TALK 8: SEMI-INFINITE ORBITS

SIMON RICHE

In these notes we explain what are the "semi-infinite orbits," some very important locally-closed sub-ind-schemes of the affine Grassmannian. We also explain a proof of the dimension estimates for intersections of semi-infinite orbits with spherical orbits (Proposition 4.5) different from the original proof of Mirković–Vilonen [MV07], due to Fargues–Scholze [FS21]. (For some details about the original proof of Mirković–Vilonen, see [BR18, §1.5].) This statement will be used crucially in later talks, in particular to show that (an appropriate renormalization of) the constant term functor sends perverse sheaves to perverse sheaves.

The "dynamical" approach to semi-infinite orbits that we follow here was present in [MV07] in some way, but was made more rigorous in [HR18, HR18b].

1. Preliminaries

1.1. Complements on ind-schemes.

1.1.1. *Definition.* We follow the conventions on ind-schemes developed in [Ri19]. Therefore, we denote by AffSch the category of affine schemes, which identifies with the opposite of the category Rings of unital commutative rings. Any scheme X defines a functor $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ via

$$T \mapsto \mathrm{Hom}(T, X),$$

and this assignment defines a fully faithful functor from the category of schemes to the category of functors $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$; we will often identify the category of schemes with its image under this functor. As usual, when $T = \mathrm{Spec}(R)$ for some $R \in \mathrm{Rings}$ we write $X(R) = \mathrm{Hom}(\mathrm{Spec}(R), X)$.

An ind-scheme is a functor

$$X : \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$$

such that there exists a filtered poset $(I, \leqslant)$ and an inductive system $(X_i : i \in I)$ of schemes such that[1]

$$X \cong \mathrm{colim}_{i \in I} X_i,$$

and moreover each transition morphism $X_i \to X_j$ is a closed immersion (for $i, j \in I$ with $i \leqslant j$). We denote by IndSch the full subcategory of the category of functors $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ whose objects are ind-schemes. We will call an isomorphism $X \cong \mathrm{colim}_{i \in I} X_i$ a *presentation* of X; whenever we write an ind-scheme in this way, we implicity assume that the X_i's form an inductive system of schemes with closed immersions as transition morphisms, as above. As explained in [Ri19, Lemma 1.10], IndSch is closed under fiber products.[2]

Of course, each scheme defines an ind-scheme, and this assignment defines a fully faithful functor from the category of schemes to IndSch. Note that if X is a scheme and $Y = \mathrm{colim}_{i \in I} Y_i$ is an ind-scheme, then the canonical map

$$\mathrm{colim}_{i \in I} \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, Y) \tag{1.1}$$

[1] Here, recall that colimits of functors can be computed termwise: if $(F_i : i \in I)$ is an inductive system of functors $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$, then its colimit satisfies $(\mathrm{colim}_i F_i)(T) = \mathrm{colim}_i F_i(T)$ for all $T \in \mathrm{AffSch}$.

[2] Here again, fiber products of functors can be computed termise, see [StP, Tag 0022].

is injective, but not necessarily surjective. It *is* surjective (hence an isomorphism) if X is quasi-compact, though; see [Ri19, Ex. 1.26].[3]

We will also consider ind-schemes over a fixed base scheme S. Such a datum consists of an ind-scheme X together with a morphism $X \to S$. We will denote by IndSch_S the category whose objects are ind-schemes over S and whose morphisms are morphisms of ind-schemes compatible with the given morphisms to S. In fact, if we denote by AffSch_S the category of affine schemes T endowed with a morphism $T \to S$, then the category of schemes over S embeds fully faithfully in the category of functors $\mathrm{AffSch}_S^{\mathrm{op}} \to \mathrm{Sets}$, and IndSch_S identifies with the category of functors $\mathrm{AffSch}_S^{\mathrm{op}} \to \mathrm{Sets}$ isomorphic to $\mathrm{colim}_{i \in I} X_i$ where $(X_i : i \in I)$ is a filtered inductive system of schemes over S such that the transition morphisms $X_i \to X_j$ are closed immersions (over S). Of course, in case $S = \mathrm{Spec}(R)$ for some $R \in \mathrm{Rings}$, then the category AffSch_S identifies with the opposite of the category Alg_R of unital commutative R-algebras.

1.1.2. *Immersions.* Recall that if X, Y are ind-schemes and $f : X \to Y$ is a morphism, then we say that f is *representable by schemes*[4] if for any scheme Z and any morphism $Z \to Y$ the fiber product $X \times_Y Z$ is a scheme. Similarly, we say that f is *representable by a locally closed, resp. closed, resp. open, immersion* if for any scheme Z and any morphism $Z \to Y$ the fiber product $X \times_Y Z$ is a scheme and the induced morphism $X \times_Y Z \to Z$ is a locally closed, resp. closed, resp. open, immersion of schemes. (In fact, by [Ri16, Lemma 1.7] it suffices to check these properties when Z is affine. This turns out to be very useful since (1.1) is an isomorphism in this case.)

1.1.3. *Underlying topological space and connected components.* If X is an ind-scheme, then its underlying topological space $|X|$ is defined as the colimit over the sets $X(K)$ where K runs over fields, with an appropriate topology; see [Ri19, Definition 1.11]. In fact, if $X = \mathrm{colim}_i X_i$ is a presentation, then we have a canonical identification

$$|X| = \mathrm{colim}_i |X_i|$$

where the right-hand side is equipped with the colimit topology.[5]

For any scheme X, any connected component of the underlying topological space $|X|$ admits a canonical scheme structure, which is characterized by the property that the corresponding embedding is a flat closed immersion, see [StP, Tag 04PX]. If X is an ind-scheme and $X = \mathrm{colim}_i X_i$ is a presentation, the connected components of $|X|$ are increasing unions of connected components of the spaces $|X_i|$. Hence they admit a canonical ind-scheme structure.

It is not clear to me how this structure behaves in a general setting (e.g., if the inclusion of a connected component is representable by a closed immersion), but under appropriate technical conditions that will be satisfied in all the cases we want to consider it is well behaved, as we now explain. Consider an ind-scheme X with a presentation $X = \mathrm{colim}_i X_i$ such that each X_i is Noetherian and each transition morphism $X_i \to X_j$ induces an injection on connected components.

Lemma 1.1. *Under the assumptions above, for any connected component Y of X the natural morphism $Y \to X$ is representable by an open and closed immersion.*

Proof. Our assumptions imply in particular that X_i has a finite number of connected components for any i (see [StP, Tag 0052]); in particular, these connected components are open and closed. As explained above, if Y is a connected component of X we have a presentation $Y = \mathrm{colim}_i Y_i$ where Y_i is a connected component of X_i for any i, and the closed immersion $X_i \to X_j$ restricts to a closed

[3] Idea of the proof: if $X = \mathrm{Spec}(R)$ is affine, then a morphism $\mathrm{Spec}(R) \to Y$ is the same as an element of $Y(R)$, hence factors through some Y_j by definition. In general, a quasi-compact scheme is a finite union of affine opens; on each such open the morphism must factor through some Y_j, and then one can use the assumption that I is filtered to see that one can choose j which works for all opens at the same time.

[4] One sometimes also finds the terminology "f is schematic."

[5] Concretely, this means that a subset of $|X|$ is open, resp. closed, iff its intersection with each $|X_i|$ is open, resp. closed.

immersion $Y_i \to Y_j$ for any $i \leqslant j$. Consider now an affine scheme Z and a morphism $Z \to X$. There exists i such that this morphism factors through X_i, and then we have

$$Y \times_X Z = \mathrm{colim}_{j \geqslant i} Y_j \times_{X_j} Z.$$

Now, for any $j \geqslant i$ we have

$$Y_j \times_{X_j} Z = (Y_j \times_{X_j} X_i) \times_{X_i} Z.$$

The fact that the morphism $X_i \to X_j$ induces an injection on connected components means that the underlying topological space of $Y_j \times_{X_j} X_i$ is Y_i; since the natural morphism $Y_j \times_{X_j} X_i \to X_i$ is a flat closed immersion (because so is $Y_j \to X_j$), it follows that $Y_j \times_{X_j} X_i = Y_i$; in particular,

$$Y \times_X Z = Y_i \times_{X_i} Z$$

is a scheme. Since Y_i is open and closed in X_i, we deduce that the morphism $Y \times_X Z \to Z$ is an open and closed immersion, as desired. □

1.1.4. *Additional properties.* Recall that if X, Y are ind-schemes and $f : X \to Y$ is a morphism, then f is said to be *ind-affine* if there exist presentations $X = \mathrm{colim}_i X_i$ and $Y = \mathrm{colim}_j Y_j$ such that f is represented by a pro-ind-system of morphisms $f_{i,j} : X_i \to Y_j$ which are affine.

If X is an ind-scheme over $\mathrm{Spec}(k)$ for some base field k, we will say that X is of ind-finite-type if it admits a presentation $X = \mathrm{colim}_i X_i$ (over k) where each X_i is of finite type over k.

Finally, we say that an ind-scheme X over a scheme S is separated if the diagonal morphism $X \to X \times_S X$ is representable by a closed immersion. For this condition to hold, it suffices that X admits a presentation $X = \mathrm{colim}_i X_i$ over S where each morphism $X_i \to S$ is separated, see [Ri16, Exercise 1.31]. In fact, if this property holds, given any presentation $X = \mathrm{colim}_i X_i$ over S, each scheme X_i is separated over S.[6]

1.1.5. *Sheaves on ind-schemes.* Let ℓ be a prime number, and let Λ be a finite ℓ-torsion ring. Recall that for any quasi-compact and quasi-separated scheme X on which ℓ is invertible one can consider the derived category $D_{\text{ét}}(X, \Lambda)$ of étale sheaves of Λ-modules on X, cf. Talk 2 or Talk $8\frac{1}{2}$. Let X be an ind-scheme which admits a presentation $X = \mathrm{colim}_i X_i$ where each X_i is quasi-compact and quasi-separated.[7] Then one defines the derived category of bounded[8] sheaves on X as

$$D_{\text{ét}}(X, \Lambda)^{\text{bd}} = \mathrm{colim}_i D_{\text{ét}}(X_i, \Lambda)$$

where $X = \mathrm{colim}_i X_i$ is any presentation over S where each X_i is quasi-compact and quasi-separated and the transition functor $D_{\text{ét}}(X_i, \Lambda) \to D_{\text{ét}}(X_j, \Lambda)$ is given by pushforward along the closed immersion $X_i \to X_j$ for $i \leqslant j$.[9] To justify that this definition makes sense, one needs to check that the category does not depend on the choice of presentation, up to canonical equivalence. In fact, if we are given another presentation $X = \mathrm{colim}_j Y_j$ where each Y_j is quasi-compact, then the identity morphism in

$$\mathrm{colim}_i X_i = X = \mathrm{colim}_j Y_j$$

is represented by pro-systems $f = (f_i)_{i \in I}$ and $g = (g_j)_{j \in J}$ in

$$\lim_i \mathrm{colim}_j \mathrm{Hom}(X_i, Y_j) \quad \text{and} \quad \lim_j \mathrm{colim}_i \mathrm{Hom}(Y_j, X_i)$$

respectively such that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. (Here we use our quasi-compactness assumption.) Given $i \in I$, if $j \in J$ is such that f_i is represented by a morphism $f_{i,j} : X_i \to Y_j$ then we have a canonical functor

$$(f_{i,j})_* : D_{\text{ét}}(X_i, \Lambda) \to D_{\text{ét}}(Y_j, \Lambda)$$

[6] To check this, one notes (by consideration of points over each affine scheme) that $(X_i \times_S X_i) \times_{X \times_S X} X = X_i$ where the morphism $X \to X \times_S X$ is the diagonal morphism, and then one uses the definition.

[7] If S is quasi-separated and X is a separated ind-scheme over S, given the comments in §1.1.4, here it suffices to assume that X admits a presentation over S in which all schemes are quasi-compact.

[8] Here, "bounded" refers to the fact that the complexes are supported on a scheme X_i; this is unrelated to the notion of bounded complex of objects in a category.

[9] Note that this functor is fully faithful. One can therefore safely think of an object in $D_{\text{ét}}(X, \Lambda)^{\text{bd}}$ as an object in some category $D_{\text{ét}}(X_i, \Lambda)$, where i can be replaced by a larger index whenever convenient.

which defines a functor

$$(f_i)_* : D_{\text{ét}}(X_i, \Lambda) \to \text{colim}_j D_{\text{ét}}(Y_j, \Lambda)$$

independent of the choice of j above (by compatibility of $*$-pushforward with composition of morphisms). Taken together, these functors define a functor

$$f_* : \text{colim}_i D_{\text{ét}}(X_i, \Lambda) \to \text{colim}_j D_{\text{ét}}(Y_j, \Lambda)$$

(for the same reason). We similarly get a functor

$$g_* : \text{colim}_j D_{\text{ét}}(Y_j, \Lambda) \to \text{colim}_i D_{\text{ét}}(X_i, \Lambda).$$

The fact that $g \circ f = \text{id}$ means that for fixed $i \in I$, if f_i is represented by some morphism $f_{i,j} : X_i \to Y_j$, and if g_j is represented by some morphism $g_{j,l} : Y_j \to X_l$, then $g_{j,l} \circ f_{i,j} : X_i \to X_l$ is the closed immersion given in our inductive system. From this one sees that $g_* \circ f_* = \text{id}$. One similarly checks that $f_* \circ g_* = \text{id}$, which finishes the verification of our claim.

1.2. **Affine Grassmannians.**

1.2.1. *Definition.* Let k be an algebraically closed field.[10] Given a k-algebra R, we denote by $R[\![z]\!]$ the k-algebra of power series in the indeterminate z with coefficients in R, and by $R(\!(z)\!)$ the localization of $R[\![z]\!]$ with respect to z (i.e. the algebra of formal Laurent series in z with coefficients in R). If R is a field, then $R(\!(z)\!)$ is the field of fractions of the integral ring $R[\![z]\!]$.

Recall that if G is a smooth affine group scheme over k, then the associated loop group LG is the functor $\text{Alg}_k \to \text{Sets}$ defined by

$$LG(R) = G(R(\!(z)\!)).$$

The positive loop group L^+G is the subfunctor defined by

$$L^+G(R) = G(R[\![z]\!]).$$

It is a standard fact that L^+G is represented by an affine group scheme over k, and that LG is represented by an ind-affine group ind-scheme over k; see [Zh16, Proposition 1.3.2].

The affine Grassmannian Gr_G is the fpqc sheaf[11] on the category Alg_k associated with the functor

$$R \mapsto LG(R)/L^+G(R).$$

It is known that Gr_G is represented by a separated ind-scheme of ind-finite type, see [Zh16, Theorem 1.2.2 and Proposition 1.3.6] or [Ri19, Theorem 3.4 and Proposition 3.18]. The proof of this fact in case G is reductive is reviewed in §1.2.4 below; the general case is not very different (see Talk 6 for details). We will denote by $[0] \in \text{Gr}_G(k)$ the base point.

1.2.2. *Big cell.* We will also consider the functor $L^-G : \text{Alg}_k \to \text{Sets}$ defined by[12]

$$L^-G(R) = G(R[z^{-1}]).$$

It is known that L^-G is represented by an ind-affine group ind-scheme of ind-finite type over k; see [Zh16, §2.3]. There exists a canonical morphism $L^-G \to G$ induced by the ring morphisms $R[z^{-1}] \to R$ sending z^{-1} to 0; the kernel of this morphism is denoted $L^{--}G$. The following statement is somewhat classical; a formal proof can be found in this generality in [HR18, Lemma 3.1].

Lemma 1.2. *Let $[0] \in \text{Gr}_G(k)$ be the base point. Then the orbit morphism*

$$L^{--}G \to \text{Gr}_G, \quad g \mapsto g \cdot [0]$$

is representable by an open immersion.

[10] The assumption that k is algebraically closed is unnecessary for most of the results below. It is made only because the author of these notes feels safer in this setting.

[11] Recall that a fpqc sheaf on Alg_k is a functor $X : \text{Alg}_k \to \text{Sets}$ such that for any collection $R_1, \cdots, R_n$ of k-algebras the natural morphism $X(\prod_i R_i) \to \prod_i X(R_i)$ is an isomorphism, and such that if $R \to R'$ is faithfully flat then the diagram $X(R) \to X(R') \rightrightarrows X(R' \otimes_R R')$ is an equalizer.

[12] In this formula, z^{-1} is treated as a formal variable; this element is not the inverse of anything.

1.2.3. *The case of* GL_n. Let us quickly review the description of $\mathrm{Gr}_{\mathrm{GL}_n}$ in terms of lattices, following [Ri19, §2]. (For a formal definition of what we mean by a lattice, see [Ri19, Definition 2.1].)

Writing $\Lambda_{0,R}$ for the lattice $R[\![z]\!]^n \subset R(\!(z)\!)^n$ (for any $R \in \mathrm{Alg}_k$), we know that we have a presentation $\mathrm{Gr}_{\mathrm{GL}_n} = \mathrm{colim}_{i\geqslant 0}\mathrm{Gr}_{\mathrm{GL}_n,i}$ where $\mathrm{Gr}_{\mathrm{GL}_n,i}$ is the scheme whose set of R-points is the set of $R[\![z]\!]$-lattices $\Lambda \subset R(\!(z)\!)^n$ with

$$z^i\Lambda_{0,R} \subset \Lambda \subset z^{-i}\Lambda_{0,R}.$$

For any k-vector space V, it is known that the functor $\mathrm{Grass}(V)$ sending a k-algebra R to the set of R-submodules $M \subset V \otimes_k R$ such that the quotient $(V \otimes_k R)/M$ is finite locally free is a smooth projective scheme over k (see [GW, §8.4]); in fact it is a disjoint union of the Grassmannians $\mathrm{Grass}_d(V)$ of d-dimensional subspaces in V, and for any d we have a natural closed immersion $\mathrm{Grass}_d(V) \hookrightarrow \mathbb{P}(\bigwedge^d V)$, see [GW, §8.10].

Writing $M_i := z^{-i}\Lambda_{0,k}/z^i\Lambda_{0,k}$, we then have a closed embedding of schemes

$$\mathrm{Gr}_{\mathrm{GL}_n,i} \hookrightarrow \mathrm{Grass}(M_i)$$

which is defined on R-points by $\Lambda \mapsto \Lambda/z^i\Lambda_{0,R}$, hence a closed embedding $\mathrm{Gr}_{\mathrm{GL}_n,i} \hookrightarrow \bigsqcup_d \mathbb{P}(\bigwedge^d M_i)$.

For $i \geqslant 0$, since $\mathrm{Gr}_{\mathrm{GL}_n,i}$ and $\mathrm{Gr}_{\mathrm{GL}_n,i+1}$ are proper, the natural morphism $\mathrm{Gr}_{\mathrm{GL}_n,i} \to \mathrm{Gr}_{\mathrm{GL}_n,i+1}$ is proper too, see [StP, Tag 01W6]. Since this morphism is a monomorphism, it must be a closed immersion by [StP, Tag 04XV].

From these considerations we see that $\mathrm{Gr}_{\mathrm{GL}_n}$ admits a presentation in which all schemes that appear are projective (in particular, of finite type) over k.

1.2.4. *The case of reductive groups.* Now, assume that G is a (connected) reductive group over k. A choice of a faithful representation of G provides a closed immersion $G \hookrightarrow \mathrm{GL}_n$ for some n, and the quotient GL_n/G is automatically affine by the main result of [Ric]. By [Zh16, Proposition 1.2.6], it follows that the induced morphism $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is representable by a closed immersion. In particular, if $\mathrm{Gr}_{\mathrm{GL}_n,i}$ is as above and if we set

$$\mathrm{Gr}_{G,i} := \mathrm{Gr}_G \times_{\mathrm{Gr}_{\mathrm{GL}_n}} \mathrm{Gr}_{\mathrm{GL}_n,i},$$

then $\mathrm{Gr}_{G,i}$ is a scheme, and the natural morphism $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{\mathrm{GL}_n,i}$ is a closed immersion. It is also easily seen that

$$\mathrm{Gr}_G = \mathrm{colim}_{i\geqslant 0}\mathrm{Gr}_{G,i},$$

and that for any $i \geqslant 0$ the natural morphism $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{G,i+1}$ is a closed immersion. In particular, as in the case of GL_n, Gr_G admits a presentation in which all schemes that appear are projective over k.[13]

1.2.5. *Spherical orbits and Schubert varieties.* We continue to assume that G is reductive, and fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. The choice of B determines a system of positive roots for (G, T) (consisting of the nonzero T-weights in the Lie algebra of B), hence a notion of dominant coweights in $X_*(T)$. By [Ric] again the quotient G/T is affine, so that the morphism $\mathrm{Gr}_T \to \mathrm{Gr}_G$ is representable by a closed immersion (again by [Zh16, Proposition 1.2.6]).

Any $\lambda \in X_*(T)$ determines a k-point $z^\lambda \in (LT)(k)$, namely the image under the morphism $L\mathbb{G}_m \to LT$ induced by λ of $z \in (L\mathbb{G}_m)(k) = (k(\!(z)\!))^\times$. The image of this point in $\mathrm{Gr}_T(k)$ will be denoted $[\lambda]$. We will also denote by z^λ and $[\lambda]$ the images of these points in $LG(k)$ and $\mathrm{Gr}_G(k)$ respectively.

Let us denote by $X_*(T)^+ \subset X_*(T)$ the subset of dominant cocharacters. Consider a presentation $\mathrm{Gr}_G = \mathrm{colim}_i\mathrm{Gr}_{G,i}$ as in §1.2.4, so that the L^+G-action on Gr_G is induced by compatible actions on each $\mathrm{Gr}_{G,i}$, and that the action on $\mathrm{Gr}_{G,i}$ factors through a quotient K_i of L^+G which is a smooth group scheme of finite type over k. If $\mu \in X_*(T)^+$, we can choose i such that $[\mu] \in \mathrm{Gr}_{G,i}$. Then it makes sense to consider the K_i-orbit $\mathrm{Gr}_{G,\mu}$ of $[\mu]$, which is a reduced locally closed subscheme of Gr_G. If $\mathrm{Gr}_{G,\leqslant\mu}$ is the closure of $\mathrm{Gr}_{G,\mu}$, endowed with the reduced closed subscheme structure, then $\mathrm{Gr}_{G,\leqslant\mu}$ is a projective reduced scheme over k, and the natural morphism $\mathrm{Gr}_{G,\mu} \to \mathrm{Gr}_{G,\leqslant\mu}$ is

[13] One must be aware that this property does *not* hold when G is not reductive.

an open immersion, see [StP, Tag 03DQ]. It is clear that this construction does not depend on i, nor on the choice of presentation of Gr_G. It is a standard fact that for $\mu, \nu \in X_*(T)^+$, we have $\mathrm{Gr}_{G,\leqslant\mu} \subset \mathrm{Gr}_{G,\leqslant\nu}$ iff $\mu \leqslant \nu$, where $\leqslant$ is the order on $X_*(T)$ such that $\nu \leqslant \nu'$ iff $\nu' - \nu$ is a sum of positive coroots. The varieties $\mathrm{Gr}_{G,\mu}$, resp. $\mathrm{Gr}_{G,\leqslant\mu}$, are called spherical orbits, resp. Schubert cells.

1.2.6. *Connected components.* If G is as in §1.2.5, it is a standard fact that the connected components of Gr_G are parametrized by $X_*(T)/\mathbb{Z}R^\vee$, see e.g. [PR08, Theorem 0.1]. Given a coset $\Lambda \in X_*(T)/\mathbb{Z}R^\vee$, the subscheme $\mathrm{Gr}_{G,\leqslant\nu}$ is contained in the component corresponding to Λ iff $\nu \in \Lambda$.

We will call a coweight λ *minuscule*[14] if $\langle \lambda, \alpha \rangle \in \{0, 1\}$ for any positive root α. If we denote by $X_*(T)_{\min} \subset X_*(T)$ the subset of minuscule coweights, then it is well known that the composition

$$X_*(T)_{\min} \hookrightarrow X_*(T) \twoheadrightarrow X_*(T)/\mathbb{Z}R^\vee$$

is a bijection. Moreover, for any $\Lambda \in X_*(T)/\mathbb{Z}R^\vee$, if λ_0 is the unique minuscule coweight in Λ we have $\lambda_0 \leqslant \lambda$ for any $\lambda \in \Lambda \cap X_*(T)^+$. As a consequence, for any such λ we have $\mathrm{Gr}_{G,\leqslant\lambda_0} \subset \mathrm{Gr}_{G,\lambda}$. The Schubert varieties attached with minuscule coweights will also be called minuscule.

This property implies that the assumptions of Lemma 1.1 are satisfied in this case: given any presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ such that the L^+G-action on Gr_G is induced by compatible actions on the $\mathrm{Gr}_{G,i}$'s which factor through an action of a smooth group scheme of finite type, the connected components in $\mathrm{Gr}_{G,i}$ are determined by the unique minuscule Schubert variety that they contain (because they are closed and L^+G-stable), so that the morphism $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{G,j}$ indeed induces an injection on sets of connected components if $i \leqslant j$. In particular, the embedding of any connected component in Gr_G is representable by an open and closed immersion.

1.3. **Attractors and repellers.** Here we briefly recall the main constructions of Talk 5, and give some complements that will be used below.

1.3.1. *Definitions.* Let us consider a base scheme S, and a scheme X over S. An action of $\mathbb{G}_m$ on X is the datum of a morphism of S-schemes $\mathbb{G}_{m,S} \times_S X \to X$ which satisfies the obvious axioms.[15]

Recall from Talk 5 that we define the functor X^0 of $\mathbb{G}_m$-fixed points in X as sending $T \in \mathrm{AffSch}_S$ to the set of morphisms $T \to X \times_S T$ over T such that the diagram

$$\begin{array}{ccc} \mathbb{G}_{m,T} & \longrightarrow & \mathbb{G}_{m,T} \times_T (X \times_S T) \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \times_S T \end{array}$$

commutes, where the left vertical arrow is the structure morphism, the right vertical arrow is induced by the action morphism $\mathbb{G}_{m,S} \times_S X \to X$, and the horizontal morphisms are induced by the given morphism $T \to X \times_S T$. In other words, given $T \in \mathrm{AffSch}_S$, $X^0(T)$ consists of the T-points of X such that for any affine scheme $T' \to T$ the induced morphism $T' \to X \times_S T'$ commutes with the action of elements in $\mathbb{G}_m(T')$ (where the action on the left-hand side is trivial).

Similarly, we denote by $(\mathbb{A}^1_S)^+$, resp. $(\mathbb{A}^1_S)^-$, the scheme $\mathbb{A}^1_S$ with the natural action of $\mathbb{G}_m$, resp. the opposite of the natural action. Then we define X^+ as the functor sending $T \in \mathrm{AffSch}_S$ to the set of morphisms $(\mathbb{A}^1_T)^+ \to X \times_S T$ over T such that the diagram

$$\begin{array}{ccc} \mathbb{G}_{m,T} \times_T (\mathbb{A}^1_T)^+ & \longrightarrow & \mathbb{G}_{m,T} \times_T (X \times_S T) \\ \downarrow & & \downarrow \\ (\mathbb{A}^1_T)^+ & \longrightarrow & X \times_S T \end{array}$$

[14] Note that our definition is more general than that given e.g. in Bourbaki. In particular, for us 0 is a minuscule coweight.

[15] Here, $\mathbb{G}_{m,S}$ is the group scheme over S sending $T \in \mathrm{AffSch}_S$ to $\mathscr{O}(T)^\times$. In practice, below S will be a k-scheme for some algebraically closed field k. The datum of an action of $\mathbb{G}_m$ on X is then equivalent to the datum of an action on X seen as a k-scheme, such that the structure morphism $X \to S$ is $\mathbb{G}_m$-equivariant for the trivial action on S.

commutes, where the vertical arrows are the action morphisms and the horizontal morphisms are induced by the given morphism $(\mathbb{A}^1_T)^+ \to X \times_S T$. In other words, given $T \in \mathrm{AffSch}_S$, $X^+(T)$ consists of the $\mathbb{A}^1_T$-points of X such that for any affine scheme $T' \to T$ the induced morphism $\mathbb{A}^1_{T'} \to X \times_S T'$ commutes with the action of elements in $\mathbb{G}_m(T')$. The functor X^- is defined similarly, replacing $(\mathbb{A}^1_S)^+$ by $(\mathbb{A}^1_S)^-$.

The natural morphisms relating the schemes T, $(\mathbb{A}^1_T)^+$ and $(\mathbb{A}^1_T)^-$ induce morphisms[16]

$$X^0 \to X, \qquad X^0 \leftarrow X^\pm \to X.$$

1.3.2. *Local linearizability.* If X is a scheme over S with an action of $\mathbb{G}_m$, this action is said to be *étale (resp. Zariski) locally linearizable* if there exists a $\mathbb{G}_m$-equivariant covering family $(U_i \to X : i \in I)$ where each U_i is affine over S and the maps $U_i \to X$ are étale (resp. open immersions). This condition is interesting in this context since, thanks to [Ri16, Theorem 1.8], if the $\mathbb{G}_m$-action is étale locally linearizable then X^0, X^+ and X^- are representable by schemes.

1.3.3. *Compatibility with closed immersions.* We will need the following facts below.

Lemma 1.3. *Let X be a scheme over S endowed with an étale locally linearizable action of $\mathbb{G}_m$. If $Y \subset X$ is a $\mathbb{G}_m$-stable closed subscheme, then the $\mathbb{G}_m$-action on $Y \to S$ is étale locally linearizable, and the natural morphisms $Y^0 \to X^0$ and $Y^\pm \to X^\pm$ are closed immersions. More specifically, there exist canonical isomorphisms*

$$Y^0 \xrightarrow{\sim} Y \times_X X^0, \qquad Y^\pm \xrightarrow{\sim} Y \times_X X^\pm,$$

such that the morphisms $Y^0 \to X^0$ and $Y^\pm \to X^\pm$ are induced by the closed immersion $Y \to X$, and the morphisms $Y^0 \to Y$, $Y^\pm \to Y$ and $Y^\pm \to Y^0$ and induced by the similar morphisms for X.

Proof. If $(U_i \to X : i \in I)$ is an equivariant étale covering as above, then of course $(U_i \times_X Y \to Y : i \in I)$ is an equivariant étale covering of Y, and each $U_i \times_X Y$ is affine over S since it is a closed subscheme of the affine scheme U_i. Hence the $\mathbb{G}_m$-action on Y is étale locally linearizable, so that we can consider the schemes Y^0 and $Y^\pm$.

We will construct the isomorphism $Y^+ \xrightarrow{\sim} Y \times_X X^+$; the other assertions can be obtained similarly. First, the natural morphisms $Y^+ \to Y$ and $Y^+ \to X^+$ induce a canonical morphism

$$Y^+ \to Y \times_X X^+. \tag{1.2}$$

Now, assume that $X \to S$ is affine. Checking that (1.2) is an isomorphism can be done Zariski locally over S, so that we can assume that S (hence also X) is affine. In this case for $T \in \mathrm{AffSch}_S$, a T-point of $Y \times_X X^+$ is a certain morphism of T-schemes $\mathbb{A}^1_T \to X$ whose restriction to $\mathbb{G}_{m,T}$ takes values in Y. It is clear that this morphism then factors uniquely through a morphism $\mathbb{A}^1_T \to Y$, which proves that (1.2) is an isomorphism in this case.

To treat the general case, consider an equivariant étale covering $(U_i \to X : i \in I)$ where each $U_i \to S$ is affine. Then we have an étale covering $(U_i \times_X Y \to Y : i \in I)$, hence étale coverings $(U_i^+ \to X^+ : i \in I)$ and $((U_i \times_X Y)^+ \to Y^+ : i \in I)$ by [Ri16, Theorem 1.8], and from the affine case treated above we see that for any i we have a canonical identification

$$(U_i \times_X Y)^+ \xrightarrow{\sim} (U_i \times_X Y) \times_{U_i} (U_i)^+ = Y \times_X U_i^+.$$

This shows that (1.2) is an isomorphism étale locally over the target. Hence it is an isomorphism by [StP, Tag 02L4]. □

Remark 1.4. Let ℓ be a prime number, and let Λ be a finite ℓ-torsion ring. Assume that S is quasi-compact and quasi-separated, and let X be a scheme over S such that the morphism $X \to S$ is of finite presentation (or, in other words, locally of finite presentation, quasi-compact and quasi-separated, see [StP, Tag 01TP]). Assume we are given an étale locally linearizable action of $\mathbb{G}_m$ on $X \to S$, and assume that ℓ is invertible on X. Let also $Y \subset X$ be a closed subscheme. (Note that $Y \to S$ is again of finite presentation.) Recall that we denote by $D_{\text{ét}}(X,\Lambda)^{\mathbb{G}_m\text{-mon}}$ the full

[16]Here and below, we use the symbol $\pm$ to mean either $+$ or $-$.

subcategory of $D_{\text{ét}}(X,\Lambda)$ generated (as a triangulated category) by objects in $D_{\text{ét}}(X,\Lambda)$ whose pullback to $\mathbb{G}_{m,S}\times_S X$ under the action morphism and the projection are isomorphic, see [Ri16, Definition 2.3]. We use similar notation for Y.

In this setting, the schemes X^0, $X^\pm$, Y^0 and $Y^\pm$ are quasi-compact and quasi separated over S (see [Ri16, Theorem 1.8(iii)]), hence quasi-compact and quasi separated, and we can consider the hyperbolic localization functors

$$L_{X/S}: D_{\text{ét}}(X,\Lambda)^{\mathbb{G}_m\text{-mon}} \to D_{\text{ét}}(X^0,\Lambda), \qquad L_{Y/S}: D_{\text{ét}}(Y,\Lambda)^{\mathbb{G}_m\text{-mon}} \to D_{\text{ét}}(Y^0,\Lambda),$$

see [Ri16, §2.1]. (Here we identify the canonically isomorphic functor $L^{\pm}_{X/S}$ from [Ri16], and denote them $L_{X/S}$ for simplicity.) From the base change theorem one sees that the pushforward functor $i_*: D_{\text{ét}}(Y,\Lambda)\to D_{\text{ét}}(X,\Lambda)$ associated with the embedding $i: Y\to X$ induces a functor from $D_{\text{ét}}(Y,\Lambda)^{\mathbb{G}_m\text{-mon}}$ to $D_{\text{ét}}(X,\Lambda)^{\mathbb{G}_m\text{-mon}}$. Lemma 1.3 shows that the squares of natural morphisms

$$\begin{array}{ccccc} Y & \longleftarrow & Y^{\pm} & \longrightarrow & Y^0 \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow{\scriptstyle i^0} \\ X & \longleftarrow & X^{\pm} & \longrightarrow & X^0 \end{array}$$

are Cartesian. Using the base change theorem, one deduces a canonical isomorphism

$$L_{X/S}\circ i_* \cong (i^0)_*\circ L_{Y/S}: D_{\text{ét}}(Y,\Lambda)^{\mathbb{G}_m\text{-mon}} \to D_{\text{ét}}(X^0,\Lambda).$$

1.3.4. *Points over fields.* Now we assume that $S=\operatorname{Spec}(k)$ for some field k.

Lemma 1.5. *Let X be a proper k-scheme with an étale locally linearizable action of $\mathbb{G}_m$. Then the natural morphism $X^+\to X$ induces a bijection*

$$X^+(K)\xrightarrow{\sim} X(K)$$

for all field extension $k\to K$. In particular, this morphism induces a bijection

$$|X^+|\xrightarrow{\sim}|X|$$

on the underlying topological spaces.

Proof. Since X is separated, the morphism $X^+\to X$ is a monomorphism by [Ri16, Remark 1.19(i)]. In particular, the map $X^+(K)\to X(K)$ is injective for any K. The surjectivity of this map follows from the fact that any morphism $\mathbb{G}_{m,K}\to X\otimes_k K$ can be extended to a morphism $\mathbb{A}^1_K\to X\otimes_k K$ by properness, see [StP, Tag 0BXZ]. The final claim follows from the fact that the underlying topological space of a scheme is the colimit of its points over all fields, see [StP, Tag 01J9]. ☐

Remark 1.6. The map $|X^+|\xrightarrow{\sim}|X|$ is *not* a homeomorphism in general.

1.4. **Attractors and repellers for ind-schemes.** Now we explain how to adapt the constructions of §1.3 to the setting of ind-schemes.

1.4.1. *Definitions.* We continue with our base scheme S, and now consider an ind-scheme X over S. An action of $\mathbb{G}_m$ on X is the datum of a morphism of S-ind-schemes $\mathbb{G}_{m,S}\times_S X\to X$ which satisfies the obvious axioms. In practice, we will in fact assume that there exists a presentation $X=\operatorname{colim}_i X_i$ by S-schemes such that the action morphism is defined by compatible actions of $\mathbb{G}_m$ on each X_i (in the sense of schemes). (As explained in [RS20, Lemma A.5], this is always satisfied if S is Noetherian and X is of ind-finite type over S.)

Given $X\to S$ as above, we will say that the $\mathbb{G}_m$-action is étale (resp. Zariski) locally linearizable if there exists a presentation $X=\operatorname{colim}_i X_i$ such that the action of $\mathbb{G}_m$ is induced by compatible actions on the X_i's, and the action on $X_i\to S$ is étale (resp. Zariski) locally linearizable for any i. Given such a datum, when writing a presentation $X=\operatorname{colim}_i X_i$ we will always implicitly assume that each X_i is $\mathbb{G}_m$-stable with an étale/Zariski locally linearizable action.

1.4.2. *Representability.* The following theorem is an easy extension of the main result of [Ri16], treated in [HR18, Theorem 2.1].

Theorem 1.7. *Let $X \to S$ be an S-ind-scheme with an étale locally linearizable $\mathbb{G}_m$-action, and write a presentation $X = \mathrm{colim}_i X_i$ as above.*

(1) *The functor X^0 is an ind-scheme, and we have a presentation $X^0 = \mathrm{colim}_i (X_i)^0$. Moreover, the natural morphism $X^0 \to X$ is representable by a closed immersion.*
(2) *The functor $X^\pm$ is an ind-scheme, and we have a presentation $X^\pm = \mathrm{colim}_i (X_i)^\pm$. Moreover, the natural morphism $X^\pm \to X$ is representable by schemes.*

Proof. If $X = \mathrm{colim}_i X_i$ is a presentation such that each X_i has an étale locally linearizable action, then as functors we have $X^0 = \mathrm{colim}_i (X_i)^0$ and $X^\pm = \mathrm{colim}_i (X_i)^\pm$. Hence X^0 and $X^\pm$ are ind-schemes by Lemma 1.3. To show that $X^0 \to X$, resp. $X^\pm \to X$, is representable by a closed immersion, resp. representable by schemes, one notices that if Z is an affine scheme and $Z \to X$ is a morphism, then there exists i such that this morphism is induced by a morphism $Z \to X_i$, and then we have

$$Z \times_X X^0 = Z \times_{X_i} (X_i)^0, \quad \text{resp.} \quad Z \times_X X^\pm = Z \times_{X_i} (X_i)^\pm. \tag{1.3}$$

(For instance, in the case of attractors, we have $Z \times_X X^+ = \mathrm{colim}_{j \geqslant i} Z \times_{X_j} (X_j)^+$, and for any $j \geqslant i$ we observe that $Z \times_{X_j} (X_j)^+ = Z \times_{X_i} (X_i \times_{X_j} (X_j)^+) = Z \times_{X_i} (X_i)^+$ by Lemma 1.3.) □

1.4.3. *Compatibility with immersions.* We will also need the following property, which is more technical (in the "open" case); see [HR18, Corollary 2.3].

Proposition 1.8. *Let X and Y be S-ind-schemes equipped with $\mathbb{G}_m$-actions. Assume that the action on X, resp. Y, is étale, resp. Zariski, locally linearizable, and that Y is separated. Let also $f : X \to Y$ be a $\mathbb{G}_m$-equivariant morphism. If f is represented by a closed, resp. open, immersion, then so are the morphisms $X^0 \to Y^0$ and $X^\pm \to Y^\pm$.*

Remark 1.9. In [HR18, Corollary 2.3] it is assumed that S is affine and connected. However the connectedness is not necessary for the arguments there to apply, and one can reduce to the case where S is affine by considering an open affine cover.

1.4.4. *Braden's theorem.* Now we assume that S is quasi-compact and quasi-separated, and we consider an ind-scheme $X \to S$ with an action of $\mathbb{G}_m$ which admits a presentation $X = \mathrm{colim}_i X_i$ over S where each $X_i \to S$ is of finite presentation, and such that the action on X is induced by compatible $\mathbb{G}_m$-actions on the X_i's which are étale locally linearizable. We have defined the category $D_{\text{ét}}(X, \Lambda)^{\mathrm{bd}}$ in §1.1.5. We set

$$D_{\text{ét}}(X, \Lambda)^{\mathbb{G}_m\text{-mon,bd}} := \mathrm{colim}_i D_{\text{ét}}(X_i, \Lambda)^{\mathbb{G}_m\text{-mon}},$$

where $D_{\text{ét}}(X_i, \Lambda)^{\mathbb{G}_m\text{-mon}}$ is defined in Remark 1.4. We can also consider the category $D_{\text{ét}}(X^0, \Lambda)^{\mathrm{bd}}$, which satisfies

$$D_{\text{ét}}(X^0, \Lambda)^{\mathrm{bd}} := \mathrm{colim}_i D_{\text{ét}}((X_i)^0, \Lambda).$$

The comments in Remark 1.4 show that the functors $L_{X_i/S}$ "glue" to define a triangulated functor

$$L_{X/S} : D_{\text{ét}}(X, \Lambda)^{\mathbb{G}_m\text{-mon,bd}} \to D_{\text{ét}}(X^0, \Lambda)^{\mathrm{bd}}.$$

2. Fixed points on the affine Grassmannian

From now on we fix an algebraically closed field k, and consider a fixed connected reductive algebraic group G over k. We also consider a cocharacter $\chi : \mathbb{G}_m \to G$. From the action of L^+G on Gr_G, and using the embedding $\mathbb{G}_m \subset L^+\mathbb{G}_m$ (as constant loops) and the morphism $L^+\chi : L^+\mathbb{G}_m \to L^+G$ we obtain an action of $\mathbb{G}_m$ on Gr_G.

2.1. **Local linearizability.** In order to start considering the formalism of §1.3, we need to check that the action under consideration is locally linearizable.

Lemma 2.1. *The $\mathbb{G}_m$-action on Gr_G is Zariski locally linearizable.*

Proof. First we consider the case $G = \mathrm{GL}_n$. Recall the construction of §1.2.3; we use the notation introduced there. The cocharacter χ defines a $\mathbb{G}_m$-action on $\mathbb{A}^n_k$, hence on M_i, on $\bigwedge^d M_i$ and finally on $\mathbb{P}(\bigwedge^d M_i)$ (for any i and d), such that the closed embedding $\mathrm{Gr}_{\mathrm{GL}_n,i} \hookrightarrow \bigsqcup_d \mathbb{P}(\bigwedge^d M_i)$ is $\mathbb{G}_m$-equivariant. Since the $\mathbb{G}_m$-action on $\mathbb{P}(\bigwedge^d M_i)$ is easily checked to be Zariski locally linearizable (see Talk 5), we deduce that so is the $\mathbb{G}_m$-action on $\mathrm{Gr}_{\mathrm{GL}_n,i}$, which finishes the proof in this case.

To treat the case of a general reductive group G, we choose a closed embedding $G \hookrightarrow \mathrm{GL}_n$ for some n as in §1.2.4. We then get a presentation $\mathrm{Gr}_G = \mathrm{colim}_{i\geqslant 0}\mathrm{Gr}_{G,i}$ and closed immersions $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{\mathrm{GL}_n,i}$. The composition of χ with the embedding $G \to \mathrm{GL}_n$ provides a cocharacter χ' of GL_n. This cocharacter defines a $\mathbb{G}_m$-action on each $\mathrm{Gr}_{\mathrm{GL}_n,i}$, such that the closed immersion $\mathrm{Gr}_{G,i} \to \mathrm{Gr}_{\mathrm{GL}_n,i}$ is equivariant. Since the action on $\mathrm{Gr}_{\mathrm{GL}_n,i}$ is Zariski locally linearizable by the case treated above, the same is true for the action on $\mathrm{Gr}_{G,i}$ (see Lemma 1.3), which finishes the proof. □

In view of Lemma 2.1 and Theorem 1.7, we can consider the ind-schemes $(\mathrm{Gr}_G)^0$ and $(\mathrm{Gr}_G)^\pm$, and the natural morphisms

$$(\mathrm{Gr}_G)^0 \leftarrow (\mathrm{Gr}_G)^\pm \to \mathrm{Gr}_G. \tag{2.1}$$

2.2. **Description of fixed points, attractors and repellers.**

2.2.1. *Statement.* The cocharacter χ defines via conjugation a $\mathbb{G}_m$-action on G. If we set

$$M := G^0, \quad P^+ := G^+, \quad P^- := G^-$$

(with respect that this action), then it is known that P^+ and P^- are parabolic subgroups of G, that M is a Levi factor in P^+ and P^-, and that $M = P^+ \cap P^-$, see [CGP, §2.1]. The natural maps

$$M \leftarrow P^\pm \to G$$

are the projection to the Levi quotient and the natural embeddings, respectively. We can consider the affine Grassmannians Gr_M, $\mathrm{Gr}_{P^\pm}$, and the induced morphisms

$$\mathrm{Gr}_M \leftarrow \mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_G. \tag{2.2}$$

Theorem 2.2. *There exist canonical isomorphisms*

$$\mathrm{Gr}_M \xrightarrow{\sim} (\mathrm{Gr}_G)^0, \quad \mathrm{Gr}_{P^\pm} \xrightarrow{\sim} (\mathrm{Gr}_G)^\pm$$

which identify the diagrams (2.1) *and* (2.2).

2.2.2. *Study of the big cell.* For the proof of Theorem 2.2 we will need the following preliminary. Recall the ind-affine ind-scheme $L^{--}G$, see §1.2. The $\mathbb{G}_m$-action on G (via conjugation) induces an action on $L^{--}G$, so that we can consider the (ind-affine) ind-schemes $(L^{--}G)^0$ and $(L^{--}G)^\pm$. The closed immersion $M \to G$, resp. $P^\pm \to G$, induces a morphism $L^{--}M \to L^{--}G$, resp. $L^{--}P^\pm \to L^{--}G$.

Lemma 2.3. *The morphisms above induce isomorphisms*

$$L^{--}M \xrightarrow{\sim} (L^{--}G)^0, \qquad L^{--}P^\pm \xrightarrow{\sim} (L^{--}G)^\pm.$$

Proof. [17] It suffices to prove the similar claims for L^- instead of L^{--}. We first consider L^-M and $(L^-G)^0$. By definition, for $R \in \mathrm{Alg}_k$, $(L^-G)^0(R)$ consists of the points $g \in G(R[z^{-1}])$ such that for any R-algebra S and any $\lambda \in S^\times$ we have

$$\chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g$$

[17] This proof is a corrected version of that of [HR18, Proposition 3.4], which is slightly wrong.

in $G(S[z^{-1}])$. On the other hand, $(L^-M)(R) = M(R[z^{-1}])$. Since $M = G^0$ for the $\mathbb{G}_m$-action on G, the set $M(R[z^{-1}])$ consists of the elements $g \in G(R[z^{-1}])$ such that for any $S' \in \mathrm{Alg}_{R[z^{-1}]}$ and $\lambda \in (S')^\times$ we have

$$\chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g$$

in $G(S')$. These two sets are subsets of $G(R[z^{-1}])$, and we will check that they coincide.

Given $g \in M(R[z^{-1}])$, for any $S \in \mathrm{Alg}_R$ and $\lambda \in S^\times$ we can consider the $R[z^{-1}]$-algebra $S' := S[z^{-1}]$ and the element $\lambda \in S^\times \subset (S')^\times$. From the description of $M(R[z^{-1}])$ given above we obtain that $\chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g$ in $G(S') = G(S[z^{-1}])$, so that g belongs to $(L^-G)^0(R)$. In the other direction, consider $g \in (L^-G)^0(R)$. Then if S' is an $R[z^{-1}]$-algebra, we can consider S' as an R-algebra, and the image of z^{-1} defines an element $s \in S'$. Since g belongs to $(L^-G)^0(R)$, we have

$$\chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g \tag{2.3}$$

in $G(S'[z^{-1}])$. The element $s \in S'$ defines an S'-algebra morphism $S'[z^{-1}] \to S'$, hence a group morphism $G(S'[z^{-1}]) \to G(S')$. Taking the image of the identity (2.3) in $G(S')$ we see that g belongs to $(L^-M)(R)$, which shows indeed that our two subsets coincide.

The proof of the isomorphisms involving $P^\pm$ is similar. If $R \in \mathrm{Alg}_k$, then $(L^-G)^\pm(R)$ is a subset of $(L^-G)(R[t]) = G(R[z^{-1}, t])$ determined by an appropriate equivariance condition, where t is another indeterminate (such that $\mathbb{A}^1_R = \mathrm{Spec}(R[t])$). Similarly we have $(L^-P^\pm)(R) = P^\pm(R[z^{-1}])$, which is a certain subset of $G(R[z^{-1}, t])$ determined by an a priori different equivariance condition. The same considerations as above show that these conditions are in fact equivalent. □

2.2.3. *Proof.*

Proof of Theorem 2.2. The action of $\mathbb{G}_m$ is obtained by functoriality from an action on G. Since the embedding $M \hookrightarrow G$ is $\mathbb{G}_m$-equivariant for the trivial action on M, we deduce that the induced morphism $\mathrm{Gr}_M \to \mathrm{Gr}_G$ is also $\mathbb{G}_m$-equivariant for the trivial action on Gr_M, which shows that this embedding factors through a morphism $\mathrm{Gr}_M \to (\mathrm{Gr}_G)^0$. On the other hand, the conjugation action on G stabilizes P^+, and extends to an action of the monoid $\mathbb{A}^1_k$ on this subgroup. It follows that the induced action on Gr_{P^+} also extends to an action of $\mathbb{A}^1_k$.[18] For this action, we therefore have $\mathrm{Gr}_{P^+} = (\mathrm{Gr}_{P^+})^+$. We deduce that the morphism $\mathrm{Gr}_{P^+} \to \mathrm{Gr}_G$ factors through a morphism

$$\mathrm{Gr}_{P^+} = (\mathrm{Gr}_{P^+})^+ \to (\mathrm{Gr}_G)^+.$$

We obtain similarly that the morphism $\mathrm{Gr}_{P^-} \to \mathrm{Gr}_G$ factors through a morphism $\mathrm{Gr}_{P^-} \to (\mathrm{Gr}_G)^-$. (Note that in these considerations we have used the fact that Gr_G, Gr_{P^+}, Gr_{P^-} are separated, so that the morphisms $(\mathrm{Gr}_G)^\pm \to \mathrm{Gr}_G$, $(\mathrm{Gr}_{P^\pm})^\pm \to \mathrm{Gr}_{P^\pm}$ are monomorphisms, see [Ri16, Remark 1.19(i)].)

To conclude, it remains to prove that the morphisms

$$\mathrm{Gr}_M \to (\mathrm{Gr}_G)^0, \quad \mathrm{Gr}_{P^+} \to (\mathrm{Gr}_G)^+, \quad \mathrm{Gr}_{P^-} \to (\mathrm{Gr}_G)^-$$

are isomorphisms. We will treat the case of the morphism $\mathrm{Gr}_{P^+} \to (\mathrm{Gr}_G)^+$; the case of the morphism $\mathrm{Gr}_{P^-} \to (\mathrm{Gr}_G)^-$ follows by applying the previous case to the cocharacter χ^{-1}, and the case of $\mathrm{Gr}_M \to (\mathrm{Gr}_G)^0$ can treated similarly (with some simplifications).

We fix an embedding $G \hookrightarrow \mathrm{GL}_n$, and consider the presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in the proof of Lemma 2.1, so that each $\mathrm{Gr}_{G,i}$ is projective over k and $\mathbb{G}_m$-stable. For any $g \in LM(k)$, it follows from Lemma 1.2 that the morphism $L^{--}G \to \mathrm{Gr}_G$ defined by $h \mapsto gh \cdot [0]$ is representable by an open immersion. Moreover these open subschemes form a covering of Gr_G, in the sense that for any i they induce an open covering of $\mathrm{Gr}_{G,i}$. (In fact, since these schemes are of finite type over k, using [GW, Corollary 3.36] it suffices to prove that any k-point in some $\mathrm{Gr}_{G,i}$ belongs to such an open subset, which follows e.g. from the Birkhoff decomposition; in fact it suffices to consider elements in $LT(k)$ where T is a maximal torus contained in M, see [Fal, Lemma 4].) By Proposition 1.8, for any $g \in LM(k)$ we deduce a morphism $(L^-G)^+ \to (\mathrm{Gr}_G)^+$ which is representable by an open immersion, and by [Ri16, Theorem 1.8] these open sub-ind-schemes form a covering of $(\mathrm{Gr}_G)^+$.

[18] See [HR18, p. 153] for an explicit description of this action in terms of a Rees construction.

For fixed $g \in LM(k)$, by Lemma 2.3 our morphism $\mathrm{Gr}_{P^+} \to (\mathrm{Gr}_G)^+$ induces an isomorphism between an open sub-ind-scheme of Gr_{P^+} and the open sub-ind-scheme of $(\mathrm{Gr}_G)^+$ considered above. We can therefore consider the inverse isomorphism, for any $g \in LM(k)$. Given two elements in $LM(k)$, these inverse isomorphisms (each defined on the corresponding open sub-ind-scheme of $(\mathrm{Gr}_G)^+$) coincide on the intersection of these sub-ind-schemes; in fact since all the schemes considered here are of finite type (see [GW, Example 3.45]), as above it suffices to prove that they coincide on k-points, which follows from the fact that the morphism $\mathrm{Gr}_{P^+}(k) \to \mathrm{Gr}_G(k)$ is injective, since these sets identify with $P^+(k(\!(z)\!))/P^+(k[\![z]\!])$ and $G(k(\!(z)\!))/G(k[\![z]\!])$ respectively, see [Ri19, Corollary 3.22]. These morphisms therefore glue to define a morphism $(\mathrm{Gr}_G)^+ \to \mathrm{Gr}_{P^+}$, which by construction is an inverse to our given morphism $\mathrm{Gr}_{P^+} \to (\mathrm{Gr}_G)^+$. □

2.3. **Some geometric consequences.**

Proposition 2.4.
(1) *The natural morphism $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_M$ is ind-affine with geometrically connected fibers.*[19] *In particular, it induces a bijection between the sets of connected components of $\mathrm{Gr}_{P^\pm}$ and Gr_M,*[20] *and for any connected component of $\mathrm{Gr}_{P^\pm}$ the embedding in $\mathrm{Gr}_{P^\pm}$ is representable by an open and closed immersion.*
(2) *The natural morphism $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_G$ is bijective and restricts to a morphism representable by a locally closed immersion on each connected component of $\mathrm{Gr}_{P^\pm}$.*

Proof. (1) By Lemma 2.1, we can choose a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as considered in §1.4.1–1.4.2. Then by Theorem 2.2 and Theorem 1.7 we have

$$\mathrm{Gr}_{P^\pm} = \mathrm{colim}_i (\mathrm{Gr}_{G,i})^\pm, \quad \mathrm{Gr}_M = \mathrm{colim}_i (\mathrm{Gr}_{G,i})^0,$$

and the morphism $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_M$ is induced by the canonical morphisms $(\mathrm{Gr}_{G,i})^\pm \to (\mathrm{Gr}_{G,i})^0$. Each of these morphisms is affine by [Ri16, Corollary 1.12], proving that our morphism is ind-affine. Regarding fibers, if K is a field, a morphism $\mathrm{Spec}(K) \to \mathrm{Gr}_M$ must factor through a morphism $\mathrm{Spec}(K) \to (\mathrm{Gr}_{G,i})^0$ for some i. Then we have

$$\mathrm{Gr}_{P^\pm} \times_{\mathrm{Gr}_M} \mathrm{Spec}(K) = \mathrm{colim}_{j \geqslant i} (\mathrm{Gr}_{G,j})^\pm \times_{(\mathrm{Gr}_{G,j})^0} \mathrm{Spec}(K).$$

The underlying topological space of the right-hand side is an increasing union of connected spaces by [Ri16, Corollary 1.12], with closed immersions as transition maps, hence is connected. The final claim follows, since a continuous map of topological spaces with connected fibers and which admits a section induces a bijection between connected components.

Finally, since each morphism $(\mathrm{Gr}_{G,i})^\pm \to (\mathrm{Gr}_{G,i})^0$ induces a bijection between sets of connected components, and because $(\mathrm{Gr}_{G,i})^0 \to (\mathrm{Gr}_{G,j})^0$ induces an injection between sets of connected components for any $i \leqslant j$ (by the identification in Theorem 2.2 and §1.2.6), the same property holds for the morphism $(\mathrm{Gr}_{G,i})^\pm \to (\mathrm{Gr}_{G,j})^\pm$, so that the embedding of each connected component in $\mathrm{Gr}_{P^\pm}$ is representable by an open and closed immersion by Lemma 1.1.

(2) Fix a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in the proof of Lemma 2.1; then we have a presentation $\mathrm{Gr}_{P^\pm} = \mathrm{colim}_i (\mathrm{Gr}_{G,i})^\pm$. By Lemma 1.5, for any i the map

$$|(\mathrm{Gr}_{G,i})^\pm| \to |\mathrm{Gr}_{G,i}|$$

is bijective. Passing to colimits we deduce that the morphism $|\mathrm{Gr}_{P^\pm}| \to |\mathrm{Gr}_G|$ is bijective, as claimed.

Let us now prove that for any i the morphism

$$(\mathrm{Gr}_{G,i})^\pm \to \mathrm{Gr}_{G,i}$$

restricts to a locally closed immersion on each connected component. In fact, using the notation of the proof of Lemma 2.1 we have a $\mathbb{G}_m$-equivariant closed immersion $\mathrm{Gr}_{G,i} \hookrightarrow \bigsqcup_d \mathbb{P}(\bigwedge^d M_i)$ and an

[19]By this we mean that for any field K and any morphism $\mathrm{Spec}(K) \to \mathrm{Gr}_M$, the underlying topological space of the ind-scheme $\mathrm{Gr}_{P^\pm} \times_{\mathrm{Gr}_M} \mathrm{Spec}(K)$ is connected.

[20]More specifically, this bijection sends a connected component of $\mathrm{Gr}_{P^\pm}$ to its image in Gr_M, and a connected component of Gr_M to its inverse image in $\mathrm{Gr}_{P^\pm}$.

induced closed immersion $(\mathrm{Gr}_{G,i})^{\pm} \hookrightarrow \bigsqcup_d(\mathbb{P}(\bigwedge^d M_i))^{\pm}$, see Lemma 1.3. Any connected component of $(\mathrm{Gr}_{G,i})^{\pm}$ embeds as a closed subscheme in a connected component of $\bigsqcup_d(\mathbb{P}(\bigwedge^d M_i))^{\pm}$. On the other hand, for any action of $\mathbb{G}_m$ on a finite-dimensional vector space V, for the induced action on $\mathbb{P}(V)$ the morphism $\mathbb{P}(V)^{\pm} \to \mathbb{P}(V)$ restricts to a locally closed immersion on each connected component, as can be seen explicitly (see Talk 5 for details). We deduce that the composition $(\mathrm{Gr}_{G,i})^{\pm} \to \bigsqcup_d \mathbb{P}(\bigwedge^d M_i)$ restricts to a locally closed immersion on each connected component, see [StP, Tag 02V0]. Using [StP, Tag 07RK] we deduce that $(\mathrm{Gr}_{G,i})^{\pm} \to \mathrm{Gr}_{G,i}$ restricts to a locally closed immersion on each connected component.

Now, consider a connected component Y of $\mathrm{Gr}_{P^{\pm}}$. We can write $Y = \mathrm{colim}_i Y_i$ where Y_i is a connected component of $(\mathrm{Gr}_{G,i})^{\pm}$ for any i. If Z is an affine scheme and $Z \to \mathrm{Gr}_G$ is a morphism, then this morphism factors through $\mathrm{Gr}_{G,i}$ for some i, and we have

$$Y \times_{\mathrm{Gr}_G} Z = \mathrm{colim}_{j \geqslant i} Y_j \times_{\mathrm{Gr}_{G,j}} Z.$$

Now for any $j \geqslant i$ we have

$$\begin{aligned} Y_j \times_{\mathrm{Gr}_{G,j}} Z &= (Y_j \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,i}) \times_{\mathrm{Gr}_{G,i}} Z \\ &= \big(Y_j \times_{(\mathrm{Gr}_{G,j})^+} ((\mathrm{Gr}_{G,j})^+ \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,i})\big) \times_{\mathrm{Gr}_{G,i}} Z = (Y_j \times_{(\mathrm{Gr}_{G,j})^+} (\mathrm{Gr}_{G,i})^+) \times_{\mathrm{Gr}_{G,i}} Z \end{aligned}$$

by Lemma 1.3. Now, as seen in the proof of Lemma 1.1 we have $Y_j \times_{(\mathrm{Gr}_{G,j})^+} (\mathrm{Gr}_{G,i})^+ = Y_i$, so that

$$Y \times_{\mathrm{Gr}_G} Z = Y_i \times_{\mathrm{Gr}_{G,i}} Z.$$

In particular this ind-scheme is a scheme, and the morphism $Y \times_{\mathrm{Gr}_G} Z \to Z$ is an open and closed immersion because so is the morphism $Y_i \to \mathrm{Gr}_{G,i}$. □

2.4. **Braden's theorem.** Let ℓ be a prime number invertible in k, and let Λ be a finite ℓ-torsion ring. The analysis in §2.1 shows that the $\mathbb{G}_m$-action on Gr_G satisfies the conditions in §1.4.4. In view of the identifications in §2.2, we therefore have a hyperbolic localization functor

$$L_{\mathrm{Gr}_G,\chi} : \mathrm{D}_{\text{ét}}(\mathrm{Gr}_G, \Lambda)^{(\mathbb{G}_m,\chi)\text{-mon,bd}} \to \mathrm{D}_{\text{ét}}(\mathrm{Gr}_M, \Lambda)^{\mathrm{bd}},$$

where we add χ in the notation to emphasize the dependence on the choice of the cocharacter (and suppress the indication of the base scheme $\mathrm{Spec}(k)$).

3. The relative case

Next, we need to discuss the analogue of the theory considered in Section 2 for affine Grassmannians over copies of curves. We continue with the setting of Section 2.

3.1. **Beĭlinson–Drinfeld Grassmannians.** We set $X = \mathbb{P}^1_k$.[21] We consider a smooth affine group scheme H over k. If Y is any scheme, we denote by $\mathcal{E}^{\mathrm{triv}}$ the trivial right H-torsor over Y. (The base scheme Y is not indicated in this notation, but it will be clear from the context.)

Recall that given a finite set I, we denote by $\mathrm{Gr}_{H,I}$ the functor on Alg_k sending R to the set of isomorphism classes of triples $(x, \mathcal{E}, \beta)$ where:

- x is an R-point of X^I, i.e. a collection $(x_i : i \in I)$ of R-points of X parametrized by I;
- $\mathcal{E}$ is a (right) H-torsor on $X_R := X \otimes_k R$;
- $\beta : \mathcal{E}_{|X_R \smallsetminus \Gamma_x} \xrightarrow{\sim} \mathcal{E}^{\mathrm{triv}}$ is a trivialization (in other words, an isomorphism of torsors).

Here if $x = (x_i : i \in I) \in X^I(R)$, then we set $\Gamma_x = \bigcup_{i \in I} \Gamma_{x_i}$ where $\Gamma_{x_i} \subset X_R$ is the graph of x_i.[22] It is a standard fact that $\mathrm{Gr}_{H,I}$ is a separated ind-scheme of ind-finite type over X^I. (To prove this claim, one uses a closed embedding $H \hookrightarrow \mathrm{GL}_n$ such that the quotient GL_n/H is quasi-affine to reduce the claim to the case $H = \mathrm{GL}_n$, see [HR18b, Proposition 3.10]. In the latter case, one checks the claim explicitly using the theory of Quot schemes; see [HR18b, Lemma 3.8].) In case H is reductive, $\mathrm{Gr}_{H,I}$ admits a presentation $\mathrm{Gr}_{H,I} = \mathrm{colim}_{j \geqslant 0} \mathrm{Gr}_{H,I,j}$ where each $\mathrm{Gr}_{H,I,j}$ is projective over X^I.

[21] All the geometric story holds for more general curves; see Talk 7 for details.

[22] See [StP, Tag 0C4H] for the scheme structure on a finite union of closed subschemes.

If $R \in \mathrm{Alg}_k$ and $x \in X^I(R)$, then we can consider the formal completion $\widehat{\Gamma}_x$ of X_R along Γ_x, which is an affine formal scheme, and denote by $\widehat{\Gamma}'_x$ the spectrum of its algebra of functions (a "true" affine scheme). It turns out that the natural morphism $\widehat{\Gamma}_x \to X_R$ uniquely extends to a morphism of schemes $\widehat{\Gamma}'_x \to X_R$; see [Zh16, §3.1] or [HR18b, §3.1.1] for details and references. Let also $\widehat{\Gamma}^\circ_x := \widehat{\Gamma}'_x \smallsetminus \Gamma_x$, which is again an affine scheme. We then define the functor $L_I H : \mathrm{Alg}_k \to \mathrm{Sets}$, resp. $L_I^+ H : \mathrm{Alg}_k \to \mathrm{Sets}$ sending R to the set of pairs (x, β) where $x \in X^I(R)$ and $\beta \in H(\widehat{\Gamma}^\circ_x)$, resp. $\beta \in H(\widehat{\Gamma}'_x)$. The functor $L_I^+ H$ is an affine group scheme over X^I, $L_I H$ is an ind-affine group ind-scheme over X^I, and $\mathrm{Gr}_{H,I}$ is the fpqc quotient $L_I H / L_I^+ H$; see [Zh16, Proposition 3.1.9] or [HR18b, Lemma 3.2 and Lemma 3.4].

For details on all of this, see Talk 7.

3.2. **Local linearizability.** Now we consider our reductive group G, and a cocharacter $\chi : \mathbb{G}_m \to G$. This provides a (fiberwise) action of $\mathbb{G}_m$ on $L_I G \to X^I$ by conjugation, and then on $\mathrm{Gr}_{G,I} \to X^I$.

Lemma 3.1. *The action of $\mathbb{G}_m$ on $\mathrm{Gr}_{G,I} \to X^I$ is Zariski locally linearizable.*

The proof of this lemma (given in [HR18b, Lemma 3.16], where "étale" can be replaced by "Zariski" in the special setting we consider here) is similar to that of Lemma 2.1: one uses a closed embedding $G \hookrightarrow \mathrm{GL}_n$ to reduce to the case $G = \mathrm{GL}_n$, and then checks the claim explicitly in this case (reducing to the case of the action on a Quot scheme).

In view of Lemma 3.1 and Theorem 1.7, we can consider the ind-schemes $(\mathrm{Gr}_{G,I})^0$ and $(\mathrm{Gr}_{G,I})^\pm$ over X^I, and the natural morphisms

$$(\mathrm{Gr}_{G,I})^0 \leftarrow (\mathrm{Gr}_{G,I})^\pm \to \mathrm{Gr}_{G,I}. \tag{3.1}$$

3.3. **Description of fixed points, attractors and repellers.** As in §2.2, the conjugation action of $\mathbb{G}_m$ on G via χ defines parabolic subgroups P^+, P^- of G, and a common Levi factor M. We can then consider the Beĭlinson–Drinfeld Grassmannians $\mathrm{Gr}_{P^\pm,I}$ and $\mathrm{Gr}_{M,I}$ over X^I, and the natural morphisms

$$M \leftarrow P^\pm \to G$$

induce morphisms of ind-schemes

$$\mathrm{Gr}_{M,I} \leftarrow \mathrm{Gr}_{P^\pm,I} \to \mathrm{Gr}_{G,I}. \tag{3.2}$$

Theorem 3.2. *There exist canonical isomorphisms*

$$\mathrm{Gr}_{M,I} \xrightarrow{\sim} (\mathrm{Gr}_{G,I})^0, \quad \mathrm{Gr}_{P^\pm,I} \xrightarrow{\sim} (\mathrm{Gr}_{G,I})^\pm$$

of ind-schemes over X^I, which identify the diagrams (3.1) *and* (3.2).

The proof of this theorem is based on the same ideas as that of Theorem 2.2, but is more technical. (In particular, the "fiberwise" identifications given by Theorem 2.2 are not sufficient to conclude.) One also uses a "big cell" in this context; see [HR18b, §3.2]. More specifically, paving $\mathbb{P}^1_k$ by two copies of $\mathbb{A}^1_k$ we reduce the question to the analogous claim for the version $\mathrm{Gr}_{G,(\mathbb{A}^1_k)^I}$ of Gr_G over $(\mathbb{A}^1_k)^I$. Any R-point in $(\mathbb{A}^1_k)^I$ defines an R-point in $(\mathbb{P}^1_k)^I$, which allows to define $L^-_{(\mathbb{A}^1_k)^I} G$ as the functor sending R to the set of pairs consisting of an R-point x in $(\mathbb{A}^1_k)^I$ and an element in $G(\mathbb{P}^1_R \smallsetminus \Gamma_x)$. Restriction to the point $\infty \in (\mathbb{P}^1_k)(k)$ allows to define a morphism $L^-_{(\mathbb{A}^1_k)^I} G \to G$, and the kernel $L^{--}_{(\mathbb{A}^1_k)^I} G$ of this morphism. As explained in [HR18b, Lemma 3.15], $L^{--}_{(\mathbb{A}^1_k)^I} G$ identifies with an open ind-scheme in $\mathrm{Gr}_{G,(\mathbb{A}^1_k)^I}$.

For details on the proof of Theorem 3.2, see [HR18b, Theorem 3.17].

3.4. **Braden's theorem.** If ℓ and Λ are as in §2.4, then once again we can consider the categories $\mathrm{D}(\mathrm{Gr}_{G,I}, \Lambda)^{(\mathbb{G}_m,\chi)\text{-mon,bd}}$ and $\mathrm{D}(\mathrm{Gr}_{M,I}, \Lambda)^{\mathrm{bd}}$, and we have a hyperbolic localization functor

$$L_{\mathrm{Gr}_{G,I},\chi} : \mathrm{D}_{\text{ét}}(\mathrm{Gr}_{G,I}, \Lambda)^{(\mathbb{G}_m,\chi)\text{-mon,bd}} \to \mathrm{D}_{\text{ét}}(\mathrm{Gr}_{M,I}, \Lambda)^{\mathrm{bd}}.$$

4. Dimension estimates

4.1. **Semi-infinite orbits.** Let B and T be as in §1.2.5. We consider the setting of Section 2, assuming that χ is a cocharacter of T which is dominant and regular. In this case we have $G^0 = T$, $G^+ = B$, and G^- is the Borel subgroup of G opposite to B with respect to T.

Recall from Proposition 2.4(1) that the morphism $\mathrm{Gr}_B \to \mathrm{Gr}_T$ induces a bijection between the connected components of Gr_B and Gr_T. It is a standard fact that $|\mathrm{Gr}_T|$ is discrete, with

$$|\mathrm{Gr}_T| = \{[\lambda] : \lambda \in X_*(T)\}.$$

Therefore, the map sending $\lambda \in X_*(T)$ to the connected component Gr_T^λ containing $[\lambda]$ induces a bijection between $X_*(T)$ and the set of connected components of Gr_T. For any $\lambda \in X_*(T)$ we have $\mathrm{Gr}_T^\lambda(k) = \{[\lambda]\} = |\mathrm{Gr}_T^\lambda|$. By Lemma 1.2 and commutativity of LT, the morphism $L^{--}T \to \mathrm{Gr}_T$ given by $g \mapsto g \cdot [\lambda]$ induces an isomorphism $L^{--}T \xrightarrow{\sim} \mathrm{Gr}_T^\lambda$, since it is representable by an open immersion and a bijection on k-points. (A description of $L^{--}T$ can be derived from [Ri19, Example 2.8] or [PR08, §3.a].)

For any $\lambda \in X_*(T)$ we will denote by S_λ the connected component of Gr_B corresponding to Gr_T^λ under the bijection considered above; then we have a natural morphism $S_\lambda \to \mathrm{Gr}_G$ which is representable by a locally closed immersion by Proposition 2.4(2). The setting considered in Section 2 can be made slightly more explicit in this case. Namely, choose a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in §1.2.4. Then for any i the scheme of finite type $(\mathrm{Gr}_{G,i})^0$ is discrete, hence the spectrum of a finite-dimensional k-algebra, see [EGA1, Chap. I, Prop. 6.4.4]. Moreover this algebra is a finite product of finite-dimensional local algebras (see [EGA1, Chap. I, §6.2]), hence $(\mathrm{Gr}_{G,i})^0$ is the disjoint union of the spectra of these local algebras, which are the connected components of $(\mathrm{Gr}_{G,i})^0$. If i is large enough we have $[\lambda] \in (\mathrm{Gr}_{G,i})^0$, and if we denote by $(\mathrm{Gr}_{G,i})^{0,\lambda}$ the connected component of $(\mathrm{Gr}_{G,i})^0$ containing λ, then $(\mathrm{Gr}_{G,i})^{0,\lambda}$ is the spectrum of a finite-dimension k-algebra. The fiber product

$$S_{\lambda,i} := (\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,i})^0} (\mathrm{Gr}_{G,i})^{0,\lambda}$$

is an affine connected scheme of finite type over k by [Ri16, Theorem 1.8, Corollary 1.12]. For $j \geqslant i$ we have a closed immersion $(\mathrm{Gr}_{G,i})^{0,\lambda} \to (\mathrm{Gr}_{G,j})^{0,\lambda}$ induced by a surjection of the associated finite-dimensional local k-algebras, and an induced closed immersion

$$(\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,i})^0} (\mathrm{Gr}_{G,i})^{0,\lambda} = (\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,i})^{0,\lambda} \to (\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,\lambda}.$$

Now the natural morphism $(\mathrm{Gr}_{G,i})^+ \to (\mathrm{Gr}_{G,j})^+$ is also a closed immersion (see Lemma 1.3), hence induces a closed immersion

$$(\mathrm{Gr}_{G,i})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,\lambda} \to (\mathrm{Gr}_{G,j})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,\lambda}.$$

Composing these immersions we obtain a closed immersion

$$S_{\lambda,i} \to S_{\lambda,j},$$

and we obtain a presentation

$$S_\lambda = \mathrm{colim}_i S_{\lambda,i}.$$

In particular, these considerations show that S_λ is an ind-affine ind-scheme.

Note that this fact can also be seen in a different way, by remarking that the morphism

$$L^{--}B \to S_\lambda \tag{4.1}$$

defined by $g \mapsto z^\lambda g \cdot [0]_B$ is an isomorphism, since it is representable by an open immersion (see Lemma 1.2) and a bijection on k-points.

We have

$$\mathrm{Gr}_B(k) = B(k(\!(z)\!))/B(k[\![z]\!]),$$

see [Ri16, Corollary 3.22]. Since $B \cong T \times U$, denoting by $[\lambda]_B$ the image of z^λ in Gr_B we deduce that

$$\mathrm{Gr}_B(k) = \bigsqcup_{\lambda \in X_*(T)} U(k(\!(z)\!)) \cdot [\lambda]_B,$$

and

$$S_\lambda(k) = U(k(\!(z)\!)) \cdot [\lambda]_B.$$

It follows from Proposition 2.4(2) that for any $\lambda \in X_*(T)$ the morphism $S_\lambda \to \mathrm{Gr}_G$ is representable by a locally closed immersion, and that these morphisms induce a bijection

$$\bigsqcup_{\lambda \in X_*(T)} S_\lambda(k) \xrightarrow{\sim} \mathrm{Gr}_G(k).$$

By [Ri19, Corollary 3.22] once again we have $\mathrm{Gr}_G(k) = G(k(\!(z)\!))/G(k[\![z]\!])$. These considerations therefore show that

$$G(k(\!(z)\!)) = \bigsqcup_{\lambda \in X_*(T)} U(k(\!(z)\!)) \cdot z^\lambda \cdot G(k[\![z]\!]),$$

which provides a geometric proof of the Iwasawa decomposition in this setting.

Remark 4.1. For any $\lambda \in X_*(T)$, the action of z^λ induces an isomorphism of ind-schemes $S_0 \xrightarrow{\sim} S_\lambda$. This allows to reduces many questions about semi-infinite orbits to the case of S_0.

4.2. **Affineness of intersections with spherical orbits.** We will denote by W the Weyl group of (G, T) and, for any $\lambda \in X_*(T)$, we will denote by λ^+ the unique dominant W-translate of λ. Given $\mu \in X_*(T)^+$, we set

$$\Lambda_\mu = \{\lambda \in X_*(T) \mid \lambda^+ \leqslant \mu\}.$$

It is a standard fact that Λ_μ is finite, and coincides with the set of elements $\lambda \in X_*(T)$ such that $[\lambda] \in \mathrm{Gr}_{G,\leqslant\mu}$. (This set can also be described as the intersection of $\lambda + \mathbb{Z}R^\vee$ with the convex hull of $W\lambda$ in $\mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$.)

Proposition 4.2. *For any $\lambda \in X_*(T)$ and $\mu \in X_*(T)^+$, the intersection*

$$S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu} := S_\lambda \times_{\mathrm{Gr}_G} \mathrm{Gr}_{G,\leqslant\mu}$$

is a connected affine scheme of finite type over k, such that the natural morphism

$$S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu} \to \mathrm{Gr}_{G,\leqslant\mu}$$

is a locally closed immersion. This scheme is nonempty iff $\lambda \in \Lambda_\mu$, and the natural map

$$\bigsqcup_{\lambda \in \Lambda_\mu} S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu} \to \mathrm{Gr}_{G,\leqslant\mu}$$

is a bijection on the underlying topological spaces.

Proof. [23] Since the morphism $S_\lambda \to \mathrm{Gr}_G$ is represented by a locally closed immersion, $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$ is a locally closed subscheme of $\mathrm{Gr}_{G,\leqslant\mu}$. Since the latter scheme is of finite type, so is $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$, see [GW, Example 3.45].

To prove the remaining assertions we will make this scheme more explicit. Consider a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in §4.1, and fix i such that $\mathrm{Gr}_{G,i}$ contains $\mathrm{Gr}_{G,\leqslant\mu}$ and $[\lambda]$. Then we have

$$S_\lambda \times_{\mathrm{Gr}_G} \mathrm{Gr}_{G,\leqslant\mu} = \mathrm{colim}_{j \geqslant i} S_{\lambda,j} \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,\leqslant\mu}$$

where $S_{\lambda,j}$ is as in §4.1. For $j \geqslant i$ we have

$$S_{\lambda,j} = (\mathrm{Gr}_{G,j})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,\lambda},$$

hence

$$S_{\lambda,j} \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,\leqslant\mu} = (\mathrm{Gr}_{G,j})^{0,\lambda} \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^+ \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,\leqslant\mu}.$$

Now $\mathrm{Gr}_{G,\leqslant\mu} \to \mathrm{Gr}_{G,j}$ is a closed immersion, hence using Lemma 1.3 we deduce that

$$\begin{aligned} S_{\lambda,j} \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,\leqslant\mu} &= (\mathrm{Gr}_{G,\leqslant\mu})^+ \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,\lambda} \\ &= (\mathrm{Gr}_{G,\leqslant\mu})^+ \times_{(\mathrm{Gr}_{G,\leqslant\mu})^0} (\mathrm{Gr}_{G,\leqslant\mu})^0 \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,\lambda}. \end{aligned}$$

[23] The proof of the corresponding statement in [FS21] is more complicated, using a reduction to GL_n and then an explicit proof in this case. The proof given here was explained to me by T. Richarz. It is closely related to the proof of [AGLR, Lemma 5.5].

Now $(\mathrm{Gr}_{G,\leqslant\mu})^0 \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,\lambda}$ is a closed subscheme of $(\mathrm{Gr}_{G,j})^{0,\lambda}$, hence is either empty or the spectrum of a finite-dimensional local k-algebra. More explicitly, if $[\lambda] \notin \mathrm{Gr}_{G,\leqslant\mu}$ (i.e. if $\lambda \notin \Lambda_\mu$) this scheme is empty (hence so his $S_\lambda \times_{\mathrm{Gr}_G} \mathrm{Gr}_{G,\leqslant\mu}$), and if $[\lambda] \in \mathrm{Gr}_{G,\leqslant\mu}$ then $(\mathrm{Gr}_{G,\leqslant\mu})^0 \times_{(\mathrm{Gr}_{G,j})^0} (\mathrm{Gr}_{G,j})^{0,\lambda}$ is the connected component $(\mathrm{Gr}_{G,\leqslant\mu})^{0,\lambda}$ of $(\mathrm{Gr}_{G,\leqslant\mu})^0$ containing λ. In particular, this shows that

$$S_{\lambda,j} \times_{\mathrm{Gr}_{G,j}} \mathrm{Gr}_{G,\leqslant\mu} = (\mathrm{Gr}_{G,\leqslant\mu})^+ \times_{(\mathrm{Gr}_{G,\leqslant\mu})^0} (\mathrm{Gr}_{G,\leqslant\mu})^{0,\lambda}.$$

The right-hand side is independent of j, which proves that

$$S_\lambda \times_{\mathrm{Gr}_G} \mathrm{Gr}_{G,\leqslant\mu} = (\mathrm{Gr}_{G,\leqslant\mu})^+ \times_{(\mathrm{Gr}_{G,\leqslant\mu})^0} (\mathrm{Gr}_{G,\leqslant\mu})^{0,\lambda}.$$

The morphism $(\mathrm{Gr}_{G,\leqslant\mu})^+ \to (\mathrm{Gr}_{G,\leqslant\mu})^0$ is affine with connected fibers by [Ri16, Corollary 1.12]. Our considerations above show that $(\mathrm{Gr}_{G,\leqslant\mu})^{0,\lambda}$ is affine and connected; as e.g. in the proof of Proposition 2.4(1), we deduce that $S_\lambda \times_{\mathrm{Gr}_G} \mathrm{Gr}_{G,\leqslant\mu}$ is affine and connected too.

Finally, using the identification above, the fact that the natural morphism

$$\bigsqcup_{\lambda\in\Lambda_\mu} S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu} \to \mathrm{Gr}_{G,\leqslant\mu}$$

is a bijection on the underlying topological spaces follows from Lemma 1.5. □

Remark 4.3. The fact that $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$ is affine is not explicitly proved in [MV07]. The crucial Proposition 4.5 below is derived in another way in this paper, using an embedding in an infinite-dimensional projective space; see e.g. [Zh16, Comments after Corollary 5.3.8] for a discussion.

The choice of B determines a system of Coxeter generators in W; we will denote by $w_\circ$ the longest element for this structure, i.e. the unique element which sends each dominant cocharacter to an antidominant cocharacter. We will also need the following fact below.

Lemma 4.4. *For any $\mu \in X_*(T)^+$ we have $|S_{w_\circ(\mu)} \cap \mathrm{Gr}_{G,\leqslant\mu}| = \{[w_\circ(\mu)]\}$. In particular, we have* $\dim(S_{w_\circ(\mu)} \cap \mathrm{Gr}_{G,\leqslant\mu}) = 0$.

Proof. It is well known that

$$(S_{w_\circ(\mu)} \cap \mathrm{Gr}_{G,\leqslant\mu})(k) = S_{w_\circ(\mu)}(k) \cap \mathrm{Gr}_{G,\leqslant\mu}(k) = \{[w_\circ(\mu)]\};$$

see [Zh16, (5.3.11)] or [MV07, (3.6)]. Since $S_{w_\circ(\mu)} \cap \mathrm{Gr}_{G,\leqslant\mu}$ is a scheme of finite type over k, its k-points are dense (see e.g. [GW, Corollary 3.36]). We deduce that $|S_{w_\circ(\mu)} \cap \mathrm{Gr}_{G,\leqslant\mu}| = \{[w_\circ(\mu)]\}$, as desired. □

4.3. **Application to the dimension estimate.** In this subsection we fix $\mu \in X_*(T)^+$. In the following statement we denote by $\rho \in \frac{1}{2}X^*(T)$ the halfsum of the positive roots of (G,T).

Proposition 4.5. *For any $\lambda \in \Lambda_\mu$, the scheme $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$ is equidimensional,[24] of dimension $\langle\rho, \mu+\lambda\rangle$.*

The proof will require a number of (easy, or at least standard) preliminary results.

Lemma 4.6. (1) *Let X be a noetherian topological space, and let $U \subset X$ be an open subset. The assignments $Z \mapsto Z \cap U$ and $Y \mapsto \overline{Y}$ induce a bijection between the set of irreducible components of X intersecting U and the set of irreducible components of U.*

(2) *If X is an irreducible scheme of finite type over k and $U \subset X$ an open subscheme, then* $\dim(X) = \dim(U)$.

Proof. The claim in (1) is classical; its proof is left to the reader. For (2), see [StP, Tag 0A21]. □

Lemma 4.7. *Let X be a separated k-scheme of finite type, and let $Z \subset X$ be a closed subscheme such that $X \smallsetminus Z$ is affine. Let $X' \subset X$ be an irreducible component[25] not contained in Z and such that $X' \cap Z \neq \varnothing$, and let Z' be an irreducible component of $X' \cap Z$. Then* $\dim(Z') = \dim(X') - 1$.

[24] By this we mean that all irreducible components have the same dimension.

[25] We will use the convention that irreducible components are always endowed with the reduced subscheme structure, as e.g. in [StP, Tag 01IZ].

Proof. The scheme $X' \smallsetminus (X' \cap Z)$ is a closed subscheme of $X \smallsetminus Z$, hence is affine. Since X' is separated as a closed subscheme of a separated scheme (see e.g. [StP, Tag 01L7]) and Z' is an irreducible component in

$$X' \cap Z = X' \smallsetminus \big(X' \smallsetminus (X' \cap Z)\big),$$

we can use [StP, Tag 0BCV] and [StP, Tag 02IZ] to conclude that $\mathrm{codim}(Z', X') = 1$. Now $\mathrm{codim}(Z', X') = \dim(X') - \dim(Z')$ by [GW, Proposition 5.30], which allows to conclude. ☐

For $\lambda \in \Lambda_\mu$ we set

$$Y^\circ_{\lambda,\mu} := \big(S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}\big)_{\mathrm{red}},$$

a reduced locally closed subscheme of $\mathrm{Gr}_{G,\leqslant\mu}$, and denote by $Y_{\lambda,\mu}$ the closure of $Y^\circ_{\lambda,\mu}$, endowed with the reduced closed subscheme structure. Then we have morphisms

$$Y^\circ_{\lambda,\mu} \hookrightarrow Y_{\lambda,\mu} \hookrightarrow \mathrm{Gr}_{G,\leqslant\mu}$$

where the first arrow is an open immersion and the second one a closed immersion, see [StP, Tag 03DQ].

Lemma 4.8. *Let $\lambda, \lambda' \in \Lambda_\mu$. If $Y^\circ_{\lambda,\mu} \cap Y_{\lambda',\mu} \neq \varnothing$, then $\lambda \leqslant \lambda'$.*

Proof. For any $\nu \in \Lambda_\mu$, using e.g. the isomorphism $L^{--}B \xrightarrow{\sim} S_\nu$ (see §4.1) and the fact that $S_\nu \cap \mathrm{Gr}_{G,\leqslant\mu}$ is a quasi-compact scheme, one sees that for $\gamma \in X_*(T)$ sufficiently dominant, we have $z^\gamma \cdot Y^\circ_{\lambda,\mu} \subset \mathrm{Gr}_{G,\lambda+\gamma}$. Given weights λ and λ' as in the statement, we can choose γ such that this condition holds both for λ and λ'. Then we obtain that $\mathrm{Gr}_{G,\leqslant\lambda'+\gamma}$ contains a point in $\mathrm{Gr}_{G,\lambda+\gamma}$, which implies that $\lambda + \gamma \leqslant \lambda' + \gamma$ (see §1.2.5), hence $\lambda \leqslant \lambda'$. ☐

Note that the integers $\langle 2\rho, \lambda\rangle$ for $\lambda \in \Lambda_\mu$ all have the same parity, namely that of $\langle 2\rho, \mu\rangle$, and vary between $-\langle 2\rho, \mu\rangle$ and $\langle 2\rho, \mu\rangle$. For $d \in \{-\langle 2\rho, \mu\rangle, -\langle 2\rho, \mu\rangle + 2, \cdots, \langle 2\rho, \mu\rangle\}$ we set

$$X_d := \bigcup_{\substack{\lambda \in \Lambda_\mu \\ \langle 2\rho,\lambda\rangle \leqslant d}} Y_{\lambda,\mu},$$

where the scheme structure on the union is as in [StP, Tag 0C4H]. Then each X_d is a reduced closed subscheme of $\mathrm{Gr}_{G,\leqslant\mu}$, hence is projective and thus separated, see [StP, Tag 07RL] and [StP, Tag 01VX]. Using Proposition 4.2 and Lemma 4.4 we see that

$$\{[w_\circ(\mu)]\} = X_{-\langle 2\rho,\mu\rangle} \subset X_{-\langle 2\rho,\mu\rangle+2} \subset \cdots \subset X_{\langle 2\rho,\mu\rangle-2} \subset X_{\langle 2\rho,\mu\rangle} = \mathrm{Gr}_{G,\leqslant\mu}.$$

Moreover, Lemma 4.8 implies that for any d as above we have

$$X_d \smallsetminus X_{d-2} = \bigsqcup_{\substack{\lambda \in \Lambda_\mu \\ \langle 2\rho,\lambda\rangle = d}} Y^\circ_{\lambda,\mu}, \tag{4.2}$$

where the right-hand side is affine by Proposition 4.2, and its set of irreducible components is the (disjoint) union of the sets of irreducible components of the schemes $Y^\circ_{\lambda,\mu}$, i.e. of the schemes $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$. These considerations and Lemma 4.6 show that Proposition 4.5 will follow once we prove the following claim.

Lemma 4.9. *For any $d \in \{-\langle 2\rho, \mu\rangle, -\langle 2\rho, \mu\rangle + 2, \cdots, \langle 2\rho, \mu\rangle\}$, the scheme X_d is equidimensional of dimension $\frac{d+\langle 2\rho,\mu\rangle}{2}$.*

Proof. First we prove by descending induction on d that any irreducible component Z of X_d satisfies $\dim(Z) \geqslant \frac{d+\langle 2\rho,\mu\rangle}{2}$. If $d = \langle 2\rho, \mu\rangle$ we have $Z = \mathrm{Gr}_{G,\leqslant\mu}$, hence the result is satisfied. Now if the claim is known for $d + 2$, if Z is still an irreducible component of X_{d+2} then by induction we have

$$\dim(Z) \geqslant \frac{d + 2 + \langle 2\rho, \mu\rangle}{2} \geqslant \frac{d + \langle 2\rho, \mu\rangle}{2}.$$

Otherwise there exists an irreducible component Z' of X_{d+2} containing Z strictly. By Lemma 4.7 applied to X_{d+2} and its closed subscheme X_d we then have

$$\dim(Z) = \dim(Z') - 1,$$

from which we deduce the claim since $\dim(Z') \geqslant \frac{d+2+\langle 2\rho,\mu\rangle}{2}$.

We now note that if $d \in \{-\langle 2\rho, \mu\rangle + 2, \cdots, \langle 2\rho, \mu\rangle\}$, no irreducible component of X_d is contained in $X_d \smallsetminus X_{d-2}$. Indeed, otherwise this irreducible component would be affine (as a closed subscheme in the affine scheme $X_d \smallsetminus X_{d-2}$) and proper (as a closed subscheme of $\mathrm{Gr}_{G,\leqslant\mu}$) hence finite (see [StP, Tag 01WN]), hence of dimension 0, which would contradict the inequality we have just proved.

Finally we prove by (ascending) induction on d that for any irreducible component Z of X_d we have $\dim(Z) \leqslant \frac{d+\langle 2\rho,\mu\rangle}{2}$. If $d = -\langle 2\rho, \mu\rangle$ then X_d is a point by Lemma 4.4, hence the claim is clear. Assume the claim is known for $d-2$, and let Z be an irreducible component of X_d. If $Z \subset X_{d-2}$, then we conclude by induction. Otherwise we consider $Z \cap X_{d-2}$, which is nonempty as explained above. If Z' is an irreducible component of this intersection, by Lemma 4.7 (applied to X_d and its closed subscheme X_{d-2}) we have $\dim(Z') = \dim(Z) - 1$. On the other hand Z' is contained in an irreducible component of X_{d-2}; using the induction hypothesis we deduce that

$$\dim(Z) - 1 = \dim(Z') \leqslant \frac{d-2+\langle 2\rho,\mu\rangle}{2},$$

hence that $\dim(Z) \leqslant \frac{d+\langle 2\rho,\mu\rangle}{2}$, as desired. □

Remark 4.10. The claim about the dimension of $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$ in Proposition 4.5 is extremely important in the proof of the Satake equivalence. On the other hand, the equidimensionality is not used.

4.4. **Comments.** The arguments of §4.3 give a bit more than Proposition 4.5. In particular, since the irreducible components of X_{d-2} have dimension strictly less than those of X_d, no irreducible component of X_d can be contained in X_{d-2}; this implies that $X_d \smallsetminus X_{d-2}$ is dense in X_d. Moreover, in view of Lemma 4.6(1) there exists a canonical bijection between the sets of irreducible components of X_d and of $X_d \smallsetminus X_{d-2}$, i.e. the union of the connected components of the schemes $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$ where λ runs over the set of weights in Λ_μ which satisfy $\langle 2\rho, \lambda\rangle = d$.

Similarly, let $\mathrm{Gr}_{G,<\mu}$ be the complement of $\mathrm{Gr}_{G,\mu}$ in $\mathrm{Gr}_{G,\leqslant\mu}$ (endowed with the reduced subscheme structure). Then we have a decomposition

$$S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu} = \big(S_\lambda \cap \mathrm{Gr}_{G,\mu}\big) \sqcup \big(S_\lambda \cap \mathrm{Gr}_{G,<\mu}\big)$$

where the first term on the right-hand side is open and the second one is closed. Here $\mathrm{Gr}_{G,<\mu}$ is the union of the varieties $\mathrm{Gr}_{G,\leqslant\mu'}$ where μ' runs over the dominant weights which satisfy $\mu' < \mu$; each irreducible component in $S_\lambda \cap \mathrm{Gr}_{G,<\mu}$ therefore has dimension strictly smaller than that of irreducible components of $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$. It follows that all the irreducible components of $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$ meet $S_\lambda \cap \mathrm{Gr}_{G,\mu}$. In conclusion, we have proved that $S_\lambda \cap \mathrm{Gr}_{G,\mu}$ is equidimensional of dimension $\langle \rho, \lambda + \mu\rangle$, and that there exists a canonical bijection between the sets of irreducible components of $S_\lambda \cap \mathrm{Gr}_{G,\mu}$ and of $S_\lambda \cap \mathrm{Gr}_{G,\leqslant\mu}$.

Continuing on this topic, we have proved in Lemma 4.8 that for $\lambda \in \Lambda_\mu$ we have

$$Y_{\lambda,\mu} \smallsetminus Y^\circ_{\lambda,\mu} \subset \bigcup_{\substack{\lambda' \in \Lambda_\mu \\ \lambda' < \lambda}} Y_{\lambda',\mu},$$

where the left hand side is endowed with the reduced subscheme structure. It seems reasonable (though maybe optimistic?) to expect that this inclusion is an equality. The following weaker claim is well known and easy to obtain.

Lemma 4.11. *Consider a presentation $\mathrm{Gr}_G = \mathrm{colim}_i \mathrm{Gr}_{G,i}$ as in §4.1. If $\lambda, \lambda' \in X_*(T)$ are such that $\lambda' \leqslant \lambda$, then for any i there exists $j \geqslant i$ such that*

$$(S_{\lambda'} \cap \mathrm{Gr}_{G,i})_{\mathrm{red}} \subset \overline{S_\lambda \cap \mathrm{Gr}_{G,j}}.$$

Proof. It suffices to prove the claim in case $\lambda' = \lambda - \alpha^\vee$ for some simple root α. The construction sketched above [BR18, (1.3.4)] or explained more precisely in [Zh16, Proof of Proposition 2.1.5(2)] shows that for l sufficiently large there exists a closed immersion $\mathbb{P}^1_k \to \mathrm{Gr}_{G,l}$ sending 0 to $[\lambda]$, ∞ to $[\lambda - \alpha^\vee]$, and such that the image of $\mathbb{A}^1_k$ is contained in $S_\lambda \cap \mathrm{Gr}_{G,l}$. In this case we have $[\lambda - \alpha^\vee] \in \overline{S_\lambda \cap \mathrm{Gr}_{G,l}}$.

Next, we consider the identification (4.1) for the weight $\lambda - \alpha^\vee$, and a presentation $L^{--}B = \mathrm{colim}_n L^{--}_n B$. Via this identification, the morphism $S_{\lambda-\alpha^\vee} \cap \mathrm{Gr}_{G,i} \to S_{\lambda-\alpha^\vee}$ factors through a morphism $S_{\lambda-\alpha^\vee} \cap \mathrm{Gr}_{G,i} \to L^{--}_n B$ for some n; in other words we have $(S_{\lambda-\alpha^\vee} \cap \mathrm{Gr}_{G,i})_{\mathrm{red}} \subset (L^{--}_n B) \cdot [\lambda - \alpha^\vee]$. Then if $j \geqslant \max(i,l)$ is such that $(L^{--}_n B) \cdot (S_\lambda \cap \mathrm{Gr}_{G,l}) \subset \mathrm{Gr}_{G,j}$, we have

$$(S_{\lambda-\alpha^\vee} \cap \mathrm{Gr}_{G,i})_{\mathrm{red}} \subset \overline{S_\lambda \cap \mathrm{Gr}_{G,j}},$$

as desired. □

Lemma 4.11 is often loosely stated as

$$\overline{S_\lambda} = \bigsqcup_{\lambda' \leqslant \lambda} S_{\lambda'}.$$

4.5. The case $G = \mathrm{SL}_2$.

4.5.1. *Description on the tree.* As explained in Talk 1, the k-points of the affine Grassmannian $\mathrm{Gr}_{\mathrm{PGL}_2}$ can be described as the vertices of a regular tree, where each vertex has a collection of neighbors parametrized by $\mathbb{P}^1(k)$. On this picture, the spherical orbits (which are parametrized by $X_*(T)^+ = \mathbb{Z}_{\geqslant 0}$) are spheres centered at the point corresponding to $[0]$ (with respect to the distance given by the length of paths from one point to another), and the corresponding Schubert variety is the set of points in the associated ball at distance from the sphere an even integer.

On such a picture, the semi-infinite orbits can be described as "spheres centered at ∞." For a graphical illustration, see [BR18, Fig. 1.5 in §1.3.2].

4.5.2. *Description of k-points.* Now we assume that $G = \mathrm{SL}_2$, with T the standard maximal torus consisting of diagonal matrices and B the Borel subgroup of upper triangular matrices. In this case $X_*(T)$ identifies naturally with $2\mathbb{Z}$, in such a way that 2ℓ corresponds to the class of the matrix

$$\begin{pmatrix} z^\ell & 0 \\ 0 & z^{-\ell} \end{pmatrix}.$$

In these terms we have

$$S_{2\ell}(k) = \left\{ \begin{pmatrix} 1 & Q \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z^\ell & 0 \\ 0 & z^{-\ell} \end{pmatrix} \cdot G(k[\![z]\!]) : Q \in k(\!(z)\!) \right\}.$$

This set is in bijection with $z^{\ell-1}k[z^{-1}]$ via the assignment

$$P \mapsto \begin{pmatrix} z^\ell & P \\ 0 & z^{-\ell} \end{pmatrix} \cdot G(k[\![z]\!]).$$

We now explain in which spherical orbit the point corresponding to P lies.

- If $P = 0$, this point belongs to $\mathrm{Gr}_{\mathrm{SL}_2, 2|\ell|}$.
- If $P \neq 0$, we write $P = az^m + \cdots + bz^{\ell-1}$ with $m \leqslant \ell - 1$ and $a \in k^\times$. If $\ell + m \leqslant 0$, using the identity

$$\begin{pmatrix} z^\ell & P \\ 0 & z^{-\ell} \end{pmatrix} \cdot \begin{pmatrix} 0 & -z^{-m}P \\ Q & z^{\ell-m} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-\ell-m}Q & 1 \end{pmatrix} \cdot \begin{pmatrix} z^m & 0 \\ 0 & z^{-m} \end{pmatrix}$$

  where $Q \in k[\![z]\!]$ is the inverse of $z^{-m}P$ we obtain that the point belongs to $\mathrm{Gr}_{\mathrm{SL}_2, 2|m|}$. If $\ell + m > 0$, using the equality

$$\begin{pmatrix} z^\ell & P \\ 0 & z^{-\ell} \end{pmatrix} = \begin{pmatrix} 1 & z^\ell P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z^\ell & 0 \\ 0 & z^{-\ell} \end{pmatrix}$$

we see that the point belongs to $\mathrm{Gr}_{\mathrm{SL}_2,2|\ell|}$. So, in all cases the point belongs to the orbit $\mathrm{Gr}_{\mathrm{SL}_2,2\max(|\ell|,|m|)}$.

From this analysis we deduce that for $\ell \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geqslant 0}$ such that $|\ell| \leqslant n$ we have

$$(S_{2\ell} \cap \mathrm{Gr}_{\mathrm{SL}_2,\leqslant n})(k) = S_{2\ell}(k) \cap \mathrm{Gr}_{\mathrm{SL}_2,\leqslant n}(k) = \left\{ \begin{pmatrix} z^\ell & P \\ 0 & z^{-\ell} \end{pmatrix} : P \in kz^{-n} \oplus \cdots \oplus kz^{\ell-1} \right\}.$$

This is in accordance with Proposition 4.5, which says that $S_{2\ell} \cap \mathrm{Gr}_{\mathrm{SL}_2,\leqslant n}$ is affine of dimension $n+\ell$.

Remark 4.12. It is likely that a refinement of the computation above can be used to construct an isomorphism of schemes $\mathbb{A}_k^{n+\ell} \xrightarrow{\sim} S_{2\ell} \cap \mathrm{Gr}_{\mathrm{SL}_2,\leqslant n}$.

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Université Clermont Auvergne, CNRS, LMBP, F-63000 Clermont-Ferrand, France.
Email address: simon.riche@uca.fr