Talk 4: Relative Perverse t-structure

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These notes are for the workshop on the Geometric Satake Equivalence which took place at Clermont-Ferrand in January 2022. Following [HS21] we introduce the relative perverse t-structure. In particular, there is no originality. Lemmata from [HS21] used in proofs we consider here can be found at the end of the document without proof for ease of reading.

1 Absolute Perverse t-structure

Let k be an algebraically closed field. For a scheme X of finite type over k, one can define the perverse t-structure on $\mathsf{D}_c(X_{\acute{e}t}, \overline{\mathbb{Q}}_\ell)$, the derived category of constructible étale $\mathbb{Q}_\ell$ sheaves, for $\ell \neq \operatorname{char} k$ as follows:
$A \in {}^p\mathsf{D}_c{}^{\leq 0}(X_{\acute{e}t}, \overline{\mathbb{Q}}_\ell)$ if and only if for every geometric point $i_{\overline{x}} : \overline{x} \to X$, A satisfies

$$i^*_{\overline{x}} \in \mathsf{D}^{\leq -\dim(\overline{x})}(\overline{\mathbb{Q}}_\ell)$$

where dim denotes the local dimension at $\overline{x}$. Using Artin vanishing theorem it is equivalent to the condition that for every U étale over X

$$R\Gamma(U, A) \in \mathsf{D}^{\leq 0}(\overline{\mathbb{Q}}_\ell).$$

Similarly, $A \in {}^p\mathsf{D}_c{}^{\geq 0}(X_{\acute{e}t}, \overline{\mathbb{Q}}_\ell)$ if and only if for every geometric point $\overline{x}$ of X

$$Ri^!_{\overline{x}}A \in \mathsf{D}^{\geq -\dim(\overline{x})}(\overline{\mathbb{Q}}_\ell).$$

Details regarding this t-structure and more can be found in [BBDG]. In [Gab04], Gabber shows how this t-structure can be extended to $\mathsf{D}(X_{\acute{e}t}, \overline{\mathbb{Q}}_\ell)$ as well.

Scholze and Hansen in [HS21] have provided a way of extending this t-structure to more general bases than a point, under slight finiteness conditions. It allows one to consider families of perverse sheaves parametrized by a base scheme. Their method relies heavily on descent techniques thus they prefer to work with ∞-categories. However as the t-structures depend only on homotopy category, previous literature applies.

2 Relative Perverse t-structure

Let Λ be a ring killed by ℓ^n for some natural number n. $\mathsf{D}(X)$ will denote the derived ∞-category of Λ-modules on $X_{\acute{e}t}$ and likewise $\mathsf{D}_c(X) \subset \mathsf{D}(X)$ will denote the full ∞-subcategory of perfect constructible complexes. We will assume that $X_{\acute{e}t}$ is locally of finite ℓ cohomological dimension. This implies that the standard t-structure on $\mathsf{D}(X)$ is left-complete and $\mathsf{D}(X)$ is compactly generated with compact objects $\mathsf{D}_c(X)$.

Theorem 2.1. *Let $f : X \to S$ be a finitely presented map of qcqs $\mathbb{Z}[\frac{1}{\ell}]$-schemes. There is a t-structure $({}^{p/S}\mathsf{D}^{\le 0}(X), {}^{p/S}\mathsf{D}^{\ge 0}(X))$ on $\mathsf{D}(X)$, called the relative perverse t-structure, with the following properties.*

1. *An object $A \in \mathsf{D}(X)$ lies in ${}^{p/S}\mathsf{D}^{\le 0}(X)$ (resp. ${}^{p/S}\mathsf{D}^{\ge 0}(X)$) if and only if for all geometric points $\bar{s} \to S$ with fibre $X_{\bar{s}} = X \times_S \bar{s}$, the restriction $A|_{X_{\bar{s}}} \in \mathsf{D}(X_{\bar{s}})$ lies in ${}^{p}\mathsf{D}^{\le 0}$ (resp. ${}^{p}\mathsf{D}^{\ge 0}$), for the usual (absolute) perverse t-structure.*
2. *For any map $S' \to S$ of schemes with pullback $X' = X \times_S S' \to X$, the pullback functor $\mathsf{D}(X) \to \mathsf{D}(X')$ is t-exact with respect to the relative perverse t-structures.*

Proof. Property 2 follows from 1 therefore it suffices to prove 1. Consider the full subcategory ${}^{p/S}\,\mathsf{D}^{\ge}(X)$ consisting of $A \in \mathsf{D}(X)$ such that the restriction $A|_{X_s} \in {}^{p}\,\mathsf{D}^{\ge}(X_{\bar{s}})$ for the fiber $X_{\bar{s}}$ over every geometric point $\bar{s}$ of S. This subcategory is presentable, closed under extensions and colimits because taking stalks is an exact functor and commutes with all colimits. Then [Lur17, Proposition 1.4.4.11] immediately implies that there exist a t-structure with connective part ${}^{p/S}\,\mathsf{D}^{\ge}(X)$.

Next, we will observe that the coconnective part also satisfies the desired property. This will involve several reduction steps. The crucial case to analyze is $S = \operatorname{Spec} V$ where V is an absolutely integrally closed valuation ring (meaning integrally closed in $\overline{\operatorname{Frac}(V)}$) of rank 1. Denote by $j : X_\eta \to X$, $i : X_s \to X$ the open and closed immersions of generic and special fibres respectively.

Proposition 2.2. *Theorem 2.1 holds when $S = \operatorname{Spec} V$ as above.*

Proof.

Lemma 2.3. *$A \in {}^{p/S}\,\mathsf{D}^{\ge 0}(X)$ if and only if $A|_{X_\eta} \in {}^{p}\,\mathsf{D}^{\ge 0}(X_\eta)$ and $Ri^!A \in \mathsf{D}^{\ge 0}(X_s)$.*

Proof. Notice that, the t-structure obtained by gluing the absolute perverse t-structures on the special fiber and the generic fiber has the same connective part as the one we have constructed. Since a t-structure is determined by its connective part, these t-structures must be equivalent. Now, the characterization of the coconnective part claimed in the lemma is the one obtained from gluing. □

Lemma 2.4. *$A \in {}^{p/S}\,\mathsf{D}^{\ge 0}(X)$ if and only if $A|_{X_\eta} \in {}^{p}\,\mathsf{D}^{\ge 0}(X_\eta)$ and $i^*A \in \mathsf{D}^{\ge 0}(X_s)$.*

Proof. We will show the equivalence of these assumptions with the assumptions in Lemma 2.3. Therefore, assume $A|_{X_\eta} \in {}^{p}\,\mathsf{D}^{\ge 0}(X_\eta)$. Consider the exact triangle

$$Ri^!A \to i^*A \to i^*Rj_*(A|_{X_\eta}).$$

We will be done if $i^*Rj_*(A|_{X_\eta}) \in {}^{p}\,\mathsf{D}^{\ge 0}(X_s)$. This is implied by the perverse t-exactness of the nearby cycles functor and is the content of Lemma 3.1 below. □

Above lemma is exactly property 1. □

It is straightforward to check that the proposed relative perverse t-structure is compatible with base change in S. Therefore it suffices to prove the existence locally on S with respect to a topology for which the assignment of the categories of interest are sheaves. Similarly, one can apply noetherian approximation as well.

Lemma 2.5. *Theorem 2.1 holds when $S = \operatorname{Spec} V$, where V is an absolutely integrally closed valuation ring.*

Proof. Using noetherian approximation, we may assume valuation of V is of finite rank. By [HS21, Theorem 2.2], the functor $X \mapsto \mathsf{D}_c(X)$ is an arc-sheaf. As we can find an arc-cover by absolutely integrally closed valuation rings of rank 1, there is a t-structure on $\mathsf{D}_c(X)$ by Lemma 2.2. Then the fact $\operatorname{Ind}(\mathsf{D}_c(X)) = \mathsf{D}(X)$ gives the desired t-structure on $\mathsf{D}(X)$. □

By arguments in [BS17], we can find an arc-cover of S whose connected components S are spectra of absolutely integrally closed valuation rings. Then using arc-descent as in the proof above, we may reduce to this case. By above, there is a relative perverse t-structure on each connected component. All that is left to do is to globalize this structure by spreading out from the connected components. Note that whether a map is 0 can be checked on connected components. This implies, for $A \in \mathsf{D}_c(X)$ whose geometric fibers are in ${}^{\mathrm{p}}\mathsf{D}^{\leq 0}$ and $B \in \mathsf{D}_c(X)$ whose geometric fibers are in ${}^{\mathrm{p}}D^{\geq 1}$, we have $\operatorname{Hom}(A, B) = 0$. By definition of of t-structures, if we can construct the truncation functors these subcategories will then give the desired t-structure. Fix some connected component S_c of S. There is a relative perverse t-structure on $X_c = X \times_S S_c$. Denoting by $A_c = A|_{X_c}$, we have a triangle

$$ {}^{p/S_c}\tau^{\leq 0} A_c \longrightarrow A_c \longrightarrow {}^{p/S_c}\tau^{\geq 1} A_c. $$

By constructibility of A, this triangle extends to a similar triangle over a constructible subset E which contains S_c.

Proposition 2.6. *E contains an open subset which contains S_c.*

Proof. By [BS17], the connected components of S are in bijection with its closed points and each connected component contains a closed point. Furthermore, if c is the closed point of S_c, then $S_c = \operatorname{Spec} \mathcal{O}_{S,c}$. In particular, S_c is the intersection of all standard opens $\mathrm{D}(f)$ of S for which $f \notin m_c$ where m_c is the maximal ideal corresponding to c. This means that S_c is pro-constructible. Then [Gro64, Theorem 1.9.7] shows E contains an open subset which contains S_c. □

By Lemma 3.2 below, this triangle is still the perverse truncation triangle over the fibers. This constructs the truncation functors locally and by uniquness they glue, concluding the proof. □

In the proposition that follows, we will require that the category of perfect complexes of Λ modules is stable under truncation. For instance, this is satisfied for noetherian rings of finite Krull dimension which have regular local rings at each prime. This follows straightforwardly from Serre's theorem that regularity of a noetherian local ring is equivalent to finite global dimension. As far as we are aware, there is no commonly accepted definition of regularity for non-local rings. Following Scholze and Hansen in [HS21] we call such rings regular.

Proposition 2.7. *In the setting of 2.1, if furthermore Λ is regular then relative perverse truncation preserves $\mathsf{D}_c(X)$.*

Proof. Using arc-descent and noetherian approximation one once again reduces to the case of an absolutely integrally closed valuation ring of finite rank. But in this case constructibility can be checked fiberwise which reduces the question to the well known case of absolute perverse truncation. □

3 Lemmata

The lemmata stated below hold more generally than the setting considered in these notes. However, for greater generality their formulation requires more care and they may not hold exactly as stated below.

Lemma 3.1. *[HS21, Lemma 6.3] Let $S = \operatorname{Spec} V$ be the spectrum of an absolutely integrally closed valuation ring V of rank 1, and let X be a finite type S-scheme. Let $j : X_\eta \in X$ and $i : X_s \in X$ be the open and closed immersion of generic and special fibre. Then for any torsion $\mathbb{Z}_\ell$-algebra Λ, the nearby cycles functor*

$$R\psi = i^* Rj_* : \mathsf{D}(X_\eta, \Lambda) \to \mathsf{D}(X_s, \Lambda)$$

is t-exact with respect to the absolute perverse t-structures on source and target.

Lemma 3.2. *[HS21, Lemma 6.4] Let $f : X \to S$ be a finitely presented map of qcqs $\mathbb{Z}[\frac{1}{\ell}]$-schemes, and let $A \in \mathsf{D}_c(X, \Lambda)$. The subset $S^{\leq 0} \subset S$ (resp. $S^{\geq 0} \subset S$) of all points $s \in S$ for which $A|_{X_s} \in {}^{\mathrm{p}}\mathsf{D}^{\leq 0}$ (resp. $A|_{X_s} \in {}^{\mathrm{p}}\mathsf{D}^{\geq 0}$) is a constructible subset of S.*

References

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