On the Beilinson-Drinfeld Grassmannians

Yassine El Kaouni

Abstract

We introduce the Bellinson-Drinfeld Grassmannian and we discuss some of its basic geometric properties and their relation to Hecke stacks. These are lecture notes from the 2022 workshop on geometric Satake that took place in Clermont-Ferrant.

1 Introduction

The Beilinson-Drinfeld Grassmannians are a generalization of the classic affine Grassmannians. First introduced by Beilinson and Drinfeld in [AV91], it upgrades the underlying algebro-geometric object of the affine Grassmannians to a family $\mathrm{Gr}_{G,I} \to X^I$ subject to a fusion structure where G is a group, X a sufficiently nice curve and I a finite set.

In the first section, we introduce the Beilinson-Drinfeld Grassmannians, then using a result due to Beauville-Laszlo, we relate it to the classical affine grassmannians Gr_G in the case where $I = \{*\}$. Furthermore, we discuss its respresentability and we show that it carries the structure of an ind-scheme.

In the second section, we define the loop groups LG and L^+G and we show that they induce a presentation of $\mathrm{Gr}_{G,I}$ as the fppf-quotient $[LG/L^+G]_{fppf}$.

As for the third section, we present a stratification of the Beilinson-Drinfeld Grassmannians in term of schemes called the Schubert cells, which are parameterized by the coweights of the group G. The Beilinson-Drinfeld Grassmannians Schubert cells arise from the Schubert cells of the affine Grassmannians through "twisting" with the curve X, thus taking in account the new parameters.

To conclude, we present in the last section a glimpse of the possible applications such as studying the Hecke stack, needed in order to define the Satake category, which is done in [FS] sections 1-6.

2 Notations

Throughout this paper, X denotes a smooth separated geometrically connected curve over a field k and $G \to X$ smooth affine group scheme. Fix a finite set I and denote by X^I the I-fold product of X. For a k-algebra R and $x \in X(R)$ denote by Γ_x its graph in X_R. For a point $(x_i)_{i\in I} \in X^I(R)$ define $\Gamma_x = \bigcup_{i\in I} \Gamma_{x_i}$. Let $E \to X$ be a G-torsor and Y a k-scheme with a G-action, define the twisted product $E \times^G Y := [E \times Y/G]$, where G acts on $E \times Y$ diagonally.

3 Definition and first properties

Definition 3.1. The Beilinson-Drinfled Grassmannian is the functor $\mathrm{Gr}_{G,I} : (Sch/k)^{op} \longrightarrow$ Sets such that for a k-algebra R, we have

$$\mathrm{Gr}_{G,I}(R) = \left\{ (x, \mathcal{E}, \alpha) \,\middle|\, \begin{array}{c} x \in X^I(R), \mathcal{E} \text{ is a } G_R\text{-torsor over } X_R \\ \alpha : \mathcal{E}_{|X_R\setminus\Gamma_x} \simeq \mathcal{E}^0{}_{|X_R\setminus\Gamma_x} \text{ a trivialization} \end{array} \right\}$$

Remark 3.1. The finite flatness hypothesis of G over X implies that a torsor $\mathcal{E}$ over X_R is representable by a principal G-bundle ($\pi : P \to X_R$, $\rho : G_R \times_{X_R} P \to P$) (we refer to [MOl16] section 4.5 for the definition of a principal G-bundle). Let Aff denote the category whose objects are affine morphisms of schemes ($f : X \to Y$), if $s : S' \to S$ is an fppf covering, by [MOl16, proposition 4.4.9] s is an effective descent morphism for Aff. The morphism π is fppf locally trivial, by descent theory we deduce that π is affine. As a consequence we can write $P = \underline{spec_{\mathcal{O}_{X_R}}}\mathcal{F}$ for some quasi-coherent $\mathcal{O}_{X_R}$-module $\mathcal{F}$. The previous discussion yields that $\mathcal{E}$ can be identified with a quasi-coherent $\mathcal{O}_{X_R}$-module satisfying certain compatibility conditions.

Theorem 3.1. *[AV91, Theorem 2.12.1] Let $p : \widetilde{S} \to S$ be a morphism of schemes, $D \subset S$ an effective Cartier divisor. Suppose that $\widetilde{D} := p^{-1}(D)$ is a cartier divisor $\widetilde{S}$ and that the induced morphism $\widetilde{D} \to D$ is an isomorphism. Set $U := S \setminus D, \widetilde{U} := \widetilde{S} \setminus \widetilde{D}$. Denote by $\mathcal{C}$ the category of quasi-coherent modules that have no non-zero local section supported in D, in other words, if $\mathcal{F}$ is an object of $\mathcal{S}$, then any section of $\mathcal{F}$ doesn't vanish on D entirely. Denote by $\widetilde{C}$ the similar category for $(\widetilde{S}, \widetilde{D})$. Denote by $\mathcal{C}'$ the category of tuples $(\mathcal{M}_1, \mathcal{M}_2, \varphi)$ where $\mathcal{M}_1$ is a quasi-coherent $\mathcal{O}_U$-module, $\mathcal{M}_2 \in \widetilde{C}$, together with an isomorphism $\varphi : \mathcal{M}_{1|\widetilde{U}} \simeq \mathcal{M}_{2|\widetilde{U}}$.*

We have a functor

$$\begin{aligned} F : \mathcal{C} &\longrightarrow \widetilde{\mathcal{C}} \\ \mathcal{M} &\longrightarrow (\mathcal{M}_{|U}, p^*\mathcal{M}, \varphi); \end{aligned}$$

here φ denotes the natural isomorphism $\mathcal{M}_{|\widetilde{U}} \simeq p^*\mathcal{M}_{|\widetilde{U}}$, obtained by pulling back across the induced map $p_{|\widetilde{U}} : \widetilde{U} \to U$. We have the following results:

(a) The functor F is an equivalence of categories,

(b) The module $\mathcal{M}$ is locally of finite type (resp. flat, locally of finite rank) if and only if $\mathcal{M}_{|U}$ and $p^*\mathcal{M}$ are as well.

Let $x_0 \in X(k)$ a closed point. Let $\widehat{\mathcal{O}}_{X,x_0}$ be the completion of $\mathcal{O}_{X,x_0}$ along its maximal ideal. Fix an isomorphism of k-algebras $\widehat{\mathcal{O}}_{X,x_0} \simeq k[[t_{x_0}]]$. For a k-algebra R, define the disc $\mathbb{D}_R := \operatorname{Spec} R[[t_{x_0}]]$ and the open disc $\mathbb{D}^*_R = \operatorname{Spec} R((t_{x_0})) = \operatorname{Spec} R[[t_{x_0}]] \setminus \{x_0\}$.

Corollary 3.1. *For a k-algebra R, one has*

$$\mathrm{Gr}_{G,I|x_0}(R) = \left\{ (\mathcal{E}, \alpha) \middle| \begin{array}{c} \mathcal{E} \text{ is a } G_R\text{-torsor on } \mathbb{D}_R \\ \alpha : \mathcal{E}_{|\mathbb{D}^*_R} \simeq \mathcal{E}^0_{|\mathbb{D}^*_R} \text{ a trivialization} \end{array} \right\}.$$

In particular $\mathrm{Gr}_{G,I|x_0}$ can be identified with the classical affine grassmannian.

Proof. We apply the previous theorem to $\widetilde{S} = \mathbb{D}_R$, $\widetilde{D} = \{x_0\}$ $S = X_R$, $D = \{x_0\}$. Thus the category of quasi-coherent modules on X_R whose sections whose sections don't vanish on $\{x_0\}$ is equivalent to the category of elements $(\mathcal{F}_1, \mathcal{F}_2, \varphi)$, where $\mathcal{F}_1$ is a quasi-coherent $\mathcal{O}_{X_R \setminus \{x_0\}}$, $\mathcal{F}_2$ a quasi-coherent $\mathcal{O}_{\mathbb{D}_R}$ module whose sections don't vanish on $\{x_0\}$ and $\varphi : \mathcal{F}_{1|\mathbb{D}^*_R} \simeq \mathcal{F}_{2|\mathbb{D}^*_R}$. Using remark 3.1, we can translate the above result to torsors. Given a G_R-torsor $\mathcal{E}$ on X_R together with a trivialization $\alpha : \mathcal{E}_{X_R\{x_0} \simeq \mathcal{E}^0_{X_R\{x_0}}$, it is obtained by considering a G_R-torsor $\mathcal{E}'$ on $\mathbb{D}_R$ together with a trivialization $\mathcal{E}'_{|\mathbb{D}^*_R} \simeq \mathcal{E}^0_{|\mathbb{D}^*_R}$. □

Theorem 3.2. *The Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,I} \to X^I$ is representable by an ind-scheme, ind-of finite type. Moreover if we assume $G \to X$ is reductive, then it is ind-projective.*

Proof. We prove the theorem in the special case where $G = GL_n$. the general case is obtained using similar arguments as in [Zhu17, propositions 1.2.5 and 1.2.6]. The idea is to choose a faithful representation $G \to GL_n$ for some $n > 1$. For a k-algebra R, a $GL_{n,R}$-torsor can be identified with a vector bundle of rank n. Thus we can write

$$\mathrm{Gr}_{GL_n,I}(R) = \left\{ (x, \mathcal{E}) |\ x \in X^I(R) \text{ and } \mathcal{E} \subset \mathcal{O}^n_{X_R \setminus \Gamma_x} \text{an } \mathcal{O}_{X_R}\text{-lattice} \right\}.$$

By an $\mathcal{O}_{X_R}$-lattice $\mathcal{E} \subset \mathcal{O}^n_{X_R \setminus \Gamma_x}$ we mean that $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_{X_R}$-module such that $\mathcal{E}(U)$ is an $\mathcal{O}_{X_R}(U)$-lattice inside $\mathcal{O}_{X_R}(U \setminus \Gamma_x)^n$ for all opens $U \subset X_R$. For $N > 0$, define the functor $\mathrm{Gr}^{(N)}_{GL_n,I}$ such that for a k-algebra R we have

$$\mathrm{Gr}^{(N)}_{GL_n,I}(R) = \{ (x, \mathcal{E}) |\ \mathcal{O}(-N\Gamma_x) \subset \mathcal{E} \subset \mathcal{O}(N\Gamma_x) \}.$$

By the theory of Hilbert schemes ([MOl16, section 1.5]), $\mathrm{Gr}^{(N)}_{GL_n,I} \to X^I$ is representable by a projective scheme of finite type. On the other hand

$$\mathrm{Gr}_{GL_n,I} = \varinjlim_{N>0} \mathrm{Gr}^{(N)}_{GL_n,I}.$$

Which concludes the proof. □

4 Loop groups

For a k-algebra R and $x \in X^I(R)$, let $\widehat{\Gamma}_x$ denote the completion of X_R along Γ_x. Write $\widehat{\Gamma}_x = \mathrm{Spf}\ A_x$ and define $\widehat{\Gamma}'_x = \mathrm{Spec}\ A_x$. We have two natural maps $\pi : \widehat{\Gamma}_x \to X_R$ and $p : \widehat{\Gamma}_x \to \widehat{\Gamma}'_x$. By [AV91, p. 2.12.6] there is a unique map $i : \widehat{\Gamma}'_x \to X_R$ such that the following diagram

$$\begin{array}{ccc} \widehat{\Gamma}_x & \xrightarrow{\pi} & X \\ {\scriptstyle p}\downarrow & \nearrow_{\exists! i} & \\ \widehat{\Gamma}'_x & & \end{array}$$

commutes. The discussion above allows to define the loop groups as following:

Definition 4.1. The loop group is the functor $L_IG : (Sch/k)^{op} \to \mathrm{Sets}$ which assigns for a k-algebra R the set

$$L_IG(R) = \left\{ (x,g) | \, x \in X^I(R), \, g \in G(\widehat{\Gamma}'_x \setminus \Gamma_x) \right\}.$$

Similarly, define the positive loop L_I^+G such that for a k-algebra R we have

$$L_I^+G(R) = \left\{ (x,g) | \, x \in X^I(R), \, g \in G(\widehat{\Gamma}'_x) \right\}.$$

Proposition 4.1. *The functor L_I^+G is represented by an affine scheme over X^I with geometrically connected fibers.*

Proof. Fix a morphism $\mathrm{Spec}\, R \to X^I$. Let $\widehat{\Gamma}_x^{(i)}$ be the i-th infinitesimal neighborhood of $\widehat{\Gamma}_x$. It results from the definition of L_I^+G that

$$L_I^+G \times_{X^I} \mathrm{Spec}\, R = \varprojlim_i \mathrm{Res}_{\widehat{\Gamma}_x^{(i)}/R} G.$$

The proposition follows. $\square$

Lemma 4.1. *The ind-scheme L_IG represents the functor (which we still denote L_IG) which assigns for a k-algebra R the set*

$$L_IG(R) = \left\{ (x, \mathcal{E}, \alpha, \sigma) \left| \begin{array}{c} x \in X^I(R), \mathcal{E} \textit{ is a } G_R\textit{-torsor over } X_R \\ \alpha : \mathcal{E}_{|X_R \setminus \Gamma_x} \simeq \mathcal{E}^0{}_{|X_R \setminus \Gamma_x}, \sigma : \mathcal{E}_{|\widehat{\Gamma}'_x} \simeq \mathcal{E}^0_{|\widehat{\Gamma}'_x} \\ \textit{trivializations} \end{array} \right. \right\}$$

Proof. For a k-algebra R, denote $M(R)$ the functor in the RHS of the previous formula. We have a morphism

$$\begin{aligned} M(R) &\longrightarrow L_IG(R) \\ (x, \mathcal{E}, \alpha, \sigma) &\longrightarrow (x, \alpha_{|\widehat{\Gamma}'_x \setminus \Gamma_x} \circ \sigma^{-1}_{|\widehat{\Gamma}'_x \setminus \Gamma_x}). \end{aligned}$$

To prove that it is an isomorphism we construct an inverse. A morphism $g : \mathcal{E}^0_{\widehat{\Gamma}'_x \setminus \Gamma_x} \longrightarrow \mathcal{E}^0_{\widehat{\Gamma}'_x \setminus \Gamma_x} \in G(\widehat{\Gamma}'_x \setminus \Gamma_x)$ gives rise to a tuple $(\mathcal{E}^0_{\widehat{\Gamma}'_x \setminus \Gamma_x}, \mathcal{E}^0, g)$. By remark 3.1 and theorem 3.1, it is equivalent to considering $(\mathcal{E}, \alpha, \sigma)$. Here $\mathcal{E}$ is a G_R-torsor together with trivializations $\alpha : \mathcal{E}_{|X_R \setminus \Gamma_x} \simeq \mathcal{E}^0{}_{|X_R \setminus \Gamma_x}, \sigma : \mathcal{E}_{|\widehat{\Gamma}'_x} \simeq \mathcal{E}^0_{|\widehat{\Gamma}'_x}$. $\square$

A consequence of the previous lemma is that we have an L_I^+G-equivariant morphism given by

$$\begin{aligned} L_IG &\longrightarrow \mathrm{Gr}_{G,I} \\ (x, \mathcal{E}, \alpha, \sigma) &\longrightarrow (x, \mathcal{E}, \alpha). \end{aligned}$$

Theorem 4.1. *The morphism $L_IG \longrightarrow \mathrm{Gr}_{G,I}$ is an L_I^+G-torsor. In particular*

$$\mathrm{Gr}_{G,I} = [L_IG/L_I^+G]_{fppf}.$$

Lemma 4.2. *Let $\mathcal{F} \to \widehat{\Gamma}'_x$ be a G_R-torsor. There exists an fppf extension $R \to R'$ such that $\mathcal{F}_{|\widehat{\Gamma}'_x \times_R R'}$ is trivial.*

Proof. It suffices to treat the case $I = \{*\}$. The general case is obtained by induction. Write $\widehat{\Gamma}'_x = \operatorname{Spec} R[[t]]$. We have the cartesian diagram

$$\begin{array}{ccc} \mathcal{F}_{|R} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} R[[t]] \end{array}.$$

By decent for affine smooth morphisms, the map $\mathcal{F} \to \operatorname{Spec} R[[t]]$ is smooth affine. By the properties of base change, the map $\mathcal{F}_{|R} \to \operatorname{Spec} R$ is smooth affine as well. Thus there exists a factorisation

$$\begin{array}{ccccc} & \nearrow & \mathcal{F}_{|R} & \longrightarrow & \mathcal{F} \\ & & \downarrow & & \downarrow \\ \operatorname{Spec} R' & \xrightarrow[\pi_0]{} & \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} R[[t]] \end{array},$$

with $\operatorname{Spec} R' \to \operatorname{Spec} R$ étale.

$$\begin{array}{ccccc} \operatorname{Spec} R'[[t]]/t^{n+1} & \xrightarrow{\pi_{n+1}} & \mathcal{F}_{|R} & \longrightarrow & \mathcal{F} \\ \uparrow & \nearrow & \downarrow & & \downarrow \\ \operatorname{Spec} R' & \xrightarrow[\pi_0]{} & \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} R[[t]] \end{array}.$$

By smoothness of the map $\mathcal{F}_{|R} \to \operatorname{Spec} R$, we can inductively construct for $n \geq 0$ a map $\pi_{n+1} : \operatorname{Spec} R'[[t]]/t^{n+1} \to \mathcal{F}$ such that $\pi_{n+1|R'[[t]]/t^n} = \pi_n$. The tuple $(\pi_n)_n$ can thus be regarded as an element of

$$\varprojlim_n \operatorname{Hom}_{R[[t]]}(\operatorname{Spec} R'[[t]]/t^{n+1}, \mathcal{F}) = \operatorname{Hom}_{R[[t]]}(\varinjlim_n \operatorname{Spec} |R'[[t]]/t^{n+1}, \mathcal{F}) = \operatorname{Hom}_{R[[t]]}(\operatorname{Spec} R'[[t]], \mathcal{F}).$$

In a conclusion $\mathcal{F}(R'[[t]]) \neq 0$, which concludes the proof. □

Proof of the theorem: Fix a k-algebra R. Let $(x, \mathcal{E}, \alpha, \sigma) \in \operatorname{Gr}(R)$. By the previous lemma we can choose an fppf extension $R' \to R$ such that $\mathcal{E}_{\widehat{\Gamma}'_x}$ becomes trivial. In particular, the map $L_I G \to \operatorname{Gr}$ is fppf locally trivial. The theorem follows. □

5 Schubert varieties

Assume $G \to \operatorname{Spec} k$ reductive. Define the functor $\widehat{X} : (Sch/k)^{op} \longrightarrow \mathrm{Sets}$ such that for a k-algebra R we have

$$\widehat{X}(R) = \left\{(x, \alpha) | x \in X(R), \alpha : \widehat{\Gamma}'_x \simeq \mathbb{D}_R\right\}.$$

For $I = \{*\}$, $\operatorname{Gr}_{G,\{*\}}$ can be expressed as the twisted product $\operatorname{Gr}_{G,\{*\}} = \widehat{X} \times^{Aut(\mathbb{D})} \operatorname{Gr}_G$, where Gr_G is the classical affine Grassmannian. Let $T \subset G$ be a maximal torus. For a cocharacter $\mu \in X_*(T)$ let $\operatorname{Gr}_{G,\leq\mu}$ be the Schubert variety of Gr_G as in [Zhu17, section 2.1].

Definition 5.1. Given a cocharacter $\mu \in X_*(T)$, define the Schubert variety $\operatorname{Gr}_{\leq\mu,X}$ for the Beilinson-Drinfled Grassmannian $\operatorname{Gr}_{G,\{*\}}$ as the twisted product $\widehat{X} \times^{Aut(\mathbb{D})} \operatorname{Gr}_{G,\leq\mu}$.

Let $U \subset X^I$ the open subscheme where all coordinates are distinct. We have

$$\operatorname{Gr}_{G,I|U} = \prod_{i \in I} \operatorname{Gr}_{G,\{i\}|U}.$$

For $\mu = (\mu_i)_{i \in I} \in X_*(T)^I$, write

$$\operatorname{Gr}_{\leq\mu,X|U} = \prod_{i \in I} \operatorname{Gr}_{\leq\mu_i|U}.$$

Definition 5.2. For $\mu = (\mu_i)_{i\in I} \in X_*(T)^I$ define the Schubert variety parameterized by μ as the scheme theoretic closure of $\mathrm{Gr}_{\leq\mu,X|U}$ inside $\prod_{i\in I} \mathrm{Gr}_{\leq\mu_i}$.

Next, we study the factorization property in the case of $I = \{i, j\}$. For simplicity, denote $\mathrm{Gr}_{G,\{*\}}$ by Gr_X and $\mathrm{Gr}_{G,\{i,j\}}$ by Gr_{X^2}. Let $\Delta : X \to X^2$ be the diagonal morphism.

Proposition 5.1. *[Zhu17, Proposition 3.1.13] They are canonical isomorphisms*

$$\Delta : \mathrm{Gr}_X \simeq \mathrm{Gr}_{X^2} \times_{X^2,\Delta} X,\ c : \mathrm{Gr}_{X^2}|_{X^2\setminus\Delta} \simeq (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_{X^2\setminus\Delta}.$$

In addition, there is an S_2-action on Gr_{X^2} such that c is S_2-equivariant.

Proposition 5.2. *[Zhu17, Proposition 3.1.14] Assume $\underline{G} = G\times X$ is constant and let p be a prime such that $p \nmid \pi_1(G_{der})$. Regard $\mathrm{Gr}_{\leq\mu,I|U} \times \mathrm{Gr}_{\leq\lambda,I|U}|_{X^2\setminus\Delta}$ as a subset of $\mathrm{Gr}_{X^2|X^2\setminus\Delta}$. Via the above identification we have $\mathrm{Gr}_{\leq(\mu,\lambda),X|\Delta} = \mathrm{Gr}_{\leq\mu+\lambda,X}$*

6 Application to Hecke stack

Definition 6.1. Define the Hecke stack as the functor $(Sch/k)^{op} \to$ Sets which assigns for a k-algebra R the set

$$\mathrm{Hecke}_{G,I}(R) = \left\{ (x, \mathcal{E}, \mathcal{E}', \alpha) \middle| \begin{array}{c} x \in X^I(R) \\ \mathcal{E} \text{ is a } G_R\text{-torsor on } \widehat{\Gamma}'_x \\ \alpha : \mathcal{E}_{\widehat{\Gamma}'_x\setminus\Gamma_x} \simeq \mathcal{E}'_{\widehat{\Gamma}'_x\setminus\Gamma_x} \end{array} \right\}$$

Similarly to the case of Beilinson-Drinfeld Grassmannians, we prove using 3.1 that for $x_0 \in X(k)$, $\mathrm{Hecke}_{G,I|x_0} = \mathrm{Hecke}_G$ where Hecke_G is the usual Hecke stack. We have an L_I^+G equivariant map

$$\begin{aligned} \mathrm{Gr}_{G,I} &\longrightarrow \mathrm{Hecke}_{G,I} \\ (x, \mathcal{E}, \alpha) &\longrightarrow (x, \mathcal{E}_{|\widehat{\Gamma}'_x}, \mathcal{E}_{0|\widehat{\Gamma}'_x}, \alpha_{|\widehat{\Gamma}'_x\setminus\Gamma_x}). \end{aligned}$$

Theorem 6.1. *The map $\mathrm{Gr}_{G,I} \to \mathrm{Hecke}_{G,I}$ is an L_I^+G torsor. In particular*

$$\mathrm{Hecke}_{G,I} = [L_I^+G \setminus L_IG/L_I^+G]_{fppf}.$$

Proof. As in the proof of theorem 4.1, using smoothness and Grotehendieck algebrization lemma, we choose for a k-algebra R an fppf-extension $R \to R'$ such that $\mathcal{E}_{|\widehat{\Gamma}'_x}$ is trivial. As a consequence, the map above becomes an isomorphism. □

References

[AV91] A.Beilinson and V.Drinfled. *Quantization of hitchin's integrable system and hecke eigensheaves.* Mathematics Subject Classification. 1991.

[BT84] F. Bruhat and J. Tits. "Groupes réeductifs sur un corps local II. Sch´ema en groupes. Existence d'une donnée radicielle valuée ". In: *Inst. Hautes ´Etudes Sci. Publ. Math.* 60 (1984), pp. 197–376.

[FS] L. Fargues and P. Scholze: "Geometrization of the local Langlands correspondence ". In: ().

[MOl16] M.Olsson. *Algebraic Spaces and Stacks.* Colloquium Publications. 2016.

[Ric19] Timo Richarz. "A new approach to geometric Satake equivalence ". In: *Documenta Mathematica* 19 (2019), pp. 209–246.

[Zhu17] Xinwen Zhu. "An introduction to affine Grassmannians and the geometric Satake equivalence ". In: *In: Geometry of Moduli Spaces and Representation Theory. IAS/Park City Mathematics Series* 24 (2017), pp. 59–154.