WORKSHOP ON GEOMETRIC SATAKE

PROGRAM BY ARNAUD MAYEUX, TIMO RICHARZ AND SIMON RICHE

In [FS21, Chapter VI], Fargues–Scholze prove the geometric Satake equivalence for the B_{dR}^{+}-affine Grassmannian, using the theory of diamonds and v-sheaves. The aim of this workshop is to translate (elements of) the proof to the more classical world of schemes and explore further directions.

1. Introduction

The central topic of the workshop is the geometric Satake equivalence:

Theorem. *Let k be an algebraically closed field, G a reductive k-group, and put $\Lambda = \mathbb{Z}/\ell^n$ for a prime ℓ invertible in k. The Satake category $\mathrm{Sat}_{G,\Lambda}$ of flat perverse étale Λ-sheaves on the local Hecke stack is equipped with a symmetric monoidal structure $\star$, called convolution, and a fiber functor*

$$\mathrm{F}\colon \mathrm{Sat}_{G,\Lambda} \to \mathrm{Mod}_{\Lambda}^{\mathrm{fg,proj}} \tag{1.1}$$

to the category of finite projective Λ-modules that induces a Tannakian type equivalence

$$\mathrm{Sat}_{G,\Lambda} \cong \mathrm{Rep}(\widehat{G}_{\Lambda}), \tag{1.2}$$

with the category of algebraic representations of the Langlands dual group $\widehat{G}_{\Lambda}$ on finite projective Λ-modules.

By passing in $\Lambda = \mathbb{Z}/\ell^n$ to the inverse limit over n, there is an analogous statement for coefficients in $\mathbb{Z}_\ell$ and, by inverting ℓ, also for coefficients in $\mathbb{Q}_\ell$. The proof proceeds, roughly, in three steps:

1. Construct the triple $(\mathrm{Sat}_{G,\Lambda}, \star, \mathrm{F})$.
2. Check the tensor compatibilities of $\star$ and F.
3. Identify $\check{G} := \mathrm{Aut}^{\star}(\mathrm{F})$ with $\widehat{G}_{\Lambda}$.

(1): The definition of the category $\mathrm{Sat}_{G,\Lambda}$ involves, among other things, the local Hecke stack $\mathrm{Hk}_G \to \mathrm{Spec}\, k$ whose geometry can be understood through the equivariant geometry of the affine Grassmannian:

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \bigcup_{\mu} \mathrm{Gr}_{G,\leqslant\mu}. \tag{1.3}$$

This is an infinite union (on the underlying reduced locus) of projective k-varieties $\mathrm{Gr}_{G,\leqslant\mu}$, called Schubert varieties and enumerated by the dominant cocharacters $\mu \in X_*^+$. The varieties $\mathrm{Gr}_{G,\leqslant\mu}$ are singular for $\mu \gg 0$. The perverse Λ-sheaves supported on these varieties encode the singularities and generate the abelian category $\mathrm{Sat}_{G,\Lambda}$. If $\Lambda = \mathbb{Q}_\ell$, then $\mathrm{Sat}_{G,\Lambda}$ is semi-simple with simple objects the intersection cohomology complexes $\mathrm{IC}_{\mu,\mathbb{Q}_\ell}$ on $\mathrm{Gr}_{G,\leqslant\mu}$. If $\Lambda = \mathbb{Z}/\ell^n$ or $\mathbb{Z}_\ell$, then the situation is more complicated. For $A, B \in \mathrm{Sat}_{G,\Lambda}$, their convolution $A \star B$ is modeled on the convolution of functions in the spherical Hecke algebra, and the functor F is defined by taking cohomology $\mathrm{F}(A) = \mathrm{H}^*(\mathrm{Gr}_G, A)$.

(2): Next, one aims to equip the pair $(\mathrm{Sat}_{G,\Lambda}, \star)$ with a symmetric monoidal structure and the functor $\mathrm{F}\colon \mathrm{Sat}_{G,\Lambda} \to \mathrm{Mod}_{\Lambda}^{\mathrm{fg,proj}}$ with functorial isomorphisms $\mathrm{F}(A \star B) \cong \mathrm{F}(A) \otimes \mathrm{F}(B)$ that are compatible with the symmetric monoidal structures. This is surprisingly subtle! A central tool will be the study of constant term functors attached to the choice of Levi subgroups in G.

(3): Using a version of Tannaka duality, one can then build a Hopf algebra in Mod_Λ whose spectrum is a flat affine Λ-group scheme $\check{G}$ so that $\mathrm{Sat}_{G,\Lambda}$ is given by its category of representations. With additional work, including an induction on the semi-simple rank of G, one can construct a pinned isomorphism $\check{G} \cong \widehat{G}_\Lambda$, giving the desired geometric Satake equivalence. Again, the final step is surprisingly subtle and requires to prove it for $\Lambda = \mathbb{Q}_\ell$ first, then to deduce the case $\Lambda = \mathbb{Z}_\ell$ and finally the case $\Lambda = \mathbb{Z}/\ell^n$ by reduction modulo ℓ^n.

List of Talks. Every talk is 90 minutes long. As a consequence, speakers are advised to prepare 60–75 minutes of actual talk, leaving time for discussions. We encourage all participants to ask many questions during the seminar and also before while preparing! The outline above leads to the following structure for our workshop:

- *Preliminaries:* Talks 0–1 are a short Leitfaden and a motivational talk on the classical Satake isomorphism.
- *Sheaf-theoretic background:* Talks 2–5 provide the necessary results on étale sheaves.
- *Geometric background:* Talks 6–8 study the geometry of affine Grassmannians.
- *Construction of the Satake category:* Talks 9–10 combine the previous talks to construct the triple $(\mathrm{Sat}_{G,\Lambda}, \star, \mathrm{F})$ and check the tensor compatibilities.
- *Proof of the equivalence:* Talks 11–12 sketch the construction of $\mathrm{Aut}^\star(\mathrm{F}) = \check{G}$ as a group scheme over Λ and its identification with $\widehat{G}_\Lambda$.

In particular, Talks 2–5 and Talks 6–8 are mostly independent of each other whereas the results need to be combined in Talks 9–12.

Main references. These include [HS21] for the sheaf-theoretic background, [Zh16, Ri19] for affine Grassmannians and [FS21, Chapter VI], [Sch20, Lectures 22–25], [BR18] for the geometric Satake equivalence.

2. Preliminaries

Talk 0: Leitfaden. There will be a short talk (around 15 minutes) given by one of the organizers which provides a Leitfaden for the workshop.

Talk 1: Definition of the dual group and the Satake isomorphism [AuP, Tag 00IZ]**,** [Yu15, Sections 3–4]**,** [Gr98, Chapters 1–5]**.** Let F be a non-Archimedean local field with ring of integers O and finite residue field k of cardinality q. Let G be a split reductive group over O (for example, $G = \mathrm{Gl}_n$). Define the Langlands dual group $\widehat{G}$ and give examples, see [AuP, Tap 00IZ] and also [Sch20, Lecture 25]. Put $\mathrm{Hk}_G(k) = G(O)\backslash G(F)/G(O)$, $\Lambda = \mathbb{Z}$. Define the spherical Hecke algebra $\mathrm{H}_G = \mathrm{Fun}_c(\mathrm{Hk}_G(k), \Lambda)$ of finitely supported functions and explain its convolution structure. Then give an overview of the Satake isomorphism $\mathrm{H}_G \otimes \Lambda[q^{\pm\frac{1}{2}}] \cong \mathrm{R}(\widehat{G}) \otimes \Lambda[q^{\pm\frac{1}{2}}]$, see [Gr98, Chapters 1–5] and also [Yu15, Sections 3–4][1] for the example of $G = \mathrm{GL}_n$. If time permits, then make for $G = \mathrm{SL}_2$ the connection to functions on the Bruhat–Tits tree whose vertices correspond to $\mathrm{Gr}_G(k) = G(F)/G(O)$.

Comment. There is a version of Theorem 1 for general fields k. When taking a finite field k and letting $F = k(\!(t)\!)$, the equivalence (1.2) recovers the Satake isomorphism after passing to Grothendieck rings and using the sheaf-function dictionary. So (1.2) can be seen as a categorical enrichment.

3. Sheaf-theoretic background

Let X be a scheme and Λ a (commutative, unital) ring. Whenever convenient, we assume that Λ is a finite local ℓ-torsion ring (e.g., $\Lambda = \mathbb{Z}/\ell^n$) for some prime number ℓ invertible on X and that X has finite ℓ-cohomological dimension. This includes all finite type schemes over an algebraically closed field or a finite field.

[1] The isomorphism in Theorem 4.1.1 is only defined over $\mathbb{Z}[\sqrt{q}^{\pm 1}]$.

Talk 2: Categories of étale sheaves [BS13, Section 6], [StP], [Ga21]. Follow [BS13, Sections 6.3, 6.4]: Introduce the derived category[2] of étale sheaves $\mathrm{D}(X, \Lambda) = \mathrm{D}(X_{\text{ét}}, \Lambda)$ on a qcqs scheme X and a coefficient ring Λ and its subcategory $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ of perfect-constructible complexes. (If $X = *$ is a geometric point, then $\mathrm{D}_{\mathrm{cons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda)$ is the full subcategory of perfect complexes of Λ-modules in the derived category of all Λ-modules.) Under the cohomological finiteness assumption on (X, Λ), the category $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is exactly the full subcategory of compact objects in $\mathrm{D}(X, \Lambda)$, see [BS13, Proposition 6.4.8]. Recall the six functors[3] [BS13, Section 6.7]: for a morphisms $f: Y \to X$ pairs of adjoint functors (f_*, f^*) and, if f is separated of finite type, $(f^!, f_!)$ and $(\underline{\mathrm{Hom}}(\text{-},\text{-}), \text{-}\otimes_\Lambda\text{-})$, all to be understood on the level of derived categories. State that they satisfy a six functor formalism: base change, projection formula, duality, etc., see also [Ga21] for a general introduction to six functor formalisms. By construction, $\mathrm{D}(X, \Lambda)$ carries the standard t-structure (being the derived category of the abelian category of sheaves of Λ-modules on $X_{\text{ét}}$) and $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ carries a t-structure if $\Lambda = \mathbb{F}_\ell$ is regular, see also [HRS21, Proposition 3.32].

Talk 3: Universally locally acyclic sheaves [BS13, Section 6.3], [HS21, Sections 3–4], [LZ20], [Sch20, Lecture 18]. Focus on settings[4] (A), (B) in [HS21] and assume that (X, Λ) is as in Talk 2. Discuss ULA (=universally locally acyclic) sheaves and nearby cycles following [HS21, Sections 3–4]. You probably will not have time to discuss arc descent; rather focus on the statements of Theorems 1.6 and 1.7 from the introduction. See also [LZ20] for a slightly different view on ULA sheaves and [Sch20, Lecture 18].

Talk 4: Relative perverse sheaves [BBD82], [HS21, Section 6]. Follow [HS21, Section 6] and define the perverse t-structure, give some details on its construction focussing on settings (A), (B) and the compatibility with the ULA property in [HS21, Theorem 6.7]. Scholze's talk [Sch21] might be helpful as well. For absolute perversity, see [BBD82] and also the course notes [Bh15].

Talk 5: Hyperbolic localization [Br02], [DG13], [Ri16], [Sch20, End of Lecture 22/beginning of Lecture 23]. Follow [Ri16] and introduce attractors, repellers, fixed points for schemes $X \to S$, show their representability assuming that the $\mathbb{G}_m$-action is Zariski locally linearizable and state Braden's theorem in the relative setting [Ri16, Theorem B]. (For examples of attractors/repellers, see Talk 8 and/or speak to Timo.) Show that hyperbolic localization preserves ULA sheaves, using the compatibility with nearby cycles and [HS21, Theorem 4.4 (iv)]. The proof of Braden's theorem in the relative setting follows [Br02] closely. For a more explicit description of $X^\pm \to X^0$ as a jet bundle and a different proof of Braden's theorem, see [DG13].

4. Geometric background

Throughout, we fix an algebraically closed field k and a split reductive k-group G. We also fix an auxiliary pair $T \subset B$ of a maximal split torus contained in a Borel subgroup in G. Let $X_*(T)^+$ be the B-dominant cocharacters.

Talk 6: Affine Grassmannians [Zh16, Sections 1.2, 1.3, 2.1], [Ri19]. The general outline follows [FS21, Section VI.1], but translated to equicharacteristic and the case of a single modification, first. The aim of the talk is to give an overview on the geometry of affine Grassmannians and to clarify its relation to the double quotient $G(O)\backslash G(F)/G(O)$ appearing in Talk 1. For generalities on ind-schemes, the speaker is referred to [Ri19, Section 1]. For an k-algebra R, let $\mathbb{D}_R = \operatorname{Spec} R[\![t]\!]$, called the disc, and $\mathbb{D}_R^* = \operatorname{Spec} R(\!(t)\!)$, called the punctured disc. Introduce the loop group $LG(R) = G(\mathbb{D}_R^*)$ and its subgroup functor of "contractible" loops $L^+G(R) = G(\mathbb{D}_R)$, see [Zh16, Section 1.3] and [Ri19, Section 3.3].

Define the local Hecke stack $\mathrm{Hk}_G : \mathrm{Alg}_k \to \mathrm{Gpd}$ as the groupoid valued functor on the category of k-algebras R parametrizing pairs of (right) G-torsors $\mathcal{E}_0, \mathcal{E}_1$ on $\mathbb{D}_R$ together with an isomorphism

[2] For the seminar, it is (mostly) sufficient to work with triangulated categories as opposed to stable ∞-categories.

[3] In the notation of *loc. cit.*, we have $\mathfrak{m}^n = 0$ for some $n \geqslant 0$ so that all formulas simplify: one can forget about completions.

[4] Ignore ∞-categories; we work with the underlying triangulated categories.

$\alpha\colon \mathcal{E}_0 \cong \mathcal{E}_1$ over $\mathbb{D}_R^*$. This defines a stack in the fpqc topology. If you are not familiar with stacks, we highly recommend reading [He09], see also [StP]. Similarly to [FS21, Proposition VI.1.7], Hk_G identifies with the étale stack $L^+G\backslash LG/L^+G$, see also [Zh16, Proposition 1.3.6, Lemma 1.3.7]. In particular[5], $\mathrm{Hk}_G(k)/\simeq = G(O)\backslash G(F)/G(O)$ for $F := k(\!(t)\!) \supset k[\![t]\!] =: O$, compare with Talk 1.

Define the affine Grassmannian as the functor Gr_G over Hk_G additionally parametrizing a trivialization $\beta\colon \mathcal{E}_1 \cong \mathcal{E}^{\mathrm{triv}}$ over $\mathbb{D}_R$ where $\mathcal{E}^{\mathrm{triv}}$ denotes the trivial G-torsor. The projection $\mathrm{Gr}_G \to \mathrm{Hk}_G$ is a (left) L^+G-torsor and identifies Gr_G with the étale quotient LG/L^+G. We are interested in its L^+G-equivariant geometry. The reason for passing from Hk_G to Gr_G is that the latter has a nice geometric structure: Gr_G is an ind-projective ind-scheme over k (=infinite union of projective k-schemes with transition maps closed immersions), see [Zh16, Theorem 1.2.2]. State the fundamental geometric properties on connected components $\pi_0(\mathrm{Gr}_G) = \pi_0(LG) = \pi_1(G)$ and that Gr_G is reduced if and only if G is semi-simple and $\mathrm{char}(k) \nmid \#\pi_1(G)$, see [Zh16, Theorem 1.3.11] and [HLR20, Proposition 7.7].

Follow [Zh16, Section 2.1] and introduce the Cartan decomposition, Schubert varieties/cells $\mathrm{Gr}_{G,(\leqslant)\mu}$, give the dimension formula and their closure relations. In particular, conclude that (1.3) holds true and that on topological spaces $|\mathrm{Hk}_G| \cong X_*(T)^+$ where the target is equipped with the topology induced by the dominance order. Give some examples of Schubert varieties if time permits.

Talk 7: Beilinson-Drinfeld Grassmannians [FS21, Section VI.1]**,** [Zh16, Section 3.1]**.** Again, we follow the outline [FS21, Section VI.1], but now deal with several modifications, compare with Talk 6. Fix an auxiliary smooth projective geometrically connected k-curve X, for simplicity, take $X = \mathbb{P}^1_k$. For a finite index set I, let X^I denote the I-fold product of X with itself.

Define the global (or Beilinson-Drinfeld) Hecke stack $\mathrm{Hk}_{G,I}\colon \mathrm{Alg}_k \to \mathrm{Gpd}$ as the groupoid valued functor on the category of k-algebras R parametrizing a point $x_I = \{x_i\}_{i\in I} \in X^I(R)$, pairs of (right) G-torsors $\mathcal{E}_0, \mathcal{E}_1$ on X_R together with an isomorphism $\alpha\colon \mathcal{E}_0 \cong \mathcal{E}_1$ over $X_R\backslash\Gamma_{x_I}$ where $\Gamma_{x_I} = \cup_{i\in I}\Gamma_{x_i}$ and $\Gamma_{x_i} \subset X_R$ denotes the graph of x_i, compare with [FS21, Definition VI.1.6]. If $I = \{*\}$ is a singleton, then, for any point $x_0 \in X(k)$, the fiber $\mathrm{Hk}_{G,I}|_{x_0}$ identifies with the local Hecke stack Hk_G after choosing a local coordinate at x_0, see [Zh16, Section 1.4]. Follow [Zh16, Section 3.1] and introduce the loop groups L_IG, L_I^+G, introduce the Beilinson–Drinfeld Grassmannians $\mathrm{Gr}_{G,I}$ additionally parametrizing a trivialization $\mathcal{E}_1 \cong \mathcal{E}^{\mathrm{triv}}$ over X_R, state $\mathrm{Hk}_{G,I} = L_I^+G\backslash L_IG/L_I^+G$ and $\mathrm{Gr}_{G,I} = L_IG/L_I^+G$ as étale stacks, respectively sheaves over X^I, see [Zh16, Proposition 3.1.9]. Again, $\mathrm{Gr}_{G,I} \to X^I$ is an ind-projective ind-scheme, see [Zh16, Theorem 3.1.3], and, for a tuple $\mu_I = \{\mu_i\}_{i\in I} \in X_*(T)^+$, the global Schubert variety $\mathrm{Gr}_{G,I,\leqslant\mu_I} \to X^I$ is projective. If $I = \{1,2\}$ contains two elements, then explain the factorization structure of $\mathrm{Hk}_{G,I}$, respectively $\mathrm{Gr}_{G,I}$ as in [Zh16, Proposition 3.1.13] and its behaviour on global Schubert varieties [Zh16, Proposition 3.1.14].

Talk 8: Semi-infinite orbits [BR18, Section I.3.2]**,** [FS21, Section VI.3]**,** [HR18, Section 3.3]**.** Follow [HR18, Section 3.3], and fix a cocharacter $\lambda\colon \mathbb{G}_m \to T \subset G$. Introduce the associated pair of parabolic parabolic subgroups P^+, P^- in G with $\mathrm{Lie}(P^+) = \mathrm{Lie}(G)_{\lambda\geqslant 0}$, respectively $\mathrm{Lie}(P^-) = \mathrm{Lie}(G)_{\lambda\leqslant 0}$, the Levi subgroup $M = P^+\cap P^-$ with $\mathrm{Lie}(M) = \mathrm{Lie}(G)_{\lambda=0}$, the maps of k-group schemes $M \leftarrow P^\pm \to G$ and the corresponding diagram $\mathrm{Gr}_M \leftarrow^{q^\pm} \mathrm{Gr}_{P^\pm} \to^{p^\pm} \mathrm{Gr}_G$ on affine Grassmannians. Also, λ defines the map $\mathbb{G}_m \subset L^+\mathbb{G}_m \to^{L^+\lambda} L^+G$, thus an action of $\mathbb{G}_m$ on Gr_G. Show that the action is Zariski locally linearizable and state [HR18, Proposition 3.3], identifying $\mathrm{Gr}_{P^\pm} = (\mathrm{Gr}_G)^\pm$ as the attractor/repeller and $\mathrm{Gr}_M = (\mathrm{Gr}_G)^0$ as the fixed points (compare with Talk 5). Also, explain [HR18, Lemma 3.7] on the geometry of attractors/repellers and add in i) that the map $p^\pm$ is bijective, due to the Iwasawa decomposition. Explain that the same results hold for the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{(\text{-}),I} = L_I(\text{-})/L_I^+(\text{-})$ for any finite index set I and $X = \mathbb{P}^1_k$, say, see Talk 7 and [HR18b, Theorem A]. State [FS21, Corollary VI.3.5], of course, translated to our setting for $\Lambda = \mathbb{Z}/\ell^n$, see also Talk 5. Here, the category of sheaves with bounded support on ind-schemes like $\mathrm{Gr}_{G,I}$ is simply defined as the union of the categories $\mathrm{D}(\mathrm{Gr}_{G,I,\leqslant\mu_I}, \Lambda)$; it is denoted $\mathrm{D}(\mathrm{Gr}_{G,I}, \Lambda)^{\mathrm{bd}}$, and similarly $\mathrm{D}_{\mathrm{cons}}(\mathrm{Gr}_{G,I}, \Lambda)^{\mathrm{bd}}$ for the perfect-constructible complexes with bounded support.

[5] This also holds for finite fields k and then involves the vanishing of $\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}k, L^+G)$, see [Ri19, Corollary 3.22].

If λ is B-dominant regular, then $M = T$ is the maximal torus and $P^+ = B$ is the Borel subgroup. The map $p^{\pm}$ defines the stratification into so-called semi-infinite orbits: the connected components $\mathrm{Gr}_B = \sqcup_\nu S_\nu$ are enumerated by $\pi_0(\mathrm{Gr}_T) = X_*(T)$ and each S_ν, called a semi-infinite orbit, immerses locally closed into Gr_G. More explicitly, $S_\nu(k) = LU(k)\cdot\nu(t)$ where $U \subset B$ is the unipotent radical. Follow the argument in [FS21, Corollary VI.3.8] and prove that the intersections with Schubert varieties $S_\nu \cap \mathrm{Gr}_{G,\leqslant\mu}$ are equidimensional of dimension $\langle\rho, \lambda+\nu\rangle$, whenever non-empty, see also [BR18, Section I, Theorem 5.2]. (Note that [FS21, Proposition VI.3.7] already follows from the ind-affineness of $q^{\pm}$ proven in [HR18, Lemma 3.7 ii)].) If time permits, explain the picture of semi-infinite orbits for $G = \mathrm{SL}_2$ in terms of the Bruhat-Tits building as in [BR18, Section I.3.2].

5. Construction of the Satake category

We continue to fix an algebraically closed field k and a (split) reductive k-group G. Also, fix an auxiliary pair $T \subset B$ of a maximal torus contained in a Borel subgroup in G and the auxiliary curve $X := \mathbb{P}^1_k$. Let $\Lambda = \mathbb{Z}/\ell^n$ for a prime ℓ invertible in k, used as sheaf coefficients.

Talk $8\frac{1}{2}$: Technique talk on the formalism of equivariant sheaves. This is a short talk (30–45 minutes) given by one of the organizers on the formalism of equivariant sheaves. The aim of the talk is to justify (some of) the constructions appearing in Talk 9–10. In particular, we define the categories $\mathrm{D}(\mathrm{Hk}_G, \Lambda)^{\mathrm{bd}}$ together with the pullback functors to $\mathrm{D}(\mathrm{Gr}_G, \Lambda)^{\mathrm{bd}}$ analogously to those appearing in [FS21, Section VI] and their variants for the Beilinson–Drinfeld Grassmannian.

Talk 9: Definition of the Satake category and convolution [Zh16, Section 5.1]**,** [FS21, Sections VI.7, VI.9]**.** Introduce the Satake category $\mathrm{Sat}_{G,\Lambda}$ as the full subcategory of $\mathrm{D}(\mathrm{Hk}_G, \Lambda)^{\mathrm{bd}}$ of objects whose pullback to Gr_G is flat (absolute) perverse, in analogy to [FS21, Definition VI.7.8] (the ULA property is automatic because the base scheme is the spectrum of the field k). Note that a perverse sheaf on Hk_G is the same as a perverse sheaf on Gr_G such that there exists an isomorphism $p^*A \simeq a^*A$ where $p, a\colon L^+G \times \mathrm{Gr}_G \to \mathrm{Gr}_G$ is the projection, respectively action, compare with [Zh16, Lemma A.1.2]. That is, the pullback functor $\mathrm{Perv}(\mathrm{Hk}_G, \Lambda) \to \mathrm{Perv}(\mathrm{Gr}_G, \Lambda)$ is fully faithful with essential image those A such that $p^*A \simeq a^*A$. Next, for $A, B \in \mathrm{D}(\mathrm{Hk}_G, \Lambda)^{\mathrm{bd}}$, define the convolution $A \star B = q^* m_*(A \boxtimes B)$ by pull-push[6] along the diagram

$$\mathrm{Hk}_G \times \mathrm{Hk}_G \xleftarrow{q} \mathrm{Hk}_G \tilde{\times} \mathrm{Hk}_G \xrightarrow{m} \mathrm{Hk}_G, \tag{5.1}$$

where $\mathrm{Hk}_G \tilde{\times} \mathrm{Hk}_G(R)$ is the convolution local Hecke stack parametrizing G-torsors $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$ on $\mathbb{D}_R$ together with isomorphisms $\mathcal{E}_0 \cong \mathcal{E}_1$ and $\mathcal{E}_1 \cong \mathcal{E}_2$ over $\mathbb{D}_R^*$ and the maps q, respectively m are given by mapping $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ to the pair $((\mathcal{E}_0, \mathcal{E}_1), (\mathcal{E}_1, \mathcal{E}_2))$, respectively $(\mathcal{E}_0, \mathcal{E}_2)$. Make the construction more explicit as in [Zh16, Section 5.1] or [BR18, Section I.6.2]. For an object $A \in \mathrm{Sat}_{G,\Lambda}$, define $\mathrm{F}(A) = \mathrm{H}^*(\mathrm{Gr}_G, A)$ which is a Λ-module. This constructs the triple $(\mathrm{Sat}_{G,\Lambda}, \star, \mathrm{F})$. We aim to prove that $\star$ preserves $\mathrm{Sat}_{G,\Lambda}$, that F takes values in finite projective Λ-modules and that the triple has a Tannakian type structure. The key tool is a globalization to the global Hecke stack as follows.

Introduce the variant of the Satake category for the global Hecke stack $\mathrm{Hk}_{G,I} \to X^I$, see [FS21, Section VI.9]. Namely, for any finite index set I, the category $\mathrm{Sat}^I_{G,\Lambda}$ is the full subcategory of $\mathrm{D}(\mathrm{Hk}_{G,I}, \Lambda)^{\mathrm{bd}}$ of objects whose pullback to $\mathrm{Gr}_{G,I} \to X^I$ is flat (relatively) perverse and ULA[7]. Mention that the global convolution $\star_I$ and F_I can be defined analogously for $\mathrm{Hk}_{G,I}$, leading to the triple $(\mathrm{Sat}^I_{G,\Lambda}, \star_I, F_I)$. If $I = \{*\}$ is a singleton and $x_0 \in X(k)$, then, under the identification $\mathrm{Hk}_{G,I}|_{x_0} \simeq \mathrm{Hk}_G$, pullback to x_0 induces an equivalence $\mathrm{Sat}^I_{G,\Lambda} \cong \mathrm{Sat}_{G,\Lambda}$ compatible with the structures.

[6] Some of the technical details needed to make this precise appear in Talk $8\frac{1}{2}$.

[7] We ignore the subtlety arising in [FS21, Proposition VI.6.2].

Talk 10: Constant term functor and compatibility with tensor structures [FS21, Section VI.6, VI.7, VI.8, VI.9]**.** Give an overview on the results in [FS21, Section VI.9]: $\mathrm{Sat}^I_{G,\Lambda}$ is stable under convolution $\star_I$; it has the structure of a symmetric monoidal category; F_I takes values in $\mathrm{LocSys}(X^I) = \mathrm{Mod}_\Lambda^{\mathrm{fg,proj}}$ (here we use $X = \mathbb{P}^1_k$ and that k is separably closed) and has the structure of a tensor functor (beware the sign issue); for each parabolic subgroup P with Levi M the constant term functor $\mathrm{CT}_{P,I}[\deg P]\colon \mathrm{Sat}^I_{G,\Lambda} \to \mathrm{Sat}^I_{M,\Lambda}$ has the structure of a tensor functor. The main tool is the study of the constant term functor $\mathrm{CT}_{B,I}$ for the Borel, reducing many statement to the (easy) case of the split maximal torus T, [FS21, Sections VI.6, VI.7, VI.8]. Taking $I = \{*\}$, we obtain that $(\mathrm{Sat}_{G,\Lambda}, \star, \mathrm{F})$ has a Tannakian type structure.

6. Proof of the equivalence

Talk 11: Tannakian reconstruction [FS21, Section VI.10]**,** [BR18, Section I.13]**.** Explain the results from [FS21, Section VI.10], see also [BR18, Section I.13]. In particular, explain the construction of the flat Λ-group scheme $\check{G} = \mathrm{Aut}^\star(F)$.

Talk 12: Identification of the dual group [FS21, Section VI.11]**,** [BR18, Section I.14]**.** Follow [FS21, Section VI.11] and prove that $\check{G} = \widehat{G}_\Lambda$ as pinned groups.

References

[AuP] The automorphic project, https://automorphic.jh.edu/. 2

[BR18] P. Baumann, S. Riche: *Notes on geometric Satake*, https://arxiv.org/abs/1703.07288. 2, 4, 5, 6

[BBD82] A. Beilinson, J. Bernstein, P. Deligne: *Faisceaux Pervers*, http://www.numdam.org/item/AST_1982__100__1_0/. 3

[Bh15] B. Bhatt: *Perverse sheaves*, lecture notes, https://web.math.princeton.edu/~takumim/MATH731.pdf. 3

[BS13] B. Bhatt, P. Scholze: *The pro-étale topology for schemes*, https://arxiv.org/abs/1309.1198. 3

[Br02] T. Braden: *Hyperbolic localization of intersection cohomology*, https://arxiv.org/abs/math/0202251. 3

[DG13] V. Drinfeld, D. Gaitsgory: *On a theorem of Braden*, https://arxiv.org/abs/1308.3786. 3

[FS21] L. Fargues, P. Scholze: *Geometrization of the local Langlands correspondence*, . 1, 2, 3, 4, 5, 6

[Ga21] M. Gallauer: *An introduction to six functor formalisms*, lecture notes, https://guests.mpim-bonn.mpg.de/gallauer/docs/m6ff.pdf. 3

[Gr98] B. Gross: *On the Satake isomorphism*, in Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser. **254**. 2

[HLR20] T. Haines, J. Lourenço, T. Richarz: *On the normality of Schubert varieties: remaining cases in positive characteristic*, https://arxiv.org/abs/1806.11001. 4

[HR18] T. Haines, T. Richarz: *The test function conjecture for parahoric local models*, https://arxiv.org/abs/1801.07094. 4, 5

[HR18b] T. Haines, T. Richarz: *The test function conjecture for local models of Weil restricted groups*, https://arxiv.org/abs/1805.07081. 4

[HRS21] T. Hemo, T. Richarz, J. Scholbach: *Constructible sheaves on schemes and a categorical Künneth formula*, https://arxiv.org/pdf/2012.02853.pdf. 3

[HS21] D. Hansen, P. Scholze: *Relative perversity*, https://arxiv.org/abs/2109.06766. 2, 3

[He09] J. Heinloth: *Lectures on the moduli stack of vector bundles on a curve*, https://www.uni-due.de/~hm0002/Artikel/StacksCourse_v2.pdf 4

[LZ20] Q. Lu, W. Zheng: *Categorical traces and a relative Lefschetz–Verdier formula*, https://arxiv.org/abs/2005.08522. 3

[Ri16] T. Richarz: *Spaces with $\mathbb{G}_m$-action, hyperbolic localization and nearby cycles*, https://arxiv.org/abs/1611.01669. 3

[Ri19] T. Richarz: *Notes on affine Grassmannians*, https://timo-richarz.com/wp-content/uploads/2020/02/BoAG_02.pdf. 2, 3, 4

[Sch20] P. Scholze: *Geometrization of the local Langlands correspondence*, lecture notes and videos, https://www.math.uni-bonn.de/people/scholze/. 2, 3

[Sch21] P. Scholze: *Relative perversity*, talk at Oberwolfach 2021, Hessenbox. 3

[StP] The Stacks project, https://stacks.math.columbia.edu. 3, 4

[Yu15] Z. Yun: *The Langlands correspondence for global function fields*, lecture notes by T. Feng, https://www.mit.edu/~fengt/FFL.pdf. 2

[Zh16] X. Zhu: *An introduction to affine Grassmannians and the geometric Satake equivalence*, https://arxiv.org/abs/1603.05593 2, 3, 4, 5

Email address: arnaud.mayeux@uca.fr
Email address: richarz@mathematik.tu-darmstadt.de
Email address: simon.riche@uca.fr