# Talk 6: Affine Grassmanian

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These notes are mostly based on [Zhu16]. Throughout, let $k$ be an algebraically closed field and $G$ a (connected) reductive group over $k$. For a $k$-algebra $R$, let $D_R$ and $D_R^*$ denote $\operatorname{Spec}(R[[t]])$ and $\operatorname{Spec}(R((t)))$, respectively. One central player in the geometric Satake equivalence is the *Hecke stack* $\mathrm{Hk}_G$ over $k$, defined by sending a $k$-algebra $R$ to the groupoid whose objects consist of the following data:

- two $G$-torsors $\mathcal{E}_1, \mathcal{E}_2$ on $D_R$.
- an isomorphism $\varphi\colon \mathcal{E}_1 \otimes_{D_R} D_R^* \xrightarrow{\sim} \mathcal{E}_2 \otimes_{D_R} D_R^*$.

Recall that for a group scheme $G$ and a scheme $X$, a $G$-torsor over $X$ is a scheme map $Y \to X$ together with a $G$-action on $Y$ such that Zariskii locally (equivalently, étale locally) on $X$ one has $Y = X \times G$, equivariantly for the canonical $G$-action on the right. The $G$-torsors on $X$ form a category and for any map of schemes $X \to S$ one gets a pullback functor from the $G$-torsors on $S$ to those on $X$, compatibly with composition of maps of schemes. The principal example in the world of affine Grassmanians is that of $\mathrm{GL}_n$-torsors — those are just vector bundles of rank $n$.

In order to describe $\mathrm{Hk}_G$ and in particular the étale sheaves on it, we will look at a simpler object, additionally parametrizing a trivialization of the second torsor:

**Definition 6.1.** The *affine Grassmanian* $\mathrm{Gr}_G$ is the presheaf sending a $k$ algebra $R$ to the set of isomorphism classes of the following data:

- a $G$-torsor $\mathcal{E}$ on $R[[t]]$.
- an isomorphism $\mathcal{E} \otimes_{D_R} D_R^* \xrightarrow{\sim} \mathcal{E}_0 \otimes_{D_R} D_R^*$, where $\mathcal{E}_0$ denotes the standard $G$-torsor on $R[[t]]$.

So, for instance, an element in $\mathrm{Gr}_{GL_n}(R)$ would consist of a finite projective $R[[t]]$-module of rank $n$ (or rather its isomorphism class) together with a trivialization after inverting $t$ — or, equivalently, an $R[[t]]$ sublattice of $R((t))$. Note that, by fixing the trivialization of $\mathcal{E}_2$, these data (even before taking isomorphism classes) have no non-trivial automorphisms, so we can expect $\mathrm{Gr}_G$ to be a reasonable set-valued sheaf. In fact $\mathrm{Gr}_G$ even has a nice geometric structure:

**Proposition 6.2.** $\mathrm{Gr}_G$ *is an ind-projective ind-scheme over* $k$.

By an ind-scheme, we will mean an fpqc sheaf on the category of $k$-algebras which can be written as a filtered colimit $\varinjlim X_i$, where the $X_i$ are schemes and the transition maps are closed immersions. It is called ind-projective if all the $X_i$ can be chosen to be projective.

*Proof.* See [Zhu16, theorem 1.1.3] for $G = \mathrm{GL}_n$ and [Zhu16, theorem 1.2.2] for the general case. Let us present the idea: For the $\mathrm{GL}_n$ case, one can always find some $r \in \mathbb{Z}_{\geq 0}$ such that

$$t^r \mathcal{E}_0 \subseteq \mathcal{E} \subseteq t^{-r} \mathcal{E}_0. \tag{1}$$

Some work is done to show that projectivity of $\mathcal{E}$ as an $R[[t]]$-module and of $\mathcal{E}/t^r\mathcal{E}_0$ as an $R$-module align, after which one can realize all $R$-points satisfying (1) as a closed subscheme of $\mathbb{P}^{(2r+1)n}$. Taking the filtered colimit over the different values of $r$ then yields the ind-projectivity of $\mathrm{Gr}_{GL_n}$.

For general $G$, one finds a embedding $G \to \mathrm{GL}_n$ with affine quotient. It turns out that this induces a closed embedding $\mathrm{Gr}_G \to \mathrm{Gr}_{GL_n}$, so $\mathrm{Gr}_G$ is ind-projective as well. □

As mentioned before, there is a map $\mathrm{Gr}_G \to \mathrm{Hk}_G$, simply forgetting that $\mathcal{E}_0$ is the trivial torsor. This map has a nice desription in terms of so-called loop groups:

**Definition + Proposition 6.3.** *Define the* loop group *and* positive loop group *as the presheaves on $k$-algebras*

$$LG\colon R \mapsto G(R((t))), \qquad L^+G\colon R \mapsto G(R[[t]]).$$

*Then $L^+G$ is represented by a scheme (of infinite type over $k$), while $LG$ is represented by an ind-affine ind-scheme. Furthermore, $LG$ and $L^+G$ inherit the structure of a group object in their respective category (equivalently, in the category of sheaves) from the group structure of $G$.*

*Proof.* For the first part, see [PR08, 1.a]. For the group structure, note that $LG(R) = G(R((t)))$ has the structure of a group (since $G$ is a group scheme), compatible with maps $R \to R'$, and similarly for $L^+G$. □

**Proposition 6.4.** $\mathrm{Gr}_G = [LG/L^+G]$ *and* $\mathrm{Hk}_G = [L^+G\backslash \mathrm{Gr}_G] = [L^+G\backslash LG/L^+G]$ *as étale (equivalently, fpqc) quotients.*

*Proof.* See [Zhu16, Proposition 1.3.6] for the first claim. The second one can be proven very similarly. □

If one is willing to picture $D_R^* = \mathrm{Spec}(R((t)))$ as a punctured formal disc, then in a way $LG$ describes the space of loops in $G$, while the subsheaf $L^+G$ are the loops whose homotopy class is trivial. The following result reinforces that impression:

**Theorem 6.5.**
1. *One has isomorphisms $\pi_0(LG) \cong \pi_1(G)$ and $\pi_0(LG) \cong \pi_0(\mathrm{Gr}_G)$.*
2. *If $G \otimes_k k((t))$ is semisimple and $p \nmid \pi_1(G \otimes_k k((t)))$, then $\mathrm{Gr}_G$ is reduced.*

*Proof.* [Zhu16, Theorem 1.3.11]; In (1), we can omit the Galois group mentioned in loc. cit. since for us, $G$ is always constant over $k = \overline{k}$ and thus the Galois group of $k((t))$ acts trivially on $\pi_1(G)$. $\square$

We want to further understand the geometry of $\mathrm{Gr}_G$. From now on, let $\mathcal{O} = k[[t]]$ and $F = k((t))$. For now, fix embeddings $T \subset B \subset G$, where $T$ is a maximal torus and $B$ a Borel subgroup. The group theoretical data $X_\bullet, X^\bullet, X_\bullet^+, X_\bullet^\bullet, W$ are defined as usual. Recall the Cartan decomposition:

**Proposition 6.6.**

$$G(F) = \bigsqcup_{\mu \in X_\bullet(T)^+} G(\mathcal{O}) t^\mu G(\mathcal{O}).$$

*Proof.* The proof in full generality can be found e.g. in [BT72, Proposition 4.4.3]. Let's look at a simpler one for $G = \mathrm{GL}_n$. Let $M_0 \subset F^n$ be the standard lattice with basis $(e_i)$. Now every $g \in \mathrm{GL}_n(F)$, identified with its matrix with respect to $e_i$, defines a translated lattice $g(M_0) \subset F^n$ and we can find some $N \in \mathbb{Z}$ such that $t^N g(M_0) \subset M_0$. By the theory of finitely generated lattices over a DVR, we can now find a basis $e_1' \dots e_n'$ of $M_0$ and integers $r_0 \geq \cdots \geq r_n$ such that

$$t^{r_1} e_1', \dots, t^{r_n} e_n'$$

forms a basis of $t^N g(M_0)$, and hence

$$t^{r_1 - N} e_1', \dots, t^{r_n - N} e_n'$$

forms a basis of $g(M_0)$. Let $B$ be the matrix of $g$ with respect to the basis $e_1' \dots, e_n'$ of $M_0$. Since $(e_i)$ and $(e_i')$ are both $\mathcal{O}$-bases of $M_0$, there is some $S \in \mathrm{GL}_n(\mathcal{O})$ such that $g = SBS^{-1}$. Similarly, $g(e_1), \dots, g(e_n)$ and $t^{r_1-N}e_1', \dots, t^{r_n-N}e_n'$ are both $\mathcal{O}$-bases of $g(M_0)$, so there exists some $C \in \mathrm{GL}_n(\mathcal{O})$ such that $B = \mathrm{diag}(t^{r_1-N}, \dots, t^{r_n-N}) \cdot C$, so

$$g = S \,\mathrm{diag}(t^{r_1-N}, \dots, t^{r_n-N}) C S^{-1} \in G(\mathcal{O}) t^\mu G(\mathcal{O}).$$

$\square$

**Corollary 6.7.**

$$G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}) = X_\bullet(T)^+.$$

Note that while the Cartan decomposition depends on both the choice of a uniformizer $t \in F$ and an embedding $T \hookrightarrow G$, Corollary 6.7 depends on neither (if one views $X_\bullet(T)^+$ as a quotient of $X_\bullet(T)$ instead of a subgroup).

We now want to dissect $\mathrm{Gr}_G$ into simpler geometric objects (actual varieties). This will be done by bounding the so-called relative position — roughly speaking, how far our torsor $\mathcal{E}_0$ is from the trivial one.

**Construction 6.8.** Let $\mathcal{E}_1, \mathcal{E}_2$ be two $G$-torsors over $D = D_k$ and let $\beta\colon \mathcal{E}_1|_{D^*} \cong \mathcal{E}_2|_{D^*}$ be an isomorphism. Since $k$ is an algebraically closed field, the $\mathcal{E}_i$ are in fact trivial, so one can choose trivializations $\Phi_i\colon \mathcal{E}_i \to \mathcal{E}_0$ and obtain an automorphism of the trivial $G$-torsor $\Phi_2\beta\Phi_1^{-1} \in \operatorname{Aut}(\mathcal{E}^0|_{D^*})$. This is an element of $G(F)$, and changing the trivializations $\Phi_i$ amounts to left- and right multiplication by elements of $G(\mathcal{O})$, so we get a well-defined element

$$\operatorname{Inv}(\beta) \in G(\mathcal{O})\backslash G(F)/G(\mathcal{O}) \cong X_\bullet(T)^+,$$

called the *relative posistion of* $\beta$.

Even if we replace $k$ by a not necessarily algebraically closed $k$-algebra $K$, we can still define $\operatorname{Inv}(\beta)$ by base changing the $\mathcal{E}_i$ to an algebraic closure $\overline{K}$; the resulting $\operatorname{Inv}(\beta)$ will be independent of the choice of $\overline{K}$.

Finally, replace $K$ by any $k$-algebra $R$. Then we can still define the relative position of $\beta$ at any $x \in \operatorname{Spec}(R)$ by base changing the $\mathcal{E}_i$ to $D_{k(x)}$, where $k(x)$ is the residue field at $x$.

**Proposition 6.9.** *[Zhu16, Proposition 2.1.4] Let* $X = \operatorname{Spec}(R)$ *and* $\mu \in X_\bullet(T)^+$. *Consider a morphism* $\beta\colon \mathcal{E}_1 \to \mathcal{E}_2$ *between two* $G$*-torsors* $\mathcal{E}_1, \mathcal{E}_2$. *Then the set of points*

$$X_{\leq\mu} := \{x \in X \,|\, \operatorname{Inv}_x(\beta) \leq \mu\}$$

*is Zariski-closed in* $X$.

**Definition 6.10.** Let $|\operatorname{Gr}_{\leq\mu}| \subset |\operatorname{Gr}_G|$ be the closed subset of points $\mathcal{E}, \beta$ with $\operatorname{Inv}(\beta) \leq \mu$, endowed with the reduced subscheme structure and call it the *Schubert variety* (of $\mu$). Similarly, the open subscheme

$$\operatorname{Gr}_\mu = \operatorname{Gr}_{\leq\mu} \setminus \cup_{\lambda<\mu} \operatorname{Gr}_\lambda$$

will be called a *Schubert cell*

**Proposition 6.11.**

1. $\operatorname{Gr}_{\text{red}} = \cup_\mu \operatorname{Gr}_{\leq\mu}$
2. $\operatorname{Gr}_\mu$ *forms a single* $L^+G$*-orbit inside* $\operatorname{Gr}_G$ *and is quasi-projective and smooth of dimension* $(2\rho, \mu)$.
3. $\operatorname{Gr}_{\leq\mu}$ *is the Zariski closure of* $\operatorname{Gr}_\mu$. *In particular, it is a projective (non-smooth) variety.*

*Proof.* For the first statement, note that we have an obvious map $\cup_\mu \operatorname{Gr}_{\leq\mu} \to \operatorname{Gr}_{\text{red}}$, which is an isomorphism on topological spaces since every field-valued point has relative position bounded by some $\mu$. Since $\operatorname{Gr}_{\leq\mu}$ is defined as having the reduced subscheme structure of $\operatorname{Gr}_G$, this map is in fact an isomorphism. For (ii) and (iii), see [Zhu16, Proposition 2.1.5]. □

**Corollary 6.12.** $|\operatorname{Hk}_G| = X_\bullet(T)^+$, *where the right hand side is equipped with the topology induced by the dominance order.*

*Proof.* A point in $|\operatorname{Hk}_G|$, represented by some field valued point $(\mathcal{E}_1, \mathcal{E}_2, \beta)$, is uniquely determined by $\operatorname{Inv}(\beta)$, so

$$|\operatorname{Hk}_G| = G(\mathcal{O})\backslash G(F)/G(\mathcal{O}) = X_\bullet(T)^+$$

as sets. But now, the left side is homeomorphic to $|\operatorname{Gr}_G|$, on which the closed subsets are described by Proposition 6.11: They are exactly the $\operatorname{Gr}_{\leq\mu}$, which correspond to the closed subsets in the dominance order on the right side. □

## References


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