TALK 5: HYPERBOLIC LOCALIZATION

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We present some results of [Ri16] about $\mathbb{G}_m$-actions on algebraic spaces and Braden Theorem.

1. Introduction

Let X be an algebraic variety over $\mathbb{C}$ and let $\mathbb{C}^\times$ act algebraically on it. Let

$$X^0 = \{x \in X | \lambda.x = x \ \forall \lambda \in \mathbb{C}^\times\}$$
$$X^+ = \{x \in X | \lim_{\lambda \to 0} \lambda.x \text{ exists } \}$$
$$X^- = \{x \in X | \lim_{|\lambda| \to +\infty} \lambda.x \text{ exists } \}.$$

It is obvious that $X^0 \subset X^+$ and $X^0 \subset X^-$. We have a map $X^+ \to X^0$ given by $x \mapsto \lim_{\lambda \to 0} \lambda.x$. Similarly, $x \mapsto \lim_{|\lambda| \to +\infty} \lambda.x$ gives a map $X^- \to X^0$. The sets X^+ and X^- are algebraic and are called attractor and repeller of the action on X. For example, if $X = \mathbb{C}$ with $\mathbb{C}^\times$-action $\lambda.x = \lambda x$, we get $X^0 = X^- = \{0\}$ and $X^+ = X$.

Let S be an arbitrary scheme. Given an algebraic action of $\mathbb{G}_{m,S}$ on an algebraic space X/S (satisfying some conditions), we are going to introduce algebraic spaces X^0, X^+, X^- in a purely algebraic way. The algebraic spaces X^0, X^+ and X^- will come with morphisms $X^0 \to X^+$, $X^0 \to X^-$, $X^+ \to X$, $X^- \to X$, $X^+ \to X^0$ and $X^- \to X^0$. In the case $S = \mathrm{Spec}\mathbb{C}$, these objects agree with the topological ones given before. In the context of algebraic spaces with $\mathbb{G}_m$-actions, we are going to state a general version of a Theorem of Braden [Br02], following Richarz [Ri16] who produced the most general version (cf also [DG13]). This Theorem is widely used in geometric representations theory. From now on Braden Theorem means the Richarz extended version.

2. Algebraic spaces

Braden Theorem is stated in the context of algebraic spaces. Let us recall StacksProject's [StP] definition of this generalization of schemes. Let S be a scheme. Let $(Sch/S)_{\mathrm{fppf}}$ be the big fppf-site.

Definition 2.1. Let $Y, X : (Sch/S)_{\mathrm{fppf}}^{op} \to Sets$ be functors. Let $Y \xrightarrow{a} X$ be a transformation of functors.

(1) We say that a is representable if for every $U \in Ob((Sch/S)_{\mathrm{fppf}})$ and any $\xi \in X(U)$ the fiber product $h_U \times_X Y$ is representable by an object V_ξ in $(Sch/S)_{\mathrm{fppf}}$ (the transformation $h_U \to X$ is defined by $(T \to U)/S \mapsto im^{X(U) \to X(T)}(\xi)$). The projection $h_{V_\xi} \to h_U \times_X Y \to h_U$ comes from a unique morphism of schemes $V_\xi \xrightarrow{a_\xi} U$, using Yoneda.
(2) Let P be a property of morphisms of schemes which is preserved under base change and is fppf local on the base. In this case we say that a representable transformation $Y \xrightarrow{a} X$ has property P if for every $U \in Ob((Sch/S)_{\mathrm{fppf}})$ and any $\xi \in X(U)$ the resulting morphism of schemes $V_\xi \to U$ has property P.

In (1), assume $Y \xrightarrow{a} X$ is representable and X is (representable by) a scheme, then Y is (representable by) a scheme: take $U = X$ and $\xi = \mathrm{Id}_X$.

Definition 2.2. An algebraic space X over S is a sheaf on the big fppf-site

$$X : (Sch/S)_{\mathrm{fppf}}^{op} \to Set$$

with representable diagonal and which admits a surjective étale map from a scheme. A morphism $X \to Y$ of algebraic spaces $/S$ is a transformation of functors from X to Y.

Example 2.3. Given a scheme $T \in Ob(Sch/S)_{\text{fppf}}$, the representable functor h_T is an algebraic space.

If X is an algebraic space over S and if $T \to S$ is a morphism we denote by X_T the base change to T.

We have a natural notion of an action of an algebraic group on an algebraic space. In particular, an action of the multiplicative group $\mathbb{G}_{m,S}$ on an algebraic space X/S is a morphism $\mathbb{G}_{m,S} \times_S X \to X$ satisfying the usual axioms of algebraic actions. In this situation, we say that X is a $\mathbb{G}_m$-space over S. Given a morphism $T \to S$ and a $\mathbb{G}_m$-space X over S, we get a $\mathbb{G}_m$-space X_T by base change, since $\mathbb{G}_{m,T} = \mathbb{G}_{m,S} \times_S T$.

3. Attractors and repellers

Let X and Y be two algebraic spaces over S, $\underline{\text{Hom}}_S(Y, X)$ is the functor

$$(Sch/S)_{\text{fppf}} \to Set$$
$$T \mapsto \text{Hom}_T(Y_T, X_T).$$

It is a sheaf. When X and Y are both $\mathbb{G}_m$-spaces over S, we consider $\mathbb{G}_m$-equivariant morphisms:

$$\underline{\text{Hom}}_S^{\mathbb{G}_m}(Y, X) : T \mapsto \left\{ f \in \text{Hom}_T(Y_T, X_T) \;\middle|\; \begin{array}{ccc} \mathbb{G}_{m,T} \times_T Y_T & \longrightarrow & \mathbb{G}_{m,T} \times_T X_T \\ \downarrow & & \downarrow \\ Y_T & \longrightarrow & X_T \end{array} \text{ commutes} \right\}.$$

Then $\underline{\text{Hom}}_S^{\mathbb{G}_m}(Y, X)$ is a subsheaf of $\underline{\text{Hom}}_S(Y, X)$ on $(Sch/S)_{\text{fppf}}$. We now define some spaces with $\mathbb{G}_m$-action.

Definition 3.1. (1) Let $S^\bullet$ be the $\mathbb{G}_m$-space where the space is S and $\mathbb{G}_m$ acts trivially.
(2) Let $(\mathbb{A}^1_S)^+$ be the $\mathbb{G}_m$-space where the underlying space is $\mathbb{A}^1_S$ and $\mathbb{G}_m$ acts by multiplication: i.e. for any S-scheme T and $\lambda \in \mathbb{G}_m(T)$, $x \in \mathbb{A}^1(T)$, the action is given by $\lambda.x = \lambda x$.
(3) Let $(\mathbb{A}^1_S)^-$ be the $\mathbb{G}_m$-space where the underlying space is $\mathbb{A}^1_S$ and $\mathbb{G}_m$ acts by division: i.e. for any S-scheme T and $\lambda \in \mathbb{G}_m(T)$, $x \in \mathbb{A}^1(T)$, the action is given by $\lambda.x = \frac{x}{\lambda}$.

Now let X be a fixed $\mathbb{G}_m$-space over S.

Definition 3.2. [Ri16, Definition 1.3] Put

$$X^0 = \underline{\text{Hom}}_S^{\mathbb{G}_m}(S^\bullet, X) \qquad \text{Fixed points}$$
$$X^+ = \underline{\text{Hom}}_S^{\mathbb{G}_m}((\mathbb{A}^1_S)^+, X) \qquad \text{Attractor}$$
$$X^- = \underline{\text{Hom}}_S^{\mathbb{G}_m}((\mathbb{A}^1_S)^-, X) \qquad \text{Repeller.}$$

Remark 3.3. We identify

$$X = \underline{\text{Hom}}_S^{\mathbb{G}_m}(\mathbb{G}_m, X)$$

via: for all T/S we send a map $\mathbb{G}_m(T) \xrightarrow{f} X(T)$ to $f(1) \in X(T)$.

Remark 3.4. Repellers are attractors for opposite actions of $\mathbb{G}_m$. So the repeller notion is interesting only when considering both X^+ and X^-, this will be the case for Braden Theorem.

Remark 3.5. To our knowledge, the definition of attractors as functors using equivariant morphisms and pioneer studies appeared first in the works of Hesselink [He80, §4] and Jurciewick [Ju85, §1.2.10]. Drinfeld [Dr13] introduced attractors in the context of algebraic spaces.

Definition 3.6. We have natural maps.

(1) The structural $\mathbb{G}_m$-morphism $(\mathbb{A}^1_S)^+ \to S^\bullet$ induces a morphism $X^0 \xrightarrow{i^+} X^+$.
(2) The structural $\mathbb{G}_m$-morphism $(\mathbb{A}^1_S)^- \to S^\bullet$ induces a morphism $X^0 \xrightarrow{i^-} X^-$.
(3) The $\mathbb{G}_m$-morphism $S^\bullet \to (\mathbb{A}^1_S)^+$ corresponding to 0 induces a morphism $X^+ \xrightarrow{q^+} X^0$.
(4) The $\mathbb{G}_m$-morphism $S^\bullet \to (\mathbb{A}^1_S)^-$ corresponding to 0 induces a morphism $X^- \xrightarrow{q^-} X^0$.
(5) The natural $\mathbb{G}_m$-morphism $\mathbb{G}_m \to (\mathbb{A}^1_S)^+$ induces a morphism $X^+ \xrightarrow{p^+} X$.
(6) The natural $\mathbb{G}_m$-morphism $\mathbb{G}_m \to (\mathbb{A}^1_S)^-$ induces a morphism $X^- \xrightarrow{p^-} X$.

Definition 3.7. [Dr13] [Ri16, Definition 1.6] A $\mathbb{G}_m$-action on an algebraic space X/S is called étale locally linearizable if there exists a $\mathbb{G}_m$-equivariant covering family $\{U_i \to X\}$, where U_i are S-affine schemes with $\mathbb{G}_m$-action and the maps $U_i \to X$ are étale.

Definition 3.8. A $\mathbb{G}_m$-action on an algebraic space X/S is called Zariski locally linearizable if there exists a $\mathbb{G}_m$-equivariant covering family $\{U_i \to X\}$, where U_i are S-affine schemes with $\mathbb{G}_m$-action and the maps $U_i \to X$ are open immersions.

Remark 3.9. Assume that S is quasi-separated. By [AHR21, §1.1], we know that any $\mathbb{G}_m$-action on a quasi-separated algebraic space X/S locally of finite presentation is étale locally linearizable.

Theorem 3.10. [Ri16, Theorem 1.8 i) ii)] *Let X be an algebraic space over S with an étale locally linearizable $\mathbb{G}_m$-action. Fix a covering family $\{U_i \to X\}$.*

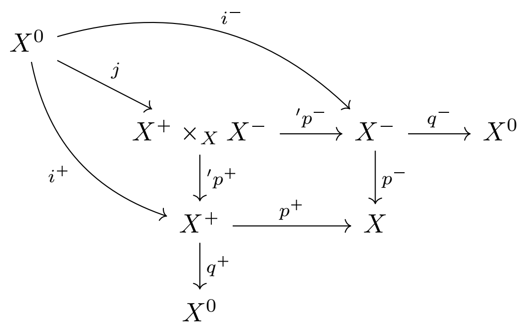
(1) *The subfunctor X^0 is representable by a closed subspace, and the induced famlly $\{U_i^0 \to X^0\}$ is S-affine, étale and covering.*
(2) *The subfunctor X^+ is representable by an algebraic space, and the induced family $\{U_i^+ \to X^+\}$ is S-affine, étale and covering.*

Proposition 3.11. [Ri16, Corollary 1.12] *Under the assumptions of Theorem 3.10, the map $X^+ \to X^0$ is affine, has geometrically connected fibers and induces a bijection on the set of connected components $\pi_0(|X^+|) \simeq \pi_0(|X^0|)$ of the underlying topological spaces.*

Proposition 3.12. [Ri16, Theorem 1.8 iii)] *If X/S is locally of finite presentation (resp. quasi-compact; resp. quasi-separated; resp. separated; resp. smooth; resp. is a scheme), so are X^0 and X^+.*

Remark 3.13.
(1) (Around the proof of Theorem 3.10) The first step is to prove that if X is S-affine, then X^+ and X^0 are representable by S-affine schemes. In fact one can write explicitly the quasi-coherent O_X-algebras of X^+ and X^0. We note that the connected assumption on S in [Ri16, § 1.3. The affine case.] is unnecessary (cf [SGA3, Exp I Prop 4.7.3.1]). It turns out that these affine considerations work for any action of a diagonalizable group scheme $D(M)_S$ on an S-affine scheme (cf [Ma22]). To explain the next steps of Richarz's proof, let us fix an algebraic space X over S with an étale locally linearizable $\mathbb{G}_m$-action. Fix a covering family $\{U_i \to X\}$. Richarz shows that $U_i^0 = X^0 \times_X U_i$ [Ri16, Lemma 1.10] and $U_i^+ = X^+ \times_{X^0} U_i^0$ [Ri16, Lemma 1.11]. By the affine case, U_i^+ and U_i^0 are S-affine schemes. In this way Richarz obtains étale atlas of the form $\{U_i^+ \to X^+\}$ and $\{U_i^0 \to X^0\}$ and obtains that X^+ and X^0 are representable using [StP, TAG 03I2] (cf [Ri16] for details). Propositions 3.11 and 3.12 are proved by Richarz using similar or direct methods.
(2) After [Dr13] and [Ri16], several generalizations and related works were realized. The works [JS18] and [JS20] study actions of reductive groups on varieties and produce a Bialynicki-Birula decomposition in this context. The work [AHR21, Corollary 14.10] gives a powerful representability result on functors of equivariant morphisms in the context of stacks. The work [HP19] gives also general representability results (cf [HP19, Introduction and Theorem 5.1.1]). The work [AHR20, §5.4, §5.5] studies $\mathbb{G}_m$-actions on Deligne-Mumford stacks over fields.

Definition 3.14. Let X/S be an algebraic space with a $\mathbb{G}_m$-action. The diagram



is called the hyperbolic localization diagram and is denoted $\mathrm{Hyploc}(X)$.

Proposition 3.15. [Ri16, 1.16,1.17]

(1) *The map $X^0 \overset{j}{\to} X^+ \times_X X^-$ is an open and closed immersion, moreover if X is S-affine then it is an isomorphism.*
(2) *Let $S' \to S$ be a morphism. Then $X_{S'}$ is a $\mathbb{G}_m$-space and $Hyploc(X_{S'}) = Hyploc(X) \times_S S'$.*
(3) *The map $X^0 \overset{i^+}{\to} X^+$ is a closed immersion.*
(4) *If X/S is separated, $X^+ \overset{p^+}{\to} X$ is a monomorphism.*

Example 3.16. (1) Let $\mathbb{G}_m$ act on $\mathbb{G}_m$ by multiplication, then $X^0 = \varnothing = X^+ = X^-$.
(2) Let $\mathbb{G}_m$ act on $\mathbb{A}^1$ by multiplication, then $X^0 = \{0\}$. We have $X^+ = \mathbb{A}^1$ and $X^- = \{0\}$.

Example 3.17. Let k be a field and $S = \mathrm{Spec}(k)$. Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ for some n. Then we get an action of $\mathbb{G}_m$ on $X = \mathbb{P}(\mathbb{A}^n)$. The action of $\mathbb{G}_m$ on X is Zariski locally linearizable. Moreover the restriction of the map $X^+ \to X$ on each component of X^+ is locally closed (cf [DG13, Theorem 1.6.8] and references given there for a more general statement). We now give an example. Let $\mathbb{G}_m$ act on $\mathbb{A}^3$ via $\lambda.(x,y,z) = (\lambda x, \lambda y, z)$. Then $X^0 = \{[0,0,1]\} \coprod \{[\overset{\simeq\mathbb{P}^1}{x,y},0]\}$ and $X^+ = \{[\overset{\simeq\mathbb{A}^2}{x,y},1]\} \coprod \{[\overset{\simeq\mathbb{P}^1}{x,y},0]\}$.

4. Braden Theorem

Let X/S be an algebraic space. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with $n > 1$ invertible on S. Let $D(X,\Lambda)$ be the unbounded derived category of $(X_{\text{ét}},\Lambda)$-modules, where $X_{\text{ét}}$ denotes the étale topos associated with X (cf Talk 2). Let $f : Y \to X$ be a morphism of S-spaces, assume it is locally of finite type.

Let

$$\begin{aligned} f^* &: D(X,\Lambda) \to D(Y,\Lambda)\\ f_* &: D(Y,\Lambda) \to D(X,\Lambda)\\ \otimes_X &: D(X,\Lambda) \times D(X,\Lambda) \to D(X,\Lambda)\\ \mathrm{Hom}_X &: D(X,\Lambda)^{op} \times D(X,\Lambda) \to D(X,\Lambda)\\ f^! &: D(X,\Lambda) \to D(Y,\Lambda)\\ f_! &: D(Y,\Lambda) \to D(X,\Lambda) \end{aligned}$$

be the six operations (cf Talk 2). The map $q^+ : X^+ \to X^0$ induces a map

$$D(X^+,\Lambda) \overset{(q^+)_!}{\to} D(X^0,\Lambda).$$

The map $p^+ : X^+ \to X$ induces a map

$$D(X,\Lambda) \overset{(p^+)^*}{\to} D(X^+,X).$$

By composition we get a map $D(X,\Lambda) \overset{L^+_{X/S}:=(q^+)_!\circ(p^+)^*}{\longrightarrow} D(X^0,\Lambda)$. The map $q^- : X^- \to X^0$ induces a map

$$D(X^-,\Lambda) \overset{(q^-)^*}{\to} D(X^0,\Lambda).$$

The map $p^- : X^- \to X$ induces a map

$$D(X,\Lambda) \overset{(p^-)^!}{\to} D(X^-,X).$$

By composition we get a map $D(X,\Lambda) \overset{L^-_{X/S}:=(q^-)_*\circ(p^-)^!}{\longrightarrow} D(X^0,\Lambda)$. Braden found a transformation of functors $L^-_{X/S} \to L^+_{X/S}$. Let us explain this. By adjunction we have a transformation

$$\mathrm{Id} \to (p^+)_*(p^+)^*.$$

Composing with the functor $(i^-)^*(p^-)^!$ we get a transformation

$$(i^-)^*(p^-)^! \to (i^-)^*(p^-)^!(p^+)_*(p^+)^*.$$

By base change $(p^-)^!(p^+)_* \simeq ({}'p^-)_*({}'p^+)^!$ so we get a transformation

$$(i^-)^*(p^-)^! \to (i^-)^*({}'p^-)_*({}'p^+)^!(p^+)^*.$$

Since j is an open and closed immersion, $(j^!, j_*)$ are adjoint so we get a transformation

$$(i^-)^*(p^-)^! \to (i^-)^*({}'p^-)_*j_*j^!({}'p^+)^!(p^+)^*.$$

Because of $({}'p^+)\circ j = i^+$, $({}'p^-)\circ j = i^-$ and $(i^-)^*(i^-)_* \simeq \mathrm{Id}$, we get a transformation

$$(i^-)^*(p^-)^! \to (i^+)^!(p^+)^*.$$

The unit transformation $\mathrm{Id} \to (i^-)_*(i^-)^*$ induces a transformation $(q^-)_* \to (q^-)_*(i^-)_*(i^-)^* = (i^-)^*$, we thus get a transformation

$$(q^-)_*(p^-)^! \to (i^+)^!(p^+)^*.$$

Analogously we have a transformation $(i^+)^! \to (q^+)_!$, we thus get a transformation

$$L^-_{X/S} = (q^-)_*(p^-)^! \to L^+_{X/S} = (q^+)_!(p^+)^*.$$

Definition 4.1. The arrow $L^-_{X/S} \to L^+_{X/S}$ introduced above is called Braden transformation.

Let X/S be a $\mathbb{G}_m$-space. Let $a : \mathbb{G}_m \times_S X \to X$ be the map associated to the action of $\mathbb{G}_m$ on X. Let $p : \mathbb{G}_m \times_S X \to X$ be the projection. Let $\mathcal{A} \in D(X,\Lambda)$ be a complex.

Definition 4.2. The complex $\mathcal{A}$ is $\mathbb{G}_m$-equivariant if there exists an isomorphism $a^*\mathcal{A} \simeq p^*\mathcal{A}$ in $D(\mathbb{G}_m \times_S X,\Lambda)$.

Definition 4.3. Let $D(X,\Lambda)^{\mathbb{G}_m\text{-mon}}$ be the full category of $D(X,\Lambda)$ strongly generated by $\mathbb{G}_m$-equivariant complexes. The objects of $D(X,\Lambda)^{\mathbb{G}_m\text{-mon}}$ are called $\mathbb{G}_m$-monodromic complexes.

Let X/S be an algebraic space locally of finite presentation with an étale locally linearizable $\mathbb{G}_m$-action. Let $n > 1$ invertible on S and $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Let $D^+(X,\Lambda)$ be the full subcategory of $D(X,\Lambda)$ of bounded below complexes. The following is called Braden Theorem.

Theorem 4.4. [DG13] [Ri16, Theorem B]

(1) *For $\mathcal{A} \in D^+(X,\Lambda)^{\mathbb{G}_m\text{-mon}}$, the arrow*

$$L^-_{X/S}\mathcal{A} \to L^+_{X/S}\mathcal{A}$$

of $D(X^0,\Lambda)$ associated to Braden transformation is an isomorphism.

(2) *For any morphism of schemes $f : S' \to S$, the isomorphism in (1) is compatible with base change along f_* and f^*. If f is locally of finite type, it is also compatible with $f_!$ and $f^!$.*

We refer to Talk 3 for the definition of being ULA.

Proposition 4.5. *Let X/S be a separated scheme of finite presentation with a $\mathbb{G}_m$-action. Let $\mathcal{A} \in D(X, \Lambda)$ be a $\mathbb{G}_m$-monodromic complex that is ULA. Then $L^+_{X/S}(\mathcal{A})(= L^-_{X/S}(\mathcal{A}))$ is ULA relatively to $X^0 \to S$.*

Proof. Using [HS21, Theorem 4.4 (iv)], we reduce to the case where S is a valuation ring with algebraically closed fraction field. Then [HS21, Theorem 4.1 (iv)] shows that if $j : X_\eta \to X$ is the inclusion of the generic fiber, then we have an equivalence $D(X_\eta, \Lambda)_{\text{cons}} \to D^{\text{ULA}}(X/S, \Lambda)$ given by j_* and j^*. Moreover we have $\mathcal{A} = j_* j^*(\mathcal{A})$. Using Theorem 4.4 (2), we have

$$L^+_{X/S}(\mathcal{A}) = L^+_{X/S} j_* j^*(\mathcal{A}) = j^0_* L^+_{X_\eta} j^*(\mathcal{A})$$

where j^0 is the morphism $X^0_\eta \to X^0$. Since $\mathcal{A}$ is ULA, $j^*\mathcal{A}$ is constructible. We have $L^+_{X_\eta} = (q^+_\eta)_! \circ (p^+_\eta)^*$. Pullbacks always preserve constructibility (cf Talk 2), so $(p^+_\eta)^*$ preserves constructibility. The morphism q^+_η is separated and finitely presented, and so $(q^+_\eta)_!$ preserves constructibility (cf Talk 2). So $L^+_{X_\eta} j^*(\mathcal{A})$ is constructible. Now $j^0_* L^+_{X_\eta} j^*(\mathcal{A})$ is ULA. □

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