OVERVIEW OF THE PAPER "COHERENT SPRINGER THEORY AND THE CATEGORICAL DELIGNE-LANGLANDS CONJECTURE

Warning: These notes are very much preliminary and evolving (hopefully...), and probably full of mistakes and misunderstandings, so use them at your own risk!

1. Hellmann's and Zhu's conjecture

1.1. **The setting.** Let F be a local non-archimedean field with residue field $\mathbb{F}_q$ of characteristic p and let $\mathbf{G}$ be a split reductive group over F, with dual group $\check{G}$ (say over $\mathbb{C}$; recall that the root datum of $\check{G}$ is dual to that of $\mathbf{G}$). Fix a split maximal torus $\mathbf{T}$ inside a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Let $\mathbf{N}$ the unipotent radical of $\mathbf{B}$. If $\mathbf{H}$ is an algebraic group over F let $H = \mathbf{H}(F)$. Finally, fix[1] a generic character $\psi : N \to \mathbb{C}^*$.

Let $\mathrm{Rep}(G)$ be the category of smooth representations of G over $\mathbb{C}$ (we could replace $\mathbb{C}$ by any field of characteristic 0 containing a square root of q). The conjectural local Langlands correspondence (LLC) is a map with finite fibers from (isomorphism classes of) irreducible objects of $\mathrm{Rep}(G)$ to L-parameters for G, satisfying a bunch of compatibilities. The philosophy of "families of representations" suggests that there should be a family version of this correspondence, but it is not at all obvious how to formulate this. Hellmann and Zhu recently made some striking conjectures, which were further refined in the groundbreaking paper of Fargues and Scholze. We review them in the next paragraphs, but we need some preliminaries.

1.2. **The stack of L-parameters.** A key player in the story is the *stack of L-parameters*

$$\mathbf{L}_{\check{G}} = X_{\check{G}}^{WD}/\check{G},$$

where $X_{\check{G}}^{WD}$ is the scheme such that $X(R)$ (for R a $\mathbb{C}$-algebra) is the set of pairs (ρ, N), where $\rho : W_F \to \check{G}(R)$ is a smooth representation (i.e. continuous for the discrete topology on the target, or equivalently trivial on an open subgroup of the inertia I_F; here W_F is the Weil group of F) and $N \in \mathrm{Lie}(\check{G})(R)$ such that $\mathrm{Ad}(\rho(w))(N) = q^{-||w||}N$, where $||\cdot|| : W_F \to \mathbb{Z}$ is the natural projection. The adjoint action of $\check{G}$ (on both ρ and N) induces an action of $\check{G}$ on $X_{\check{G}}^{WD}$.

Remark 1.1. (1) It is not difficult to check that the functor describing $X_{\check{G}}^{WD}$ is indeed representable by a scheme, which is an infinite disjoint union of affine schemes. One can prove that $X_{\check{G}}^{WD}$ is reduced and local complete intersection (this is proved in the references made below). It follows that if we formulate the moduli problem in the world of derived algebraic geometry, the resulting derived scheme is actually classical. On the other hand, the definition of $X_{\check{G}}^{WD}$ obviously makes sense if we

[1] More precisely, we are fixing a Whittaker datum, i.e. a G-conjugacy class of pairs $(\mathbf{B}, \psi)$ as above.

replace $\check{G}$ by any affine algebraic group over $\mathbb{C}$. However, in general the resulting schemes will no longer have such nice properties and it turns out that the right objects needed to formulate the conjectures below are those from derived algebraic geometry!

(2) In particular, if $\mathbf{P}$ is a standard parabolic of $\mathbf{G}$ (inducing a parabolic $\check{P} \subset \check{G}$), we will write $X_{\check{P}}^{WD}$ for the corresponding *derived scheme*. Note that the construction is functorial, i.e. we get natural maps (of derived stacks) $\mathbf{L}_{\check{P}} \to \mathbf{L}_{\check{G}}$ and $\mathbf{L}_{\check{P}} \to \mathbf{L}_{\check{M}}$ if $\mathbf{M}$ is the Levi quotient of $\mathbf{P}$. One can show that the morphism $\mathbf{L}_{\check{P}} \to \mathbf{L}_{\check{G}}$ is proper, while $\mathbf{L}_{\check{P}} \to \mathbf{L}_{\check{M}}$ has finite Tor dimension.

(3) Dat, Helm, Kurinczuk and Moss (DHKM from now on), Zhu, Fargues and Scholze made a very deep study of the moduli stack of Langlands parameters, working even integrally. The stack appearing above is obtained by taking the fiber over $\mathbb{C}$ of the moduli stack studied in the mentioned works. Things tend to get much easier when we drop integrality conditions, in particular we can use the language of Weil-Deligne representations to define the moduli problem.

1.3. **The main conjecture.** We can now state the conjecture, as in Hellmann's paper. See also the beautiful paper of Zhu on coherent sheaves on the stack of Langlands parameters. This conjecture is a special case of an amazing conjecture of Fargues and Scholze, whose statement would take us too far away...

Conjecture 1.2. *There is a fully faithful exact functor*

$$R_G^{\psi} : D^+(\mathrm{Rep}(G)) \to \mathrm{QCoh}^+(\mathbf{L}_{\check{G}})$$

which is compatible with local class field theory (in the case of tori), with parabolic induction and with Whittaker models.

Let me briefly explain what the above compatibilities mean:

• Suppose first that $\mathbf{G} = \mathbf{T}$ is a torus. Local class field theory shows that $X_{\check{G}}^{WD}$ is the scheme classifying smooth characters $F^* \to \check{T}(R)$ (R being a test $\mathbb{C}$-algebra) and that

$$\mathrm{Rep}(T) \simeq \mathrm{QCoh}(X_{\check{T}}^{WD}).$$

The functor R_T^{ψ} should be compatible with this identification, via the natural map $X_{\check{T}}^{WD} \to \mathbf{L}_{\check{T}}$.

• Compatibility with parabolic induction means that whenever $\mathbf{P}$ is a standard parabolic with Levi quotient $\mathbf{M}$, we should have a canonical isomorphism (all functors are implicitly derived)

$$R_G^{\psi} \circ i_{\bar{P}}^G \simeq \beta_* \alpha^* \circ R_M^{\psi_M},$$

where $\alpha : \mathbf{L}_{\check{P}} \to \mathbf{L}_{\check{M}}$ and $\beta : \mathbf{L}_{\check{P}} \to \mathbf{L}_{\check{G}}$ are the morphisms discussed above. Here ψ_M is the restriction of ψ to the unipotent radical of the Borel subgroup $B \cap M$ of M (we implicitly use a section of the projection $\mathbf{P} \to \mathbf{M}$ to realise M as a subgroup of G). Finally, $i_{\bar{P}}^G$ is the (normalized) parabolic induction with respect to the *opposite* parabolic $\bar{\mathbf{P}}$. The reason why this parabolic appears is not so clear, but it will become more natural when we discuss the case of the principal block.

- Finally, compatibility with Whittaker models is the existence of a natural isomorphism

$$R_{\check{G}}^{\psi}(c\text{--}\mathrm{Ind}_N^G \psi) \simeq \mathscr{O}_{\mathbf{L}_{\check{G}}}.$$

Remark 1.3. Assuming the existence of $R_{\check{G}}^{\psi}$, its full faithfulness induces a morphism

$$\mathscr{O}(X_{\check{G}}^{WD})^{\check{G}} \to Z(\mathrm{Rep}(G)),$$

where on the right we have the Bernstein center of $\mathrm{Rep}(G)$. Fargues and Scholze constructed a canonical such morphism, and one expects that these two morphisms are the same (in particular the first one does not depend on ψ).

1.4. **The principal block.** The category $\mathrm{Rep}(G)$ has a block decomposition (due to Bernstein), where blocks are indexed by certain equivalence classes of pairs (M, σ), where M is a Levi subgroup of G and σ is a cuspidal representation of M. Let $\mathrm{Rep}_1(G)$ be the *principal block*, i.e. the one containing the trivial representation. Equivalently, it is the block indexed by the equivalence class of the pair $(T, 1)$ (1 being the trivial representation of T), or the block consisting of representations π all of whose irreducible subquotients appear in the parabolic induction of an unramified character of T.

The choice of an integral structure on $\mathbf{G}$ yields an Iwahori subgroup $I \subset G(F)$ (those $g \in \mathbf{G}(\mathscr{O}_F)$ whose image in $\mathbf{G}(k_F)$ lands in $\mathbf{B}(k_F)$, where k_F is the residue field of F). Then $\mathrm{Rep}_1(G)$ can be further characterized as the full subcategory of $\mathrm{Rep}(G)$ of representations generated by their I-invariants. Moreover, passage to I-invariants $\pi \mapsto \pi^I = \mathrm{Hom}_G(c\text{--}\mathrm{Ind}_I^G 1, \pi)$ induces an equivalence of categories

$$\mathrm{Rep}_1(G) \simeq \mathscr{H}(G, I)\text{--mod},$$

where

$$\mathscr{H}(G, I) = \mathrm{End}_G(c\text{--}\mathrm{Ind}_I^G 1) \simeq \mathscr{C}_c^{\infty}(I \backslash G / I, \mathbb{C})$$

is the Hecke-Iwahori algebra.

For a representation $\pi \in \mathrm{Rep}(G)$ let π_1 be its projection to the factor $\mathrm{Rep}_1(G)$. It is a classical fact that if $\mathbf{P} \subset \mathbf{G}$ is a standard parabolic with Levi quotient $\mathbf{M}$, the (normalized) parabolic induction functors i_P^G and $i_{\bar{P}}^G$ from $\mathrm{Rep}(M)$ to $\mathrm{Rep}(G)$ induce functors at the level of principal blocks. Choose a section of the projection $\mathbf{P} \to \mathbf{M}$ to realize M as a subgroup of G and set $I_M = I \cap M$, an Iwahori subgroup of M. One can show that there is a canonical embedding $\mathscr{H}(M, I_M) \to \mathscr{H}(G, I)$ inducing natural isomorphisms of $\mathscr{H}(G, I)$-modules

$$i_P^G(\pi)^I \simeq \mathrm{Hom}_{\mathscr{H}(M, I_M)}(\mathscr{H}(G, I), \pi^{I_M}), \;\; i_{\bar{P}}^G(\pi)^I \simeq \mathscr{H}(G, I) \otimes_{\mathscr{H}(M, I_M)} \pi^{I_M},$$

explaining (maybe...) why it's rather $i_{\bar{P}}^G$ that appears in the formulation of compatibility with parabolic induction, and not i_P^G.

1.5. **The stack of unipotent Langlands parameters.** This is defined by

$$\mathbf{L}^u_{\check{G}} = X_{\check{G}}/\check{G},$$

where $X_{\check{G}}$ is the subscheme of $\check{G} \times \mathrm{Lie}(\check{G})$ consisting of pairs (φ, N) such that $\mathrm{Ad}(\varphi)(N) = q^{-1}N$. We can see $X_{\check{G}}$ as the subscheme parametrizing L-parameters (ρ, N) with ρ unramified (and so determined by the image of a Frobenius element, which explains the notation (φ, N) above). The scheme $X_{\check{G}}$ makes sense if we replace $\check{G}$ by any affine algebraic group, but while in the case of the reductive group $\check{G}$ the scheme is a complete intersection inside $\check{G} \times \mathrm{Lie}(\check{G})$ (see Hellmann's paper), this is not true in general and the good object is the associated derived scheme (defined by the obvious fiber product, taken now in the category of derived schemes). In particular for any standard parabolic $\mathbf{P}$ of $\check{G}$ we have the derived scheme $X_{\check{P}}$. There are obvious morphisms

$$\alpha : \mathbf{L}^u_{\check{P}} \to \mathbf{L}^u_{\check{M}}, \ \ \beta : \mathbf{L}^u_{\check{P}} \to \mathbf{L}^u_{\check{G}},$$

the map β being proper and schematic. Moreover, one can show that (derived) α^* preserves the $+$-part and also categories of coherent (complexes of) sheaves.

Conjecture 1.4. *There is a fully faithful exact embedding*

$$R^{\psi,u}_G : D^+(\mathrm{Rep}_1(G)) \to \mathrm{QCoh}^+(\mathbf{L}^u_{\check{G}})$$

compatible with class field theory for tori, with parabolic induction and with Whittaker models. Moreover, this functor sends $D^b(\mathrm{Rep}_{1,fg}(G))$ to $\mathrm{D^bCoh}(\mathbf{L}^u_{\check{G}})$ and then extends to a functor

$$D(\mathrm{Rep}_1(G)) \to \mathrm{Ind}(\mathrm{D^bCoh}(\mathbf{L}^u_{\check{G}})).$$

I will not make again explicit what these compatibilities mean, but they are very much like in the previous conjecture (in particular for the Whittaker model we should ask that the principal block component of $c - \mathrm{Ind}_N^G \psi$ is sent to the structure sheaf).

Note that the last sentence uses Bernstein's theorem that $\mathscr{H}(G, I)$ has finite cohomological dimension to deduce that $D(\mathrm{Rep}_1(G))$ is the Ind-completion of $D^b(\mathrm{Rep}_{1,fg}(G))$ (here $\mathrm{Rep}_{1,fg}(G)$ consists of representations in $\mathrm{Rep}_1(G)$ which are finitely generated, or equivalently such that their I-invariants are finitely generated $\mathscr{H}(G, I)$-modules).

Theorem 1.5. *(Ben-Zvi, Chen, Helm, Nadler) The second conjecture holds, and the first one holds for $\mathbf{G} = \mathrm{GL}_n$.*

Somewhat surprisingly, for $\mathbf{G} = \mathrm{GL}_n$ one can show that the second conjecture implies the first. The proof crucially uses the local Langlands correspondence (Harris-Taylor, Henniart, Scholze) in order to identify W_F-conjugacy classes of irreducible smooth (finite dimensional) representations of I_F with cuspidal representations of various GL_n up to torsion by unramified characters. This is used to deduce a bijection between connected components of $X^{WD}_{\mathrm{GL}_n}$ and Bernstein components of the category $\mathrm{Rep}(G)$. Moreover, one shows that each connected component of $X^{WD}_{\mathrm{GL}_n}$ is a product of schemes of the form X_{GL_d} for various d's and over various finite extensions of F. In order words, on the stack of

L-parameters side and on a fixed connected component everything reduces to the case of unipotent parameters. Similarly, work of Bushnell-Henniart on type theory for GL_n shows that each Bernstein component is described by a suitable semisimple block, and the associated Hecke algebra decomposes as a tensor product of various Hecke-Iwahori algebras. Carefully putting this together (as is done in chapter 5 of the paper under review), one deduces that the conjecture for the principal block implies the first conjecture.

From now on I will move to the BZCHN paper, and unfortunately in order to follow their notations we will have to switch the roles of G and $\check{G}$. So let G be a reductive group over $\mathbb{C}$ (or $\overline{\mathbb{Q}}_\ell$) and let $\check{G}$ be its dual group, a split reductive group over F. Let $\check{I}$ be an Iwahori subgroup in $\check{G}(F)$ and let

$$\mathscr{H}_q = \mathscr{H}(\check{G}(F), \check{I})$$

be the associated Hecke-Iwahori algebra. Thus now we are interested in the principal block for $\check{G}$, or equivalently in $\mathscr{H}_q$-modules, and try to embed them in quasi-coherent (or ind-coherent, stay tunned...) sheaves on the stack $\mathbf{L}_G^u$ of unipotent Langlands parameters, classifying pairs (g, N) in $G \times \mathrm{Lie}(G)$ with $gNg^{-1} = qN$ (I use the convention in BZCHN here, one passes from the one above to this one by $g = \varphi^{-1}$) up to G-conjugacy.

1.6. **The coherent Springer sheaf.** From now on we only focus on the second conjecture. The strategy is to construct an explicit sheaf[2] S corresponding to the $\mathscr{H}(G, I)$-module $\mathscr{H}(G, I)$, and set $R_G^\psi(M) = S \otimes_{\mathscr{H}(G,I)} M$ pour $M \in D^+(\mathscr{H}(G, I))$. One of the hardest things to check (but not the only hard thing...) is that S has endomorphism ring $\mathscr{H}(G, I)$ (here we use the natural involution on $\mathscr{H}(G, I)$ to pass from left modules to right modules when needed). This takes upon the major part of the article.

Before we go to the construction of the sheaf S, it is convenient to recall some facts from classical Springer Theory. Thus let G be a split reductive group (over $\mathbf{Z}$, say), let B be a Borel subgroup of G, and T a split maximal torus contained in B. Let W be the associated Weyl group. The choice of a square root of q induces an isomorphism of algebras (by work of Lusztig and other people)

$$\mathbf{C}[W] \simeq \mathscr{H}_q^f = \mathbf{C}[B(\mathbf{F}_q)\backslash G(\mathbf{F}_q)/B(\mathbf{F}_q)] \simeq \mathrm{End}_{G(\mathbf{F}_q)}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)}\mathbf{C}).$$

On the other hand, we can find a geometric interpretation of the group algebra $\mathbf{C}[W]$ using the geometry of the (reduced) nilpotent cone $\mathscr{N} \subset \mathfrak{g} = \mathrm{Lie}(G)$ (recall that $\mathscr{N}$ is defined as the fiber over 0 of the "characteristic polynomial" morphism $\mathfrak{g} \to \mathfrak{g}//G \simeq \mathfrak{t}//W$). More precisely, consider the Springer resolution $\mu : \tilde{\mathscr{N}} \to \mathscr{N}$, where $\tilde{\mathscr{N}}$ is the variety of pairs (x, b) with $x \in \mathscr{N}$, $b \in G/B$ and $x \in b$ (here we regard G/B as the variety of Borel subgroups of G). The map μ sends (x, b) to x. Then $\tilde{\mathscr{N}}$ is smooth (a vector bundle over G/B) and μ is a proper map and a resolution of singularities of $\mathscr{N}$. Let $d = \dim \mathscr{N}$. One

[2] More precisely complex of sheaves, though one conjectures that this complex in concentrated in degree 0.

shows that

$$S := \mu_* \mathbf{C}[d] \in \mathrm{Perv}(\mathscr{N}/G),$$

the category $\mathrm{Perv}(\mathscr{N}/G)$ is semisimple (this is due to Lusztig) and there is an isomorphism of algebras

$$\mathrm{End}(S) \simeq \mathbf{C}[W].$$

A key point here is that the Steinberg variety $Z := \tilde{\mathscr{N}} \times_{\mathscr{N}} \tilde{\mathscr{N}}$ has irreducible components in bijection with W. Combining the above facts yields an embedding

$$\mathscr{H}_q^f - \mathrm{mod} \to \mathrm{Perv}(\mathscr{N}/G)$$

induced by tensoring with the Springer sheaf S.

Now let us go back to the setting of the paper under review. Defining the Springer sheaf in this context is quite easy. Namely, consider the natural map

$$\mu : \mathbf{L}_B^u \to \mathbf{L}_G^u$$

and define (recall that everything is derived here and that μ is schematic and proper, so μ_* preserves coherence)

$$\mathscr{S}_{q,G} := \mu_* \mathscr{O}_{\mathbf{L}_B^u} \in \mathrm{Coh}(\mathbf{L}_G^u).$$

Conjecture 1.6. *The complex $\mathscr{S}_{q,G}$ is concentrated in degree* 0.

This is proved for $G = \mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$, but seems very hard in general. The hardest theorem of the paper is

Theorem 1.7. *The dg-algebra of self-ext of $\mathscr{S}_{q,G}$ is concentrated in degree* 0 *and naturally isomorphic to $\mathscr{H}_q$. Tensoring with the coherent Springer sheaf $\mathscr{S}_{q,G}$ gives a fully faithful embedding*

$$\mathrm{Perf}(\mathscr{H}_q - \mathrm{mod}) \to \mathrm{Coh}(\mathbf{L}_G^u),$$

which extends to a fully faithful embedding

$$D(\mathscr{H}_q - \mathrm{mod}) \to \mathrm{IndCoh}(\mathbf{L}_G^u),$$

compatible with parabolic induction and Whittaker models.

1.7. **The strategy.** I will only focus on the proof of the fact that the dg-algebra $\mathrm{End}(\mathscr{S}_{q,G})$ is concentrated in degree 0 and isomorphic to $\mathscr{H}_q$. The idea is to compare both these algebras to a third object, the Hochschild homology of the affine Hecke category $\mathrm{Coh}(Z/G)$, where Z is the *derived* Steinberg variety, in other words Z is the derived fiber product (i.e. fiber product in the category of derived or dg schemes)

$$Z := \tilde{\mathscr{N}} \times_{\mathfrak{g}} \tilde{\mathscr{N}}.$$

The derived stack Z/G is therefore

$$Z/G = X \times_Y X,$$

where

$$f : X := \tilde{\mathscr{N}}/G \to Y := \mathfrak{g}/G$$

is a proper morphism of *smooth* derived stacks (we work here with $\mathfrak{g}/G$ and not with $\mathfrak{N}/G$ since the first is smooth). Using this description of Z/G one can define a monoidal (but not symmetric monoidal!) structure on the (∞-) category $\mathrm{Coh}(Z/G)$, via convolution product. Roughly the convolution product is defined by the recipe

$$\mathscr{F} * \mathscr{G} = p_{13,*}(p_{12}^{!}\mathscr{F} \otimes p_{23}^{!}\mathscr{G}),$$

where p_{ij} are the natural projections from $X \times_Y \times_Y X$ to Z. One needs to be more precise and describe this in the ∞-world, and one way to make things precise is to use the work of Ben-Zvi, Francis and Nadler, who established a natural equivalence of ∞-categories (everything is derived below and π_i are the natural projections $Z/G = X \times_Y X \to X$)

$$\mathrm{Coh}(Z/G) \simeq \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X), \mathrm{Perf}(X)), K \mapsto (\mathscr{F} \mapsto \pi_{2,*}(\pi_1^{!}\mathscr{F} \otimes K))$$

and the ∞-category on the right-hand side has a natural monoidal structure via composition of functors.

The idea is that we can compute the Hochschild homology of $\mathrm{Coh}(Z/G)$ in two different ways:

- we can use categorical traces arguments (see the next section) and previous work of Ben-Zvi, Nadler and Preygel to identify the categorical trace of $\mathrm{Coh}(Z/G)$ with $\mathrm{Coh}(\mathbf{L}^u_G)$ (this is a quite difficult result!), and identify the Springer sheaf $\mathscr{S}_{q,G}$ with a very specific object of the categorical trace, whose endomorphism algebra gives the Hochschild homology of $\mathrm{Coh}(Z/G)$ by a very general result of Gaitsgory, Kazhdan, Rozenblyum and Varshavsky.
- we can use the deep work of Bezrukavnikov to compute the Hochschild homology via semi-orthogonal decompositions of the category $\mathrm{Coh}(Z/G)$. The point here is that the homotopy category of $\mathrm{Coh}(Z/G)$ is identified with $\mathrm{Coh}(\mathrm{Iw}\backslash \mathrm{Fl}_{\check{G}})$, where $\mathrm{Fl}_{\check{G}}$ is the affine flag variety for $\check{G}$. The orbits of Iw on $\mathrm{Fl}_{\check{G}}$ yield the semi-orthogonal decomposition and their geometry is well understood so that explicit computations can be made on each stratum. Of course, this takes a lot of work...

2. Traces and categorical traces

From now on we will work exclusively in the ∞-world, so I will just use "category" for ∞ ($= (\infty, 1)$)-category and "2-category" for $(\infty, 2)$-category (whatever that means...).

2.1. **∞-categories for dummies.** Here are a few key useful facts (many of the things below are very imprecise...):

- The (ordinary) category of all (small) categories is by definition a full subcategory of that of simplicial (small) sets and we use the word functor instead of simplicial map. Key examples of categories are nerves of ordinary categories and simplicial sets $\mathrm{Sing}(X)$ attached to topological spaces X. The categories $\mathrm{Sing}(X)$ are special cases of "Kan complexes" or ∞-grupoids, which are categories in which all morphisms are isomorphisms.

- If C is a category and S is a set of objects (i.e. 0-simplices), we can talk about the full subcategory $C[S]$ of C spanned by objects in S (so n-simplices in $C[S]$ are those n-simplices in C whose vertices are all in S).
- It is a highly nontrivial fact that we can organise the collection of small categories in a (large) category Cat. Its 0-simplices are small categories, 1-morphisms are functors between small categories, etc (and it's not easy to make precise what etc means...).
- Let $\mathscr{S}$ be the category of "spaces", full subcategory of Cat spanned by Kan complexes. Let $\mathrm{Cat}^{\mathrm{ord}}$ be the full subcategory of Cat spanned by nerves of ordinary categories. The inclusions of these full subcategories in Cat have left adjoints (whatever that means at this stage...) denoted $A \mapsto A^{\mathrm{top}}$ and $A \mapsto \mathrm{Ho}(A)$. The Kan complex A^{top} is obtained from A by inverting all 1-morphisms. We call $\mathrm{Ho}(A)$ the homotopy category of A, and we identify it with an ordinary category. Its objects are those of A, Hom sets are homotopy classes of maps between objects in A.
- If $A \in \mathrm{Cat}$, we say a morphism $f : X \to Y$ is an isomorphism if f induces an isomorphism in $\mathrm{Ho}(A)$. Applying this to Cat (seen as a small category after an enlargement of the universe...) yields the notion of equivalence of categories (so a functor $f : A \to B$ is an equivalence if f is an isomorphism in Cat).
- The category of simplicial sets has internal Hom's $(A, B) \mapsto \mathrm{Fun}(A, B)$, the n-simplices of $\mathrm{Fun}(A, B)$ being given by simplicial maps from $A \times \Delta^n$ to B, where Δ^n is the standard n-simplex. A fundamental fact is that $\mathrm{Fun}(A, B)$ is a category if B is a category (and A is any simplicial set).
- The category Cat has all limits and colimits (whatever that means...), in particular it has fibre products. If X, Y are objects in a category A we can in particular define

$$\mathrm{Map}_A(X, Y) = \mathrm{Fun}([1], A) \times_{A \times A} \{X, Y\},$$

where $[1]$ is the ordinary category $0 \to 1$ (seen as a category via the nerve construction), $\mathrm{Fun}([1], A) \to A \times A$ is the functor induced by evaluation at the two objects of $[1]$ and $\{X, Y\}$ is seen as a discrete object in $\mathscr{S}$. It turns out that $\mathrm{Map}_A(X, Y) \in \mathscr{S}$. Moreover, if $A, B \in \mathrm{Cat}$ then we have a natural isomorphism

$$\mathrm{Map}_{\mathrm{Cat}}(A, B) \simeq \mathrm{Fun}(A, B)^{\mathrm{top}}.$$

If X, Y are objects of a category A, we have

$$\mathrm{Hom}_{\mathrm{Ho}(A)}(X, Y) = \pi_0(\mathrm{Map}_A(X, Y)),$$

where $\pi_0 : \mathscr{S} \to \mathrm{Set}$ is the "connected components functor" (left adjoint to the inclusion $\mathrm{Set} \subset \mathscr{S}$).
- A functor $f : A \to B$ is called fully faithful if $\mathrm{Map}_A(X, Y) \to \mathrm{Map}_B(f(X), f(Y))$ is an isomorphism in $\mathscr{S}$ for all $X, Y \in A$. It is called conservative if $f(u)$ being an isomorphism in B forces the morphism $u : X \to Y$ being an isomorphism in A.

2.2. **Presheaf and Ind-categories.** Let C be a small category and let

$$\mathscr{P}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathscr{S})$$

be the category of $\mathscr{S}$-valued presheaves on C. The fundamental (and difficult in this context!) ∞-categorical Yoneda lemma gives a fully faithful embedding, preserving all small limits existing in C, $j : C \to \mathscr{P}(C)$; j sends an object $X \in C$ to the presheaf $\mathrm{Map}_C(-, X)$. Moreover, for any category D with small colimits composition with j induces an equivalence of categories

$$\mathrm{Fun}^L(\mathscr{P}(C), D) \simeq \mathrm{Fun}(C, D),$$

where on the left we have the full subcategory of $\mathrm{Fun}(C, D)$ consisting of functors having a right-adjoint.

Say $F \in \mathscr{P}(C)$ is representable if F belongs to the essential image of j. The *idempotent-completion* of C is the full subcategory of $\mathscr{P}(C)$ spanned by objects that are retracts of objects in the essential image of j. We say C is idempotent complete if j induces an equivalence between C and its idempotent-completion.

The Ind-category $\mathrm{Ind}(C)$ is the smallest full subcategory of $\mathscr{P}(C)$ which contains the essential image of j and is stable under filtered colimits. More generally, for any regular cardinal k we can define $\mathrm{Ind}_k(C)$ by considering k-filtered limits (so we recover $\mathrm{Ind}(C)$ for $k = \omega$). The Yoneda embedding lands in the k-compact objects of $\mathrm{Ind}_k(C)$, and for any category D with small k-filtered colimits composition with j induces an equivalence between k-continuous functors $\mathrm{Ind}_k(C) \to D$ and $\mathrm{Fun}(C, D)$. Moreover $j : C \to \mathrm{Ind}_k(C)$ preserves all k-small colimits existing in C. So $\mathrm{Ind}(C)$ is obtained from C by freely adjoining the colimits of all small filtered diagrams.

2.3. **Higher algebra.** This section is ridiculously imprecise, but it is impossible to make these things precise without writing an infinite amount of math I don't understand...

If A is a category with finite products, one can define the category $\mathrm{CMon}(A)$ of commutative monoids in A as the full subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, A)$ of functors $F : \mathrm{Fin}_* \to A$ such that for all pointed finite sets $(I, *)$ the natural map $F(I, *) \to \prod_{i \in I \setminus \{*\}} F(\{i, *\}, *)$ is an equivalence (here $(I, *) \to \{i, *\}$ sends i to i and everything else to $*$). Here Fin_* is the (nerve of the) category of finite pointed sets (i.e. pairs $(I, *)$ with I a finite set and $* \in I$), maps $(I, *) \to (J, *)$ being maps $I \to J$ sending $*$ to $*$. There is a natural forgetful functor $\mathrm{CMon}(A) \to A$ (evaluate F at $(\{0\}, *)$) and we will usually (and abusively!) identify $F \in \mathrm{CMon}(A)$ with its image M. Then M comes with a natural functor $\otimes M \times M \to M$.

Applying this to the category Cat of all (small) categories, we obtain the category $\mathrm{SM} = \mathrm{CMon}(\mathrm{Cat})$ of *symmetric monoidal categories*. If $A, B \in \mathrm{SM}$, a *symmetric monoidal functor* between A and B is simply a 1-morphism $A \to B$ in the category SM. One can construct a category $\mathrm{Fun}^{\mathrm{SM}}(A, B)$ of symmetric monoidal functors whose k-simplices are $\mathrm{Fun}^{\mathrm{SM}}(A, \mathrm{Fun}([k], B))$ (here $[k]$ is the category $0 \to 1 \to \ldots \to k$, and one can naturally turn $\mathrm{Fun}([k], B)$ into an object of SM).

A similar construction (with Fin_* replaced by Δ^{op}, where Δ is the usual simplicial category) yields the notion of (associative, unital) monoidal category and algebra object in such a gadget.

Take now $A \in \mathrm{SM}$. One can define a category $\mathrm{CAlg}(A)$ of *commutative algebra objects in* A by

$$\mathrm{CAlg}(A) = \mathrm{Fun}^{\mathrm{SM}}(\mathrm{Fin}, A),$$

where Fin is the category of finite sets, endowed with the symmetric monoidal structure for which tensor product is simply disjoint union. There is a natural forgetful functor $\mathrm{CAlg}(A) \to A$ and we identify (abusively, again...) an object $R \in \mathrm{CAlg}(A)$ with its image R in A. Then R comes with a functor $\otimes : R \times R \to R$ and with a *unit object* $1_R \in R$ and they satisfy the usual compatibilities up to homotopy, but they satisfy many more compatibilities... Similarly, if A is a monoidal category one can define the category $\mathrm{Alg}(A)$ of (associative, unital) objects in A. The category $A = \mathrm{Cat}$ has a natural cartesian symmetric monoidal structure, and $\mathrm{Alg}(A)$ is the category of monoidal categories, while $\mathrm{CAlg}(A) = \mathrm{SM}$.

A rather complicated construction attaches to any monoidal category A a category $A-\mathrm{mod}$ of (left) A-module categories (or "categories left-tensored over A). I will not say anything about this, just take for granted that it exists! Similarly, if $R \in \mathrm{Alg}(A)$ one can define a category $R - \mathrm{mod}(A)$ of R-module objects in A. For instance, if $A = \mathrm{Cat}$ then $R - \mathrm{mod}(A) = R - \mathrm{mod}$.

2.4. **Dualizability.** Let C be a monoidal category. We say $X \in C$ is right-dualizable if X is right dualizable in the ordinary monoidal category $\mathrm{Ho}(C)$, i.e. there is $X^{\vee,R} \in C$ and maps

$$\mathrm{ev} : X \otimes X^{\vee,R} \to 1_C, \ \mathrm{coev} : 1_C \to X^{\vee,R} \otimes X$$

such that

$$(\mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}) : X \to X \otimes X^{\vee,R} \otimes X \to X$$

projects to identity in $\pi_0(\mathrm{Map}_C(X, X))$ and similarly for

$$(\mathrm{id} \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes \mathrm{id}) : X^{\vee,R} \to X^{\vee,R} \otimes X \otimes X^{\vee,R} \to X^{\vee,R}.$$

Let C^{R-d} be the full subcategory spanned by right-dualizable objects of C. Similarly define the notion of left-dualizable and the subcategory C^{L-d}. If $X \in C^{R-d}$ there are natural isomorphisms

$$\mathrm{Map}_C(Y, X^{\vee,R}) \simeq \mathrm{Map}_C(X \otimes Y, 1_C)$$

and the pair $(X^{\vee,R}, \mathrm{ev})$ is uniquely determined (up to contractible choice). See GRI, 4.1.3 (book of Gaitsgory-Rozenblyum) for the precise meaning of uniqueness here. See also GRI, 4.1.6, 4.1.7 for

Proposition 2.1. *a) If $X \in C^{L-d}$ is an A-module object for some $A \in \mathrm{Alg}(C)$, then $X^{\vee,L}$ has a natural structure of right A-module object.*

b)

2.5. **Traces and their functoriality.** An excellent reference is the paper "A toy model for the Drinfeld-Lafforgue shtuka construction" by Gaitsgory, Kazhdan, Rozenblyum and Varshavsky (it contains in particular proofs for the statements made below...).

2.5.1. *The basics.* Let A be a symmetric monoidal category with unit 1_A. The notions of right-dualizable and left-dualizable are the same in A, so we simply say dualizable in this case and write $A^d = A^{R-d}$ for the full subcategory spanned by dualizable objects.

Now take $X \in A^d$ and $F : X \to X$ an endomorphism of X. We get a morphism

$$\mathrm{Tr}(F, X) \in \mathrm{End}_A(1_A)$$

by composing

$$1_A \to X \otimes X^\vee \to X \otimes X^\vee \to 1_A,$$

the first map being the unit, the second $F \otimes \mathrm{id}_{\mathrm{X}^\vee}$ and the third the co-unit. This construction has some very nice features:

• it is symmetric monoidal: if $X_1, X_2 \in A^d$ and $F_i : X_i \to X_i$, then $X_1 \otimes X_2 \in A^d$ and there is a canonical isomorphism (endowed with higher compatibilities)[3]

$$\mathrm{Tr}(F_1 \otimes F_2, X_1 \otimes X_2) \simeq \mathrm{Tr}(F_1, X_1)\mathrm{Tr}(F_2, X_2).$$

• cyclicity: if $F : X \to Y, G : Y \to X$ are morphisms between dualizable objects of A, then there is a canonical isomorphism

$$\mathrm{Tr}(G \circ F, X) \simeq \mathrm{Tr}(F \circ G, Y).$$

• compatibility with units ($\mathrm{Tr}(\mathrm{Id}_{1_A}, 1_A) = \mathrm{id}_{1_A} = 1_{\mathrm{End}_A(1_A)}$) and with duality (there is a canonical isomorphism $\mathrm{Tr}(F, X) \simeq \mathrm{Tr}(F^\vee, X^\vee)$.

2.5.2. *Extra functoriality of traces for 2-categories.* Now suppose that A is a symmetric monoidal 2-category. Say $X_i \in A^d$ $(i = 1, 2)$are endowed with endomorphisms $F_i : X_i \to X_i$ and that we are given a morphism $T : X_1 \to X_2$ such that

• T is compatible with F_i up to a 2-morphism α, i.e. we are *given* a 2-morphism $\alpha : T \circ F_1 \to F_2 \circ T$.

• T has a right adjoint, i.e. there is a morphism $T^R : X_2 \to X_1$ and 2-morphisms $\mathrm{Id}_{X_1} \to T^R \circ T, T \circ T^R \to \mathrm{Id}_{X_2}$ satisfying the standard conditions (GR, chapter 12, section 1).

The above datum induces a natural map

$$\mathrm{Tr}(T, \alpha) : \mathrm{Tr}(F_1, X_1) \to \mathrm{Tr}(F_2, X_2)$$

in the ∞-category $\mathrm{End}_A(1_A)$. Indeed, one easily checks that if $F \to G$ is a morphism in $\mathrm{Map}(X, X)$ (so $F, G : X \to X$ and $X \in A^d$), then we naturally get an induced map $\mathrm{Tr}(F, X) \to \mathrm{Map}(G, X)$, i.e. for fixed X the trace construction is functorial in the

[3] The product on the right is simply composition in the commutative monoid (since A is symmetric monoidal) $\mathrm{End}_A(1_A)$.

endomorphism F of X. Thus we get $\mathrm{Tr}(T,\alpha)$ by using this functoriality three times, plus the cyclicity of the trace construction

$$\mathrm{Tr}(F_1,X_1)\to\mathrm{Tr}(F_1\circ T^R\circ T,X_1)\simeq\mathrm{Tr}(T\circ F_1\circ T^R,X_2)\to\mathrm{Tr}(F_2\circ T\circ T^R,X_2)\to\mathrm{Tr}(F_2,X_2).$$

This may look fairly abstract and nonsensical, so let us discuss some examples.

Example 2.2. It is possible (see HSS) to define a symmetric monoidal category $L(A)$ with objects pairs (X,F) consisting of an object X of A and an endomorphism F of X, and 1-morphisms from (X_1,F_1) to (X_2,F_2) given by pairs (T,α) with $T:X_1\to X_2$ and $\alpha:T\circ F_1\to F_2\circ T$. Letting $L(A)^{\mathrm{rig}}$ the subcategory of $L(A)$ in which we allow only objects (X,F) with $X\in A^d$, and 1-morphisms only pairs (T,α) where T has a right adjoint, one can show that $L(A)^{\mathrm{rig}}$ inherits a symmetric monoidal structure and that the trace construction yields a symmetric monoidal functor $\mathrm{Tr}:L(A)^{\mathrm{rig}}\to\mathrm{End}_A(1_A)$. In particular, if $X\in A$ is an algebra (resp. commutative algebra) object of A, $F:X\to X$ is a morphism of algebra (resp. commutative algebra) objects, and if $X\in A^d$ and the map $X\otimes X\to X$ has a right adjoint, then $\mathrm{Tr}(F,X)$ lifts to an algebra (resp. commutative algebra) object in $\mathrm{End}_A(1_A)$. Moreover, if M is an X-module object in A such that $M\in A^d$ and the action map $X\otimes M\to M$ has a right adjoint, and if $F_M:M\to M$ is right-lax monoidal compatible with $F:X\to X$, then $\mathrm{Tr}(F_M,M)$ has a natural $\mathrm{Tr}(F,X)$-module structure.

Example 2.3. Take for A the category $DG=\mathrm{Vect}-\mathrm{mod}(\mathrm{Pr}^L\mathrm{St},\mathrm{k})$ of k-linear (k of characteristic 0) presentable stable categories, with k-linear, continuous (hence exact) functors as morphisms. This can be endowed with the structure of an $(\infty,2)$-category, where maps between $C,D\in DG$ form the ∞-category $\mathrm{Fun}_{\mathrm{ex,cont}}(C,D)$ of continuous, exact and k-linear functors between A and B (a full subcategory of $\mathrm{Fun}(C,D)$). It is not easy to describe the dualizable objects in A, but one key point is that *any compactly generated* object of DG is dualizable, and this suffices for all practical applications. More precisely, if $C\in DG$ is compactly generated, then $C=\mathrm{Ind}(C^c)$ (C^c being the full subcategory of compact objects of C) and the dual of C is simply $\mathrm{Ind}((C^c)^{\mathrm{op}})$ (here the unit and counit are constructed using the Yoneda pairing). Thus whenever $C\in A$ is compactly generated and F is an endofunctor of C, we obtain $\mathrm{Tr}(F,C)\in\mathrm{End}_A(1_A)$. Now $1_A=\mathrm{Vect}$ and $\mathrm{End}_A(1_A)\simeq\mathrm{Vect}$ as monoidal ∞-categories. Thus we identify $\mathrm{Tr}(F,C)$ with an object of Vect. Let $C_i\in DG^d$, $F_i:C_i\to C_i$ endomorphisms and let $T:C_1\to C_2$ be a morphism having a *continuous* right adjoint. Any natural transformation $\alpha:T\circ F_1\to F_2\circ T$ induces a map $\mathrm{Tr}(T,\alpha):\mathrm{Tr}(F_1,C_1)\to\mathrm{Tr}(F_2,C_2)$.

Example 2.4. Now take for A the Morita category of DG, so its objects are $\underline{C}$ (which should be thought of as $C-\mathrm{mod}$) with C a monoidal dg-category, and

$$\mathrm{Map}_A(\underline{C},\underline{D})=D\otimes C^{\mathrm{op}}-\mathrm{mod}\in\mathrm{Cat}.$$

Here C^{op} is C with reversed multiplication (not to be confused with the opposite of C as simplicial set!). The unit of A is $1_A=\underline{\mathrm{Vect}}$ and its endomorphism ∞-category is identified with DG (as symmetric monoidal categories), so we will see traces of endomorphisms of dualizable objects of A as objects of DG. All objects of A are dualizable and we have natural isomorphisms $\underline{C}^\vee \simeq \underline{C^{\mathrm{op}}}$, the unit and co-unit maps being induced by the $C \otimes C^{\mathrm{op}}$-module C.

Now take $\underline{C} \in A$ and $Q \in C \otimes C^{\mathrm{op}} - \mathrm{mod}$, seen as an endomorphism of $\underline{C}$. We have an identification

$$\mathrm{Tr}(Q, \underline{C}) = C \otimes_{C \otimes C^{\mathrm{op}}} Q.$$

In particular, suppose that $F : C \to C$ is a monoidal endofunctor of C. We can then define a $C \otimes C^{\mathrm{op}}$-module $Q = C_F$, which is simply C where the left action of C is twisted by F and the right action is induced by right multiplication. We call

$$\mathrm{Tr}^{\mathrm{cat}}(F, C) = C \otimes_{C \otimes C^{\mathrm{op}}} C_F$$

the *categorical trace of F on C*. This is a dg-category, not simply an object of Vect.

Theorem 2.5. *(Gaitsgory, Kazhdan, Rozenblyum, Varshavsky) Suppose that C is a rigid compactly generated monoidal category and F is a monoidal endofunctor of C. Then $\underline{C}$ is dualizable in A and for each dualizable right C-module M endowed with a semi-linear endomorphism*[4] *$F_M : M \to M$ we can canonically attach a "character" $[M, F_M] \in \mathrm{Tr}^{\mathrm{cat}}(F, C)$ in such a way that we have a canonical equivalence of algebras*

$$HH(F, C) \simeq \mathrm{End}_{\mathrm{Tr}^{\mathrm{cat}}(F,C)}([C, F])$$

and, via this identification, natural isomorphisms

$$\mathrm{Tr}(F_M, M) \simeq \mathrm{Hom}_{\mathrm{Tr}^{\mathrm{cat}}(F,C)}([C, F], [M, F_M]).$$

The next hard result is a consequence of previous work of Ben-Zvi, Nadler and Preygel. See theorem 3.25 in the paper under review for the last sentence in the theorem.

Theorem 2.6. *One has natural identifications*

$$\mathrm{Tr}^{\mathrm{cat}}(\mathrm{Coh}(Z/G), q_*) = \mathrm{Coh}(\mathbf{L}^u_G)$$

and $[\mathrm{Coh}(Z/G), q_*]$ gets identified with the coherent Springer sheaf. Moreover $\mathrm{Coh}(Z/G)$ is rigid monoidal.

Applying the above two theorems, we obtain an isomorphism of algebras

$$\mathrm{End}(\mathscr{S}_q) \simeq \mathrm{Tr}(q_*, \mathrm{Coh}(Z/G)).$$

[4]Thus we are given homotopy coherent identifications $F_M(c \otimes m) \simeq F(c) \otimes F_M(m)$.

2.6. **Calculation of** $\mathrm{Tr}(q_*, \mathrm{Coh}(Z/G))$**.** This is quite hard an roundabout, and takes all of chapter 2 of the paper. The proof will be explained in Simon's talk, so I will be extremely brief and imprecise here. We will work over $k = \overline{\mathbb{Q}}_\ell$ with $\ell \neq p$. Let

$$\mathbf{H} = \mathrm{Coh}(Z/G)$$

the affine Hecke category and $\mathbf{H}^m = \mathrm{Coh}(Z/(G \times \mathbb{G}_m))$ its mixed version. The fundamental ingredient in the calculation is a deep theorem of Bezrukavnikov, describing $\mathrm{Ho}(\mathbf{H})$ in terms of ℓ-adic sheaves on a suitable stack. Then standard constructions yield a semi-orthogonal decomposition of $\mathbf{H}$, which upgrades to a semi-orthogonal decomposition of $\mathbf{H}^m$.

Let $G^\vee$ be the dual group of G, seen as a group over $\overline{\mathbb{F}}_q$. Associated to it we have the affine flag variety $\mathrm{Fl}_{G^\vee} = LG^\vee/I^\vee$ (étale sheafification), where $LG^\vee$ is the loop-group of $G^\vee$, a group ind-scheme representing the functor $R \mapsto G^\vee(R((z))$ and $I^\vee$ is the Iwahori subgroup.[5] One can define (with care...) a triangulated category $D_b^c(I^\vee \backslash \mathrm{Fl}_{G^\vee}, k)$.

Theorem 2.7. *(Bezrukavnikov) There is an equivalence of monoidal categories*

$$\mathrm{Ho}(\mathbf{H}) \simeq D_b^c(I^\vee \backslash \mathrm{Fl}_{G^\vee}, k)$$

intertwining pullback by Frobenius on the right-hand side and q_* on the left-hand side.

Now, the idea is that one has a good understanding of the geometry of the $I^\vee$-orbits on the affine flag variety: they are in bijection with the affine Weyl group W_{aff}, and if $j_w : \mathrm{Fl}_w \to \mathrm{Fl}_{G^\vee}$ is the corresponding embedding then one can compute the endomorphism algebra $A_w = \mathrm{End}(j_{w,!}\underline{k})$, namely by standard manipulations it is isomorphic to $R\Gamma(I^\vee \backslash \mathrm{Fl}_w, k)$. On the other hand, the stabiliser of Fl_w modulo its pro-unipotent radical is $\check{T}$, so we are reduced to the computation of the cohomology complex of a torus, which is standard. In particular this shows that $A := A_w$ is independent of w up to isomorphism. Moreover, letting C_w be the subcategory generated (in a suitable sense) by $j_{w,!}\underline{k}$, one checks that the C_w form a semi-orthogonal decomposition of $D_b^c(I^\vee \backslash \mathrm{Fl}_{G^\vee}, k)$, which yields via Bezrukavnikov's equivalence a semi-orthogonal decomposition of $\mathbf{H}$. Moreover, the computation of A_w above allows one to show that $HH(C_w, q_*) \simeq k$, in particular it is concentrated in degree 0. Next, Hochschild homology can be shown to be compatible with semi-orthogonal decompositions. This allows one to deduce that $HH(\mathbf{H}, q_*)$ is concentrated in degree 0 and has a k-basis indexed by W_{aff}. Unfortunately, this says nothing about the algebra structure, and this is quite tricky. The idea is to vary q, by considering the affine Hecke algebra $\mathscr{H}_{\mathrm{aff}}$, a $k[q, q^{-1}]$-algebra whose q-specialisation is $\mathscr{H}_q$ (the Iwahori-Hecke algebra) whenever q is a prime power. One shows that the above semi-orthogonal decomposition of $\mathbf{H}$ lifts to one of $\mathbf{H}^m$ and by similar computations (which are actually nontrivial, see Simon's talk!) one shows that $HH(\mathbf{H}^m)$ is concentrated in degree 0 and

[5] More precisely, we have the arc-group $L^+G^\vee$, a group scheme representing the functor $R \mapsto G^\vee(R[[z]])$, and $I^\vee$ is the subgroup of $L^+G^\vee$ pullback of $B^\vee$ under the $L^+G^\vee \to G^\vee$ induced by $z \mapsto 0$.

isomorphic as $k[q,q^{-1}]$-module to $k[W_{\mathrm{aff}}] \otimes_k k[q,q^{-1}]$. Next, one constructs a canonical algebra map

$$K_0(\mathbf{H}^m) \otimes_{\mathbf{Z}} k \to HH(\mathbf{H}^m),$$

using functoriality of traces and the interpretation of the left-hand side as a Grothendieck group. The claim is that this map is an isomorphism. As both sides are compatible with semi-orthogonal decompositions, one can do the computation on each stratum, where it follows from the above discussion. Finally, a classical theorem of Kazhdan-Lusztig and Ginzburg identifies $K_0(\mathbf{H}^m) \otimes_{\mathbf{Z}} k$ with $\mathscr{H}_{\mathrm{aff}}$, thus yielding an isomorphism of algebras

$$\mathscr{H}_{\mathrm{aff}} \simeq HH(\mathbf{H}^m).$$

Finally, similar arguments construct a natural isomorphism between the q-specialization of $HH(\mathbf{H}^m)$ and $HH(\mathbf{H}, q_*)$, which allows one to conclude.