly in terms of ordinary constructible complexes on spaces that carry compatible free actions of both K and $\mathcal{G}$. We will not go further in this direction.

It should be noted that $D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, P, \Bbbk)$ is *not* a triangulated category in general. (The issue is that when one tries to form the cone of a morphism, the isomorphism "β" is not well-defined in general.)

The following lemma is the analogue in our present setting of [**BL**, Lemma 2.3.2].

Lemma 10.2.2. — *Let $p : P \to X$ be an n-acyclic resolution. Then if $n \geq b - a$, the functor*

$$D^{[a,b]}_{K \ltimes \mathcal{G}}(X, P, \Bbbk) \to D^{[a,b]}_K(\bar{P}, \Bbbk)$$

sending $(\mathscr{F}_X, \bar{\mathscr{F}}, \beta)$ to $\bar{\mathscr{F}}$ is fully faithful, and its essential image consists of complexes whose pullback to P belongs to the essential image of p^*.

Proof. — Since p is an n-acyclic resolution, one checks as in [**BL**, Proposition 1.9.2] that the pullback functor $D_K^{[a,b]}(X, \Bbbk) \to D_K^{[a,b]}(P, \Bbbk)$ is fully faithful. Hence a triple $(\mathscr{F}_X, \bar{\mathscr{F}}, \beta)$ as in the definition of $D_{K \ltimes \mathcal{G}}^{[a,b]}(X, P, \Bbbk)$ is entirely determined by the object $\bar{\mathscr{F}}$, and a complex in $D_K^{[a,b]}(\bar{P}, \Bbbk)$ can be completed to such a triple iff its pullback to P belongs to the essential image of p^*. □

The following lemma is the analogue in our present setting of [**BL**, Proposition 2.2.1].

Lemma 10.2.3. — *Let $P \to X$ be an n-acyclic resolution, and assume that X is a K-equivariant principal $\mathcal{G}$-bundle over some base Z. Then if $n \geq b - a$, the canonical functor*

$$D_K^{[a,b]}(Z) \to D_{K \ltimes \mathcal{G}}^{[a,b]}(X, P, \Bbbk)$$

is an equivalence of categories.

Proof. — Since the morphism $X \to Z$ is smooth by our assumption on $\mathcal{G}$, the morphism $\bar{P} \to Z$ induced by our morphism $P \to X$ is also n-acyclic, see Remark 10.1.2(2). As in [**BL**, Proposition 1.9.2], this implies that the pullback functor $D_K^{[a,b]}(Z) \to D_K^{[a,b]}(\bar{P}, \Bbbk)$ is fully faithful. Similarly, the pullback functor $D_K^{[a,b]}(X) \to D_K^{[a,b]}(P, \Bbbk)$ is fully faithful. Moreover, since the morphism $P \to \bar{P}$ is smooth, one checks using the same considerations as for Remark 10.1.2(2) (see also [**BL**, §A.3]) that a complex in $D_K^{[a,b]}(\bar{P}, \Bbbk)$ belongs to the essential image of the former functor iff its pullback to P belongs to the essential image of the latter. This implies that the category $D_K^{[a,b]}(Z)$ identifies with the category of complexes in $D_K^{[a,b]}(\bar{P}, \Bbbk)$ whose pullback to P belongs to the essential image of the pullback functor $D_K^{[a,b]}(X) \to D_K^{[a,b]}(P, \Bbbk)$. This finishes the proof, in view of Lemma 10.2.2. □

The following lemma is our replacement for the discussion in [**BL**, §2.2.3].

Lemma 10.2.4. — *For fixed $a \leq b$, the category $D_{K \ltimes \mathcal{G}}^{[a,b]}(X, P, \Bbbk)$ does not depend (up to canonical equivalence) on the choice of n-acyclic K-equivariant $\mathcal{G}$-resolution $P \to X$ such $n \geq b - a$.*

Sketch of proof. — We will show that if $p_1 : P_1 \to X$ and $p_2 : P_2 \to X$ are two n-acyclic K-equivariant $\mathcal{G}$-resolutions and if p_2 is quasi-affine, then the categories $D_{K \ltimes \mathcal{G}}^{[a,b]}(X, P_1, \Bbbk)$ and $D_{K \ltimes \mathcal{G}}^{[a,b]}(X, P_2, \Bbbk)$ are canonically equivalent. This will imply the claim in view of Proposition 10.1.11.

As in [**BL**] we will consider the scheme

$$P := P_1 \times_X P_2$$

and the natural map $p : P \to X$. The first observation is that P is again an n-acyclic K-equivariant $\mathcal{G}$-resolution (for the diagonal K- and $\mathcal{G}$-actions). Here, using in particular Remark 10.1.2(2), the only thing which is not immediate is that P is a principal $\mathcal{G}$-bundle over some base which is separated and of finite type. However by assumption p_2 is quasi-affine, so the morphism $P \to P_1$ is quasi-affine. As in the proof of Proposition 2.1.7, we can therefore apply descent (more specifically, [**SGA1**, Exp. VIII, Corollaire 7.9]) to obtain the scheme $\bar{P}$ and the principal $\mathcal{G}$-bundle $P \to \bar{P}$ over $\bar{P}$. Moreover, since p_2 is separated and of finite type, so is $P \to P_1$. We then deduce (using [**SP**, Tag 01KV] and [**GW**, Proposition 10.7], as in Remark 2.1.10) that the morphism $\bar{P} \to \bar{P}_1$ is also separated and of finite type, and hence that $\bar{P}$ is separated and of finite type over $\mathbb{C}$.

Since P is now known to be a $\mathcal{G}$-resolution, we can consider the category $D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P, \Bbbk)$, and we have canonical functors

$$D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P_1, \Bbbk) \to D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P, \Bbbk) \leftarrow D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P_2, \Bbbk)$$

induced by pullback under the morphisms $\bar{P} \to \bar{P}_1$ and $\bar{P} \to \bar{P}_2$ respectively; so to conclude it suffices to prove that each of these functors is an equivalence of categories. The two cases are similar, and for notational simplicity we assume that $i = 1$.

By definition, an object of $D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P_1, \Bbbk)$ is a triple $(\mathscr{F}_X, \bar{\mathscr{F}}, \beta)$ where β is an isomorphism between the pullbacks to $\mathscr{F}_X$ (a complex on X) and $\bar{\mathscr{F}}$ (a complex on $\bar{P}_1$) to P_1. Applying Lemma 10.2.3 to the resolution $P \to P_1$, we see that the datum of $\bar{\mathscr{F}}$ is equivalent to the datum of a triple $(\mathscr{G}_{P_1}, \bar{\mathscr{G}}, \gamma)$ where γ is an isomorphism between the pullbacks of $\mathscr{G}_{P_1}$ (a complex on P_1) and $\bar{\mathscr{G}}$ (a complex on $\bar{P}$) to P. Here β identifies $\mathscr{G}_{P_1}$ with the pullback of $\mathscr{F}_X$, so these data are equivalent to the datum of $\mathscr{F}_X$, $\bar{\mathscr{G}}$, and an isomorphism between their pullbacks to P. Such data are exactly the objects of the category $D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P, \Bbbk)$, which finishes the proof. □

10.2.2. Definition of the equivariant derived category. — Let X be a $K \ltimes \mathcal{G}$-variety. For any integers $a \le b$, we define

$$D^{[a,b]}_{K\ltimes\mathcal{G}}(X, \Bbbk) := D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P, \Bbbk)$$

where $p : P \to X$ is any n-acyclic K-equivariant $\mathcal{G}$-resolution, with $n \ge b - a$. (By Lemma 10.2.4, this category does not depend on the choice of P up to canonical equivalence.) If $a' \le a \le b \le b'$, there is a natural fully faithful functor

$$D^{[a,b]}_{K\ltimes\mathcal{G}}(X, \Bbbk) \hookrightarrow D^{[a',b']}_{K\ltimes\mathcal{G}}(X, \Bbbk).$$

We then define

$$D^{\mathrm{b}}_{K\ltimes\mathcal{G}}(X, \Bbbk) = \varinjlim_{[a,b]} D^{[a,b]}_{K\ltimes\mathcal{G}}(X, \Bbbk).$$

We define a distinguished triangle in $D^{\mathrm{b}}_{K\ltimes\mathcal{G}}(X, \Bbbk)$ to be a triangle of the form $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to \mathscr{F}_1[1]$ where, for some choice of $a \le b$, the objects $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ and $\mathscr{F}_1[1]$ belong to $D^{[a,b]}_{K\ltimes\mathcal{G}}(X, \Bbbk) = D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P, \Bbbk)$ (here P is a resolution as above), and the triangle $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to \mathscr{F}_1[1]$ is distinguished as a triangle in $D^{[a,b]}_{K}(\bar{P}, \Bbbk)$, see Lemma 10.2.2. One can check as in [**BL**] (see also [**Ac3**, Theorem 6.4.10]) that this collection of triangles endows the category $D^{\mathrm{b}}_{K\ltimes\mathcal{G}}(X, \Bbbk)$ with the structure of a triangulated category.

We also have a natural "forgetful functor"

$$\mathsf{For}^{\mathcal{G}} : D^{\mathrm{b}}_{K\ltimes\mathcal{G}}(X, \Bbbk) \to D^{\mathrm{b}}_{K}(X, \Bbbk) \tag{10.2.1}$$

that sends a triple $(\mathscr{F}_X, \bar{\mathscr{F}}, \beta)$ to $\mathscr{F}_X$. This functor is easily seen to be triangulated.

Note that if X has the structure of a K-equivariant principal $\mathcal{G}$-bundle over some base $\bar{X}$ which is separated and of finite type, then the map $\mathrm{id} : X \to X$ is an ∞-acyclic resolution of X. As a consequence, we obtain an equivalence of categories

$$D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, \Bbbk) \cong D^{\mathrm{b}}_{K}(\bar{X}, \Bbbk).$$

10.2.3. Basic properties. — We now prove a few properties of equivariant derived categories, mainly with respect to change of groups.

Suppose we have a K-stable smooth closed subgroup scheme $\mathcal{H} \subset \mathcal{G}$. Recall that our assumptions on C allow us to invoke [**An**, Theorem 4.C], which guarantees that the fppf sheafification of the functor $R \mapsto \mathcal{G}(R)/\mathcal{H}(R)$ is represented by a C-scheme, which we will denote by $\mathcal{G}/\mathcal{H}$. As in the proof of Proposition 10.1.11, the K-action on $\mathcal{G}$ induces a K-action on $\mathcal{G}/\mathcal{H}$.

Lemma 10.2.5. — *The quotient morphism $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ is a K-equivariant principal $\mathcal{H}$-bundle over $\mathcal{G}/\mathcal{H}$, and the scheme $\mathcal{G}/\mathcal{H}$ is smooth and of finite type over C. Moreover, if $\mathcal{H}$ is a normal subgroup scheme of $\mathcal{G}$, then $\mathcal{G}/\mathcal{H}$ is an affine C-group scheme.*

Proof. — The first assertion follows from the generalities recalled in the proof of Proposition 10.1.11. By Remark 2.1.2(2) this implies that this map is smooth, and then (using [**SP**, Tag 02K5]) that $\mathcal{G}/\mathcal{H}$ is smooth over C. By [**SGA3.1**, Exp. $\mathrm{VI_B}$, Propositions 9.2(x) and 9.2(xiii)], this scheme is separated and of finite type over C.

If $\mathcal{H}$ is normal, this C-scheme has a natural structure of a group scheme. Then we note that, if η is the generic point of C, by [**SGA3.1**, Exp. $\mathrm{VI_B}$, Proposition 9.2(v)] we have

$$(\mathcal{G}/\mathcal{H})_\eta = \mathcal{G}_\eta/\mathcal{H}_\eta,$$

where the subscript η means the fiber at η, and the right-hand side denotes the fppf sheafification of the natural quotient presheaf. This quotient is an affine group scheme by [**SGA3.1**, Exp. $\mathrm{VI_B}$, Théorème 11.17]. In view of [**An**, Proposition 2.3.1], this implies that $\mathcal{G}/\mathcal{H}$ is affine over C. □

Proposition 10.2.6 **(Quotient equivalence).** — *Assume that $\mathcal{H} \subset \mathcal{G}$ is a K-stable normal subgroup scheme. Let X be a $K \ltimes \mathcal{G}$-variety, and let Y be a $K \ltimes (\mathcal{G}/\mathcal{H})$-variety. Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & C & \end{tikzcd}$$

where f is equivariant with respect to the projection $\mathcal{G} \to \mathcal{G}/\mathcal{H}$, and which makes X into a K-equivariant principal $\mathcal{H}^{\mathrm{op}}$-bundle over Y (for the right action of $\mathcal{H}^{\mathrm{op}}$ deduced from the left action of $\mathcal{G}$ via the embedding $\mathcal{H} \to \mathcal{G}$). Then there is a canonical equivalence of triangulated categories

$$D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, \Bbbk) \cong D^{\mathrm{b}}_{K \ltimes (\mathcal{G}/\mathcal{H})}(Y, \Bbbk).$$

(From the definition of a $K \ltimes \mathcal{G}$-variety, we see that the preceding proposition assumes that X and Y both admit suitable equivariant line bundles. In fact, we need only assume this for Y: the pullback of a $\mathcal{G}/\mathcal{H}$-equivariant relatively ample line bundle on Y is a $\mathcal{G}$-equivariant relatively ample line bundle on X because f is affine by Remark 2.1.2(2); see [**GW**, Proposition 13.83].)

Proof. — In view of the definitions of $D^{\mathrm{b}}_{K\ltimes\mathcal{G}}(X,\Bbbk)$ and $D^{\mathrm{b}}_{K\ltimes(\mathcal{G}/\mathcal{H})}(Y,\Bbbk)$, as in [**BL**, §2.6.2], to prove the proposition it suffices to prove that if $p : P \to X$ is a resolution, then there exists a C-scheme Q and morphisms $P \to Q \to \bar{P}$ which make P a principal $\mathcal{H}$-bundle over Q and Q a principal $\mathcal{G}/\mathcal{H}$-bundle over $\bar{P}$. (Indeed, if p is n-acyclic for some n, the morphism $Q \to Y$ provided by Remark 2.1.3 will be n-acyclic by Remark 10.1.2(2).

We construct the scheme Q as the associated bundle $P \times^{\mathcal{G}}_{C} (\mathcal{G}/\mathcal{H})$, which is possible thanks to Remark 2.1.8(1). We then have canonical smooth morphisms

$$P = P \times^{\mathcal{G}}_{C} \mathcal{G} \to Q \to \bar{P}$$

induced by the smooth morphisms $\mathcal{G} \to \mathcal{G}/\mathcal{H} \to C$. The fact that the first (resp. second) of these morphisms is a principal $\mathcal{H}$-bundle over Q, resp. a principal $\mathcal{G}/\mathcal{H}$-bundle over $\bar{P}$, follows from the compatibility of associated bundles with fiber products, see §2.1.4, and the fact that $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ is a principal $\mathcal{H}$-bundle over $\mathcal{G}/\mathcal{H}$. □

The following statement is given only for completeness. (It will not be used in the rest of the book.) Let $\mathcal{H} \subset \mathcal{G}$ be a K-stable closed subgroup scheme, and let X be a $K \ltimes \mathcal{H}$-variety. (In particular, X admits a relatively ample $\mathcal{H}$-equivariant line bundle $\mathscr{L}$.) Then we can apply Proposition 2.1.7 to the principal $\mathcal{H}$-bundle $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ and to X to obtain an "induced" scheme $\mathcal{G} \times^{\mathcal{H}}_{C} X$, such that the map $\mathcal{G} \times^{\mathcal{H}}_{C} X \to \mathcal{G}/\mathcal{H}$ is separated and of finite type by Remark 2.1.10; as a consequence, $\mathcal{G} \times^{\mathcal{H}}_{C} X$ itself is separated and of finite type over C.

This scheme is not quite a $K \ltimes \mathcal{G}$-variety, because we are (potentially) missing the equivariant relatively ample line bundle. But this line bundle is required only to construct sufficiently many acyclic resolutions with the property that the morphism to the given scheme is quasi-affine (as in Proposition 10.1.11), see the proof of Lemma 10.2.4. In the present setting, the required resolutions can be constructed as follows. Consider an n-acyclic resolution P_X as in Proposition 10.1.11, for the action of $\mathcal{H}$ on X. Since the morphism $P_X \to X$ is quasi-affine, the pullback of $\mathscr{L}$ to P_X is relatively ample for the morphism $P_X \to C$ by [**GW**, Proposition 13.83]. As above we can therefore consider the quotient $\mathcal{G} \times^{\mathcal{H}}_{C} P_X$. The morphism $\mathcal{G} \times^{\mathcal{H}}_{C} P_X \to \mathcal{G} \times^{\mathcal{H}}_{C} X$ is quasi-affine and n-acyclic by descent, and it is not difficult to check that the natural map $\mathcal{G} \times^{\mathcal{H}}_{C} P_X \to \bar{P}_X$ is a principal $\mathcal{G}$-bundle. From these considerations we obtain that the category $D^{\mathrm{b}}_{K\ltimes\mathcal{G}}(\mathcal{G} \times^{\mathcal{H}}_{C} X, \Bbbk)$ is well defined.

***Proposition 10.2.7* (Induction equivalence).** — *In the setting described above, there is a canonical equivalence of triangulated categories*

$$D^{\mathrm{b}}_{K\ltimes\mathcal{G}}(\mathcal{G} \times^{\mathcal{H}}_{C} X, \Bbbk) \cong D^{\mathrm{b}}_{K\ltimes\mathcal{H}}(X, \Bbbk).$$

Sketch of proof. — We consider the scheme $\mathcal{G} \times_C X$, with the action of $\mathcal{G} \times_C \mathcal{H}$ determined by $(g, h) \cdot (a, x) = (gah^{-1}, h \cdot x)$. Applying Proposition 10.2.6 to the projection $\mathcal{G} \times_C X \to X$ (and the normal subgroup $\mathcal{G} \subset \mathcal{G} \times_C \mathcal{H}$) we obtain an equivalence of categories

$$D^{\mathrm{b}}_{K \ltimes \mathcal{H}}(X, \Bbbk) \cong D^{\mathrm{b}}_{K \ltimes (\mathcal{G} \times_C \mathcal{H})}(\mathcal{G} \times_C X, \Bbbk).$$

Then, applying the same proposition to the projection map $\mathcal{G} \times_C X \to \mathcal{G} \times_C^{\mathcal{H}} X$ and the subgroup $\mathcal{H} \subset \mathcal{G} \times_C \mathcal{H}$ we obtain an equivalence

$$D^{\mathrm{b}}_{K \ltimes (\mathcal{G} \times_C \mathcal{H})}(\mathcal{G} \times_C X, \Bbbk) \cong D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(\mathcal{G} \times_C^{\mathcal{H}} X, \Bbbk).$$

Combining these two equivalences we obtain the desired claim. □

The following statement is a variant of [**BL**, Theorem 3.7.3].

Proposition 10.2.8. — *Let $\mathcal{H} \subset \mathcal{G}$ be a K-stable closed normal subgroup scheme with the property that for any closed point $x \in C$, the fiber $\mathcal{H}_x$ is unipotent. Then, for any $K \ltimes (\mathcal{G}/\mathcal{H})$-variety X, there is an equivalence of triangulated categories*

$$D^{\mathrm{b}}_{K \ltimes (\mathcal{G}/\mathcal{H})}(X, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, \Bbbk),$$

where in the right-hand side X is regarded as a $\mathcal{G}$-scheme via the quotient morphism $\mathcal{G} \to \mathcal{G}/\mathcal{H}$.

Proof. — Recall that $\mathcal{G}/\mathcal{H}$ is a smooth affine group scheme of finite type over C by Lemma 10.2.5. Let $(\bar{P}, q : P \to \bar{P}, p : P \to X)$ be an n-acyclic K-equivariant $\mathcal{G}$-resolution of X. Then as in the proof of Proposition 10.2.6 we can consider the "associated bundle" $Q = P \times_C^{\mathcal{G}} (\mathcal{G}/\mathcal{H})$ and the K-equivariant principal $\mathcal{G}/\mathcal{H}$-bundle $Q \to \bar{P}$. By Remark 10.1.2(1) the quotient map $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ is ∞-acyclic (since a complex unipotent algebraic group is contractible as a topological space), and using Remark 10.1.2(2) we deduce that the natural map $b : P \to Q$ is also ∞-acyclic.

If $p' : Q \to X$ is the morphism induced by p, then since b is ∞-acyclic, for a sheaf $\mathscr{F}$ on X, we have

$$p'_*(p')^*\mathscr{F} \cong p'_* b_* b^*(p')^*\mathscr{F} \cong p_* p^* \mathscr{F}.$$

Hence from the fact that p is n-acyclic we deduce that p' is n-acyclic as well. Thus, $(\bar{P}, Q \to \bar{P}, p' : Q \to X)$ is an n-acyclic K-equivariant $\mathcal{G}/\mathcal{H}$-resolution of X. Using Lemma 10.2.2 on both sides, we obtain an equivalence

$$D^{[a,b]}_{K \ltimes (\mathcal{G}/\mathcal{H})}(X, Q, \Bbbk) \cong D^{[a,b]}_{K \ltimes \mathcal{G}}(X, P, \Bbbk)$$

whenever $b - a \leq n$. The result follows. □

The following lemma only has nontrivial content for group schemes over C; it does not have an analogue purely in the context of algebraic groups.

Lemma 10.2.9. — *Let $C' \subset C$ be either a single point, or an affine open subset, and let $\mathcal{G}' = \mathcal{G} \times_C C'$. If X is a $K \ltimes \mathcal{G}$-variety whose structure map $X \to C$ factors through the inclusion $C' \hookrightarrow C$, then there is a canonical equivalence of triangulated categories*

$$D^{\mathrm{b}}_{K \ltimes \mathcal{G}'}(X, \Bbbk) \cong D^{\mathrm{b}}_{K \ltimes \mathcal{G}}(X, \Bbbk),$$

where in the left-hand side X is regarded as a C'-scheme.

Proof. — This follows from the observation that any n-acyclic $\mathcal{G}$-resolution of X seen as a scheme over C is, in fact, an n-acyclic $\mathcal{G}'$-resolution of X seen as a scheme over C'. □

10.2.4. Sheaf functors. — Let us briefly comment on how to construct pushforward and pullback functors associated with equivariant morphisms of schemes.

Let X, Y be $K \ltimes \mathcal{G}$-varieties, and let $f : X \to Y$ be a $K \ltimes \mathcal{G}$-equivariant map of C-schemes. Also let $\mathscr{F} \in D^{\mathrm{b}}_{K\ltimes\mathcal{G}}(X, \Bbbk)$, and choose $a \leq b$ such that $\mathscr{F}$ lies in $D^{[a,b]}_{K\ltimes\mathcal{G}}(X, \Bbbk)$, and such that $f_*\mathsf{For}^{\mathcal{G}}(\mathscr{F})$ and $f_!\mathsf{For}^{\mathcal{G}}(\mathscr{F})$ lie in $D^{[a,b]}_{K}(Y, \Bbbk)$, where $\mathsf{For}^{\mathcal{G}}$ is as in (10.2.1). Choose $n \geq b - a$, and an n-acyclic resolution $p_Y : P \to Y$ such that p_Y is smooth and $P' := P \times_Y X$ is a principal $\mathcal{G}$-bundle(2) (e.g. a resolution constructed as in the proof of Proposition 10.1.11). Let $\bar{P}$ and $\bar{P}'$ be the quotients of P and P', and consider the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & P' & \xrightarrow{q_X} & \bar{P}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \bar{f}} \\ Y & \xleftarrow{p_Y} & P & \xrightarrow{q_Y} & \bar{P} \end{array}$$

where both squares are cartesian. Here, the horizontal maps are all smooth. Write $\mathscr{F}$ as a triple $(\mathscr{F}_X, \bar{\mathscr{F}}, \beta) \in D^{[a,b]}_{K\ltimes\mathcal{G}}(X, P', \Bbbk)$. We then set

$$f_*\mathscr{F} = (f_*\mathscr{F}_X, \bar{f}_*\bar{\mathscr{F}}, \tilde{f}_*\beta), \qquad f_!\mathscr{F} = (f_!\mathscr{F}_X, \bar{f}_!\bar{\mathscr{F}}, \tilde{f}_!\beta).$$

Here, $\tilde{f}_*\beta : \tilde{f}_* p_X^*\mathscr{F} \xrightarrow{\sim} \tilde{f}_* q_X^* \bar{\mathscr{F}}$ can be regarded as an isomorphism $p_Y^* f_*\mathscr{F} \xrightarrow{\sim} q_Y^* \bar{f}_* \bar{\mathscr{F}}$ by smooth base change. Similarly, by (ordinary) base change $\tilde{f}_!\beta$ can be regarded as an isomorphism $p_Y^* f_!\mathscr{F} \xrightarrow{\sim} q_Y^* \bar{f}_! \bar{\mathscr{F}}$. (In this case, the smoothness of p_Y is not required in this construction.) It is obvious from the construction that the functors f_*, $f_!$ commute with $\mathsf{For}^{\mathcal{G}}$.

The construction of the functors f^* and $f^!$ is similar: given a complex $\mathscr{G} \in D^{[a,b]}_{K\ltimes\mathcal{G}}(Y, \Bbbk)$, we choose the same data as above; then $\mathscr{G}$ is represented by a triple $(\mathscr{G}_Y, \bar{\mathscr{G}}, \beta) \in D^{[a,b]}_{K\ltimes\mathcal{G}}(Y, P, \Bbbk)$, and we set

$$f^*\mathscr{G} = (f^*\mathscr{G}_Y, \bar{f}^*\bar{\mathscr{G}}, \tilde{f}^*\beta), \qquad f^!\mathscr{F} = (f^!\mathscr{F}_X, \bar{f}^!\bar{\mathscr{F}}, \tilde{f}^!\beta).$$

Here $\tilde{f}^*\beta$ can be seen as an isomorphism $p_X^* f^*\mathscr{G}_Y \xrightarrow{\sim} q_X^* \bar{f}^*\bar{\mathscr{G}}$ by compatibility of pullback functors with composition (we do not need the smoothness assumption on p_Y here), and $\tilde{f}^!\beta$ can be seen as an isomorphism $p_X^* f^!\mathscr{G}_Y \xrightarrow{\sim} q_X^* \bar{f}^!\bar{\mathscr{G}}$ by smoothness of q_X and p_X (see [**BL**, §1.8]).

For further details on the traditional "six operations," see [**BL**, §3] or [**Ac3**, §6.5].

(2) In the setting of group schemes over a field, one can check that the analogous condition is automatic in case Y is quasi-projective, using the fact that a torsor under a smooth affine group scheme over a field is locally isotrivial, see [**Ra**, Lemme XIV.1.4]. We do not know if a statement of this sort is true over a curve.

10.3. Equivariant nearby cycles and monodromy

Suppose we have a map $f : X \to \mathbb{A}^1_{\mathbb{C}}$, where X is a separated scheme of finite type over $\mathbb{C}$. Let $X^\times = f^{-1}(\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{0\})$, and let $X_0 = f^{-1}(0)$. As in Section 9.1, we have a nearby cycles functor

$$\Psi_f : D^{\mathrm{b}}_c(X^\times, \Bbbk) \to D^{\mathrm{b}}_c(X_0, \Bbbk). \tag{10.3.1}$$

In this section, we consider the problem of upgrading this to a functor between equivariant derived categories, in the setting where an algebraic group (or group scheme) acts on X.

10.3.1. The case of algebraic groups acting trivially on $\mathbb{A}^1$. — Consider first the "classical" situation: let G be an algebraic group (i.e. an affine $\mathbb{C}$-group scheme of finite type, which is automatically smooth) acting on X. Assume also that $f : X \to \mathbb{A}^1$ is G-equivariant, where G acts trivially on $\mathbb{A}^1$. Then $X^\times$ and X_0 are both G-stable, and the method from [**BL**, §3] can be used to define a functor

$$\Psi_f : D^{\mathrm{b}}_G(X^\times, \Bbbk) \to D^{\mathrm{b}}_G(X_0, \Bbbk). \tag{10.3.2}$$

The reason that we require G to act trivially on $\mathbb{A}^1$ is this: given a G-resolution $(\bar{P}, P \to \bar{P}, P \to X)$ of X, the definition of (10.3.2) involves the nonequivariant nearby cycles functor for $\bar{P}$—but there is no induced map $\bar{P} \to \mathbb{A}^1$ unless the G-action on the latter is trivial.

If G acts nontrivially on $\mathbb{A}^1$, there is in general no notion of "G-equivariant nearby cycles." However, there is an exception coming from Proposition 9.3.6: if $G = \mathbb{G}_{\mathrm{m}}$ acts on $\mathbb{A}^1$ by the natural dilation action, then we have functors

$$\Psi_f : D^{\mathrm{b}}_{\mathbb{C}^\times}(X^\times, \Bbbk) \to D^{\mathrm{b}}_{\mathbb{C}^\times\text{-mon}}(X, \Bbbk), \tag{10.3.3}$$

$$\Psi^{\mathrm{un}}_f : \mathsf{Perv}_{\mathbb{C}^\times}(X^\times, \Bbbk) \to \mathsf{Perv}(X /\!\!/ \mathbb{C}^\times, \Bbbk). \tag{10.3.4}$$

10.3.2. The case of group schemes. — Now suppose we have a smooth affine group scheme $\mathcal{G} \to \mathbb{A}^1$ of finite type. Let

$$\mathcal{G}^\times := \mathcal{G} \times_{\mathbb{A}^1} (\mathbb{A}^1 \smallsetminus \{0\}), \qquad \mathcal{G}_0 := \mathcal{G} \times_{\mathbb{A}^1} \{0\}.$$

If there is an action of the multiplicative group $\mathbb{C}^\times$ on $\mathcal{G}$ compatible with the natural dilation action of $\mathbb{C}^\times$ on $\mathbb{A}^1$, then we have explained in Section 10.2 how to define the equivariant derived category $D^{\mathrm{b}}_{\mathbb{C}^\times \ltimes \mathcal{G}}(X, \Bbbk)$, where X is a $\mathbb{C}^\times \ltimes \mathcal{G}$-variety.

On the other hand, following the construction of [**BL**] we have the equivariant derived category $D^{\mathrm{b}}_{\mathbb{C}^\times \ltimes \mathcal{G}_0}(X_0, \Bbbk)$. It is also possible to consider monodromic categories, as in Section 9.3, but keeping the $\mathcal{G}_0$-equivariance condition. In this way one obtains the $\mathbb{C}^\times$-monodromic $\mathcal{G}_0$-equivariant derived category of X_0, denoted by

$$D^{\mathrm{b}}_{\mathcal{G}_0, \mathbb{C}^\times\text{-mon}}(X_0, \Bbbk).$$

Similarly,

$$D^{\mathrm{b}}_{\mathcal{G}_0}(X_0 /\!\!/ \mathbb{C}^\times, \Bbbk) \subset D^{\mathrm{b}}_{\mathcal{G}_0, \mathbb{C}^\times\text{-mon}}(X_0, \Bbbk)$$

is defined to be the full triangulated subcategory generated by the image of the forgetful functor from $D^{\mathrm{b}}_{\mathbb{C}^\times \ltimes \mathcal{G}_0}(X_0, \Bbbk)$. The "monodromy" construction also works in the setting, providing a canonical group morphism

$$\mu_{\mathscr{F}} : X_*(T) \to \mathrm{Aut}_{D^{\mathrm{b}}_{\mathcal{G}_0}(X_0,\Bbbk)}(\mathscr{F})$$

for any $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{G}_0, \mathbb{C}^\times\text{-mon}}(X_0, \Bbbk)$.

The following statement summarizes the circumstances under which it makes sense to work with $\mathcal{G}$-equivariant nearby cycles.

Proposition 10.3.1. — *Let $\mathcal{G} \to \mathbb{A}^1$ be a smooth affine group scheme of finite type, and let X be a $\mathcal{G}$-variety. In particular, X is equipped with a structure map $f : X \to \mathbb{A}^1$.*

1. *There is a t-exact functor*
$$\Psi_f : D^{\mathrm{b}}_{\mathcal{G}^\times}(X^\times, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{G}_0}(X_0, \Bbbk)$$
that corresponds to the "traditional" nearby cycles functor (10.3.1) *under the appropriate forgetful functors. Moreover, this functor admits an automorphism $\mathsf{m}_{\mathscr{F}}$ that is mapped to the monodromy automorphism of §9.1.3 under the forgetful functor $\mathsf{For}^{\mathcal{G}_0}$.*
2. *Suppose there is an action of $\mathbb{C}^\times$ on $\mathcal{G}$ such that $\mathcal{G} \to \mathbb{A}^1$ is equivariant with respect to the natural dilation action of $\mathbb{C}^\times$ on $\mathbb{A}^1$, and suppose that X has the structure of a $\mathbb{C}^\times \ltimes \mathcal{G}$-variety. Then there is a t-exact functor*
$$\Psi_f : D^{\mathrm{b}}_{\mathbb{C}^\times \ltimes \mathcal{G}^\times}(X^\times, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{G}_0, \mathbb{C}^\times\text{-mon}}(X_0, \Bbbk)$$
which corresponds to (10.3.3) *under the appropriate forgetful functors, along with a left exact functor*
$$\Psi_f^{\mathrm{un}} : \mathsf{Perv}_{\mathbb{C}^\times \ltimes \mathcal{G}^\times}(X^\times, \Bbbk) \to \mathsf{Perv}_{\mathcal{G}_0}(X_0 /\!\!/ \mathbb{C}^\times, \Bbbk).$$
Moreover, for both of these, we have
$$\mathsf{m}_{\mathscr{F}} = \mu_{\Psi_f(\mathscr{F})}(-1).$$
3. *In the setting of part* (2)*, suppose that $\Psi_f^{\mathrm{un}}(\mathscr{F}) = \Psi_f(\mathscr{F})$ for all $\mathscr{F} \in \mathsf{Perv}_{\mathbb{C}^\times \ltimes \mathcal{G}^\times}(X^\times, \Bbbk)$. Then Ψ_f^{un} extends to a t-exact triangulated functor*
$$\Psi_f^{\mathrm{un}} = \Psi_f : D^{\mathrm{b}}_{\mathbb{C}^\times \ltimes \mathcal{G}^\times}(X^\times, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{G}_0}(X_0 /\!\!/ \mathbb{C}^\times, \Bbbk).$$

The proof consists in carrying out a construction that follows the pattern explained at the end of the preceding section, or in [**BL**, §3]. (For part (2) we have to consider the $\mathbb{C}^\times$-monodromic $\mathcal{G}$-equivariant derived category of X; this category is defined to be $D^{\mathrm{b}}_{\widetilde{\mathbb{C}^\times} \ltimes \mathcal{G}}(X, \Bbbk)$, where the latter is defined as in Section 10.2, but using the $\widetilde{\mathbb{C}^\times}$-equivariant derived categories of $\mathbb{C}^\times \ltimes \mathcal{G}$-resolutions of X, rather than their $\mathbb{C}^\times$-equivariant derived categories.) We omit further details.

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INDEX OF NOTATION