HOCHSCHILD HOMOLOGY OF THE HECKE CATEGORY (AFTER BEN-ZVI, CHEN, HELM AND NADLER)

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In this text I explain what I understand about the description of the Hochschild homology of the Hecke category, following [BZCHN, §2]. Some of the proofs sketched below are slightly different from those contained in that reference.

1. Hochschild homology of ∞-categories

1.1. **Setting.** For references on the notions considered here, we recommend [HSS, §4.1].

We consider the following ∞-categories.

- $\mathrm{Cat}^{\mathrm{perf}}$ is the ∞-category of small, stable, idempotent-complete ∞-categories and exact functors;
- $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ is the ∞-category of stable presentable ∞-categories and left adjoint functors.

In $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$, the condition that a functor is a left adjoint (i.e. admits a right adjoint) is equivalent to the condition that it preserves small colimits, see [GR, Chap. 1, Theorem 2.5.4]. We have an ind-completion functor

$$\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}. \tag{1.1}$$

This functor induces an equivalence between $\mathrm{Cat}^{\mathrm{perf}}$ and the (non full!) subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ consisting of compactly generated presentable stable ∞-categories and left-adjoint functors preserving compact objects. (Here the property of preserving compact objects is equivalent to the right adjoint commuting with small colimits, see [GR, Lemma 7.1.5].) If $\mathcal{C}$ is in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ we will denote by $\mathcal{C}^\omega$ the ∞-subcategory of compact objects; then for $\mathcal{A}$ in $\mathrm{Cat}^{\mathrm{perf}}$ we have $\mathrm{Ind}(\mathcal{A})^\omega = \mathcal{A}$.

Remark 1.1. Idempotent completeness of ∞-categories concerns existence of certain colimits, see the discussion preceding [Lu1, Corollary 4.4.5.14]. Hence presentable ∞-categories are automatically idempotent-complete. But since these colimits are *not* finite, it is not automatic in a stable ∞-category.

Recall that both $\mathrm{Cat}^{\mathrm{perf}}$ and $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ admit canonical symmetric monoidal structures such that (1.1) is symmetric monoidal. The monoidal structure on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ is given by the Lurie tensor product, and the structure on $\mathrm{Cat}^{\mathrm{perf}}$ is defined precisely so that Ind becomes monoidal; see [BGT1, §3.1]. For simplicity, the tensor product bifunctors will be denoted $\otimes$ in both cases. The unit object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ is the ∞-category Sp of spectra (denoted $\mathcal{S}_\infty$ in [BGT1]), and the unit object in $\mathrm{Cat}^{\mathrm{perf}}$ is the ∞-category of finite spectra Sp^ω (i.e. compact objects in Sp). We also have

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internal Hom objects $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A},\mathcal{B})$ in $\mathrm{Cat}^{\mathrm{perf}}$ (describing the exact functors from $\mathcal{A}$ to $\mathcal{B}$) and $\mathrm{Fun}^{\mathrm{L}}(\mathcal{A},\mathcal{B})$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ (describing the left-adjoint functors from $\mathcal{A}$ to $\mathcal{B}$).

Remark 1.2. By [BGT1, Corollary 4.25] the category $\mathrm{Cat}^{\mathrm{perf}}$ is complete and cocomplete, i.e. admits small limits and colimits.

1.2. **Linear categories.** We fix a field k.[1] In practice we will work with "k-linear" versions of the ∞-categories above; this corresponds to the setting of [HSS, §4] with $\mathcal{E}$ being the ∞-category Vect_k^{ω} of compact objects in the (compactly generated presentable stable) ∞-category Vect_k of k-vector spaces, i.e. the ∞-category of bounded complexes of finite-dimensional k-vector spaces. Here Vect_k^{ω} is a commutative algebra object in $\mathrm{Cat}^{\mathrm{perf}}$, and Vect_k is a commutative algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$. We will therefore consider the following ∞-categories.

- $\mathrm{Cat}_k^{\mathrm{perf}}$ is the ∞-category of Vect_k^{ω}-modules in $\mathrm{Cat}^{\mathrm{perf}}$.
- $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ is the ∞-category of Vect_k-modules in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$.

There are natural monoidal structures on these categories, whose monoidal products will be denoted $\otimes_k$ in both cases; the monoidal product on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ can be computed by taking the colimit of the simplicial diagram

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{C}\otimes \mathrm{Vect}_k \otimes \mathcal{D} \rightrightarrows \mathcal{C}\otimes\mathcal{D}$$

for $\mathcal{C},\mathcal{D}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$. We also have internal Hom objects $\mathrm{Fun}_k(\mathcal{A},\mathcal{B})$ in $\mathrm{Cat}_k^{\mathrm{perf}}$ (describing k-linear exact functors) and $\mathrm{Fun}_k^{\mathrm{L}}(\mathcal{C},\mathcal{D})$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ (describing k-linear left-adjoint functors). The functor (1.1) lifts to a symmetric monoidal functor

$$\mathrm{Ind} : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}.$$

Remark 1.3. (1) By [HSS, Proposition 4.9(3)], the right adjoint of any morphism in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ is a Vect_k-module functor.
(2) I guess that Remark 1.2 implies that $\mathrm{Cat}_k^{\mathrm{perf}}$ is cocomplete.

Objects in $\mathrm{Cat}_k^{\mathrm{perf}}$ are naturally enriched over Vect_k. Namely, if $\mathcal{A}\in\mathrm{Cat}_k^{\mathrm{perf}}$ and $a\in\mathcal{A}$, then the functor

$$\mathrm{Vect}_k^{\omega}\to\mathcal{A}$$

given by action on a is exact, hence we can consider the induced functor

$$\mathrm{Vect}_k\to\mathrm{Ind}(\mathcal{A}),$$

which admits a right adjoint. The restriction of this right adjoint to $\mathcal{A}$ is denoted $\mathcal{A}_k(a,-)$. This procedure defines a functor

$$\mathcal{A}^{\mathrm{op}}\to\mathrm{Fun}_k(\mathcal{A},\mathrm{Vect}_k),$$

called the "enriched Yoneda embedding." For $a,b\in\mathcal{A}$ we therefore have an object $\mathcal{A}_k(a,b)\in\mathrm{Vect}_k$. As explained in [HSS, Comments after Corollary 4.11] the enriched Yoneda embedding is fully faithful.

For the same reason, objects in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ are also enriched over Vect_k.

[1]Eventually this field will be assumed to be of characteristic 0, but this restriction is not required in this section and the next one.

Remark 1.4. Since Sp (resp. Sp^ω) is the unit object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ (resp. $\mathrm{Cat}^{\mathrm{perf}}$), any object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ (resp. $\mathrm{Cat}^{\mathrm{perf}}$) admits a canonical action of Sp (resp. Sp^ω). By the same considerations as above, for $\mathcal{A}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ or $\mathrm{Cat}^{\mathrm{perf}}$ and $a, b \in \mathcal{A}$ we therefore have an object $\mathcal{A}(a,b) \in \mathrm{Sp}$. Since Vect_k belongs to $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ we have a canonical functor

$$\mathrm{Sp} \to \mathrm{Vect}_k \tag{1.2}$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$, see e.g. [GR, Chap. 1, Corollary 6.2.6], and its right adjoint

$$\mathrm{Vect}_k \to \mathrm{Sp}. \tag{1.3}$$

If $\mathcal{A}$ belongs to $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ or $\mathrm{Cat}^{\mathrm{perf}}_k$, for $a, b \in \mathcal{A}$ the spectrum $\mathcal{A}(a,b)$ can be recovered from the object $\mathcal{A}_k(a,b)$ by taking the image under (1.3). Objects in ∞-categories have *(pointed) spaces* (or, in other words, ∞-groupoids) of morphisms. For $\mathcal{A}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ or $\mathrm{Cat}^{\mathrm{perf}}$ and $a, b \in \mathcal{A}$, the space of morphisms from a to b can be recovered from $\mathcal{A}(a,b)$ by taking the image under the functor Ω^∞ from [GR, §6.2.7] (which is the right adjoint of the canonical functor Σ^∞ from spaces to spectra, see [GR, §6.2.2]).

1.3. **Definition.** If $\mathcal{C}$ is a dualizable object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$, with dual $\mathcal{C}^\vee$, then the *Hochschild homology* of $\mathcal{C}$ is the object $\mathrm{HH}(\mathcal{C}) \in \mathrm{Vect}_k$ representing the composition

$$\mathrm{Vect}_k \xrightarrow{\mathrm{coev}} \mathcal{C} \otimes_k \mathcal{C}^\vee \xrightarrow{\mathrm{ev}} \mathrm{Vect}_k. \tag{1.4}$$

More generally, given a functor $f : \mathcal{C} \to \mathcal{C}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$, we denote by $\mathrm{HH}(\mathcal{C}, f) \in \mathrm{Vect}_k$ the object representing the composition

$$\mathrm{Vect}_k \xrightarrow{\mathrm{coev}} \mathcal{C} \otimes_k \mathcal{C}^\vee \xrightarrow{f \otimes \mathrm{id}} \mathcal{C} \otimes_k \mathcal{C}^\vee \xrightarrow{\mathrm{ev}} \mathrm{Vect}_k.$$

A particular case of this construction is when $\mathcal{C}$ is compactly generated, i.e. of the form $\mathrm{Ind}(\mathcal{A})$ for some $\mathcal{A}$ in $\mathrm{Cat}^{\mathrm{perf}}_k$; in this case we have $\mathcal{C}^\vee = \mathrm{Ind}(\mathcal{A}^{\mathrm{op}})$. For details, see [HSS, Proposition 4.10].

Remark 1.5. (1) If A is an algebra object in Vect_k, then we can consider the ∞-category $\mathrm{RMod}_A(\mathrm{Vect}_k)$ of right A-modules in Vect_k. This category is dualizable. If M is an A-bimodule we can consider the functor $f_M = (-) \otimes_A M$ on $\mathrm{RMod}_A(\mathrm{Vect}_k)$, and we have

$$\mathrm{HH}(\mathrm{RMod}_A(\mathrm{Vect}_k), f_M) = A \otimes_{A \otimes_k A^{\mathrm{op}}} M.$$

(2) As a consequence of the comments above, if $\mathcal{A} \in \mathrm{Cat}^{\mathrm{perf}}_k$ we have evaluation and coevaluation morphisms

$$\mathrm{Ind}(\mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{A}) = \mathrm{Ind}(\mathcal{A}^{\mathrm{op}}) \otimes_k \mathrm{Ind}(\mathcal{A}) \to \mathrm{Vect}_k,$$
$$\mathrm{Vect}_k \to \mathrm{Ind}(\mathcal{A}^{\mathrm{op}}) \otimes_k \mathrm{Ind}(\mathcal{A}) = \mathrm{Ind}(\mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{A}),$$

but these functors do not necessarily preserve compact objects, hence do not necessarily provide a duality datum between $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$. In fact, $\mathcal{A}$ is dualizable in $\mathrm{Cat}^{\mathrm{perf}}_k$ iff these maps preserve compact objects, and in this case the dual of $\mathcal{A}$ is $\mathcal{A}^{\mathrm{op}}$. See [HSS, §4.3] for more on this topic. If $\mathcal{A}$ is dualizable in $\mathrm{Cat}^{\mathrm{perf}}_k$, then the composition (1.4) is induced by a functor $\mathrm{Vect}^\omega_k \to \mathrm{Vect}^\omega_k$; as a consequence, $\mathrm{HH}(\mathcal{C})$ belongs to Vect^ω_k in this case.

1.4. **Functoriality.** This construction admits the following functoriality. Consider pairs $(\mathcal{C}, f)$ and $(\mathcal{D}, g)$ as above. A *morphism of pairs* from $(\mathcal{C}, f)$ to $(\mathcal{D}, g)$ is the datum of a pair (F, ψ) where $F : \mathcal{C} \to \mathcal{D}$ is a morphism in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ whose right adjoint perserves small colimits and $\psi : F \circ f \to g \circ F$ is an object in $\mathrm{Fun}^{\mathrm{L}}_k(\mathcal{C}, \mathcal{D})$. In this case we have a morphism

$$\mathrm{HH}(F, \psi) : \mathrm{HH}(\mathcal{C}, f) \to \mathrm{HH}(\mathcal{D}, g)$$

defined as the composition

$$\mathrm{HH}(\mathcal{C}, f) \to \mathrm{HH}(\mathcal{C}, F^{\mathrm{R}}Ff) \to \mathrm{HH}(\mathcal{C}, F^{\mathrm{R}}gF) \cong \mathrm{HH}(\mathcal{D}, gFF^{\mathrm{R}}) \to \mathrm{HH}(\mathcal{D}, g)$$

where:

- F^{R} is the right adjoint to F;
- the first, resp. second, resp. third, map is induced by the morphism of functors $f \to F^{\mathrm{R}}Ff$ (induced by adjunction), resp. $F^{\mathrm{R}}Ff \to F^{\mathrm{R}}gF$ (induced by ψ), resp. $gFF^{\mathrm{R}} \to g$ (induced by adjunction);
- the isomorphism follows from cyclicity of the trace.

As a first application of this construction, consider a compact object $c \in \mathcal{C}$. Then the morphism

$$\alpha_c : \mathrm{Vect}_k \to \mathcal{C}$$

given by action on c admits a right adjoint, and preserves compact objects, so that this adjoint preserves colimits (hence is a morphism in Vect_k). Any map $\psi : c \to f(c)$ defines a morphism of pairs $(\alpha_c, \psi) : (\mathrm{Vect}_k, \mathrm{id}) \to (\mathcal{C}, F)$, which provides a map

$$\mathrm{HH}(\alpha_c, \psi) : k = \mathrm{HH}(\mathrm{Vect}_k) \to \mathrm{HH}(\mathcal{C}, f).$$

The image of $1 \in k$ is denoted $[c] \in \mathrm{HH}^0(\mathcal{C}, f)$. (This vector depends on ψ; in practice we will take $f = \mathrm{id}$, and $\psi = \mathrm{id}$.)

As explained in [HSS, §2 and §4.4], this construction can be upgraded as follows. The ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ can be upgraded to a symmetric monoidal $(\infty, 2)$-category $\mathbf{Pr}^{\mathrm{L}}_{\mathrm{St},k}$. Then we have a symmetric monoidal ∞-category $\mathrm{End}(\mathbf{Pr}^{\mathrm{L}}_{\mathrm{St},k})$ whose objects are the pairs $(\mathcal{C}, f)$ with $\mathcal{C}$ a dualizable object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ and $f : \mathcal{C} \to \mathcal{C}$ is a morphism in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$, and whose 1-morphisms are morphisms of pairs (F, ψ) as above. Moreover, we have a symmetric monoidal functor

$$\mathrm{HH} : \mathrm{End}(\mathbf{Pr}^{\mathrm{L}}_{\mathrm{St},k}) \to \mathrm{Vect}_k.$$

If $\mathcal{C}$ is an algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$ which is dualizable (as an object of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$), whose unit is compact and such that the tensor product $\mathcal{C} \otimes_k \mathcal{C} \to \mathcal{C}$ sends compact objects to compact objects, then $\mathcal{C}$ defines an algebra object in $\mathrm{End}(\mathbf{Pr}^{\mathrm{L}}_{\mathrm{St},k})$, so that $\mathrm{HH}(\mathcal{C})$ becomes an algebra object in Vect_k. The same comments apply if we are moreover given a monoidal functor $f : \mathcal{C} \to \mathcal{C}$; we obtain an algebra object $\mathrm{HH}(\mathcal{C}, f)$. In particular, if $\mathcal{A}$ is an algebra object in $\mathrm{Cat}^{\mathrm{perf}}_k$, then $\mathcal{C} = \mathrm{Ind}(\mathcal{A})$ satisfies these assumptions.

1.5. **Computation via a complex.** We consider $\mathcal{A} \in \mathrm{Cat}^{\mathrm{perf}}_k$, and a functor $f : \mathcal{A} \to \mathcal{A}$. We set $\mathcal{C} = \mathrm{Ind}(\mathcal{A})$, and still denote by f the induced endofunctor of $\mathcal{C}$. In this setting, the Hochschild homology $\mathrm{HH}(\mathcal{C}, f)$ can be computed via a complex as follows. Given a set S of objects in $\mathcal{A}$, for $n \geq 0$ we set

$$\mathsf{C}_{-n}(S, f) = \bigvee_{a_0, \cdots, a_n \in S} \mathcal{A}_k(a_n, a_{n-1}) \otimes_k \cdots \otimes_k \mathcal{A}_k(a_1, a_0) \otimes_k \mathcal{A}_k(a_0, f(a_n)) \quad \in \mathrm{Vect}_k.$$

This collection of objects has face and degeneracy maps which define a simplicial object in Vect_k, and we can consider

$$\mathsf{C}(S,f) = \mathrm{colim}_n \, \mathsf{C}_n(S,f).$$

As explained in [HSS, Proposition 4.24], if S contains one object in each equivalence class in $\mathcal{A}$, there is an equivalence

$$\mathrm{HH}(\mathcal{C},f) \cong \mathsf{C}(S,f).$$

It is claimed in [BZCHN, Remark 2.8] that this property holds as soon as S is stable under F and generates $\mathcal{A}$ (i.e. $\mathcal{A}$ does not contain a strict stable, idempotent complete, full subcategory containing S.) This version does not seem to appear in the literature, but according to the authors of [HSS] their proof also applies in this setting.

In [BZCHN] the authors also use a variant of this construction, as follows. Consider the multiplicative group[2] $\mathbb{G}_{\mathrm{m},k}$, its ∞-category $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$ of representations (an object of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St},k}$), and the full subcategory $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})^\omega$ of compact objects (an object of $\mathrm{Cat}_k^{\mathrm{perf}}$). Assume that $\mathcal{A}$ is a $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})^\omega$-module category, and that f commutes with the action of $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})^\omega$. By the same considerations as in §1.2, in this case $\mathcal{A}$ is enriched over $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$: for $a,b \in \mathcal{A}$ we have an object

$$\mathcal{A}_{\mathbb{G}_{\mathrm{m}}}(a,b) \in \mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}).$$

The obvious embedding $\mathrm{Vect}_k \to \mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$ has as right adjoint the functor of fixed points $(-)^{\mathbb{G}_{\mathrm{m},k}}$; we therefore have

$$\mathcal{A}_k(a,b) = \big(\mathcal{A}_{\mathbb{G}_{\mathrm{m}}}(a,b)\big)^{\mathbb{G}_{\mathrm{m},k}}.$$

More generally, denoting by k_m the 1-dimensional $\mathbb{G}_{\mathrm{m},k}$-module of weight m, for any $m,m' \in \mathbb{Z}$ we have

$$\mathcal{A}_k(k_m \otimes a, k_{m'} \otimes b) = \big(\mathcal{A}_{\mathbb{G}_{\mathrm{m}}}(a,b) \otimes k_{m'-m}\big)^{\mathbb{G}_{\mathrm{m},k}}. \tag{1.5}$$

Let S be an F-stable collection of objects in $\mathcal{A}$ which generate $\mathcal{A}$ under the action of $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})^\omega$, i.e. such that the objects $V \otimes a$ with $V \in \mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})^\omega$ and $a \in S$ generate $\mathcal{A}$. Then we can consider the complex as above associated with the collection of objects $(k_m \otimes a : m \in \mathbb{Z}, a \in S)$, where k_m is the 1-dimensional $\mathbb{G}_{\mathrm{m},k}$-module of weight m: we find a simplicial object with n-term

$$\bigvee_{\substack{a_0,\cdots,a_n \in S \\ m_0,\cdots,m_n \in \mathbb{Z}}} \mathcal{A}_k(k_{m_n} \otimes a_n, k_{m_{n-1}} \otimes a_{n-1}) \otimes_k \cdots \otimes_k \mathcal{A}_k(k_{m_1} \otimes a_1, k_{m_0} \otimes a_0)$$
$$\otimes_k \mathcal{A}_k(k_{m_0} \otimes a_0, k_{m_n} \otimes f(a_n)).$$

Using (1.5), the latter object identifies with

$$\bigvee_{\substack{a_0,\cdots,a_n \in S \\ m_0,\cdots,m_n \in \mathbb{Z}}} \big(\mathcal{A}_{\mathbb{G}_{\mathrm{m}}}(a_n,a_{n-1}) \otimes k_{m_{n-1}-m_n}\big)^{\mathbb{G}_{\mathrm{m},k}} \otimes_k \cdots \otimes_k \big(\mathcal{A}_{\mathbb{G}_{\mathrm{m}}}(a_1,a_0) \otimes k_{m_0-m_1}\big)^{\mathbb{G}_{\mathrm{m},k}}$$
$$\otimes_k \big(\mathcal{A}_{\mathbb{G}_{\mathrm{m}}}(a_0,f(a_n)) \otimes k_{m_n-m_0}\big)^{\mathbb{G}_{\mathrm{m},k}},$$

[2] In [BZCHN] the authors in fact consider this construction for a reductive algebraic group, assuming that k has characteristic 0, but in practice they apply it only for $\mathbb{G}_{\mathrm{m},k}$. A natural setting for this construction might be to consider a linearly reductive group. (Semisimplicity of representations seems to be important.)

hence finally with

$$\bigvee_{a_0,\cdots,a_n\in S} \big(\mathcal{A}_{\mathbb{G}_\mathrm{m}}(a_n,a_{n-1})\otimes_k \cdots \otimes_k \mathcal{A}_{\mathbb{G}_\mathrm{m}}(a_1,a_0) \otimes_k \mathcal{A}_{\mathbb{G}_\mathrm{m}}(a_0,f(a_n))\big)^{\mathbb{G}_{\mathrm{m},k}} \otimes \mathscr{O}(\mathbb{G}_{\mathrm{m},k}). \tag{1.6}$$

1.6. **Hochschild homology is a localizing invariant.** In [BGT1, §5.1] the authors define what it means for a sequence

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \tag{1.7}$$

of objects and morphisms in $\mathrm{Cat}^{\mathrm{perf}}$ to be an *exact sequence*. (This involves consideration of an ∞-categorical version of the Verdier quotient of triangulated categories, see in particular [BGT1, Proposition 5.14].) This definition can in fact be checked at the level of homotopy categories: a sequence as in (1.7) is exact iff it satisfies the following properties:

- the functor $\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{B})$ induced by f is fully faithful;
- the functor $\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{C})$ induced by $g \circ f$ is zero;
- the induced functor $\mathrm{Ho}(\mathcal{B})/\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{C})$ (where the first category is the Verdier quotient) identifies $\mathrm{Ho}(\mathcal{C})$ with the idempotent completion of $\mathrm{Ho}(\mathcal{B})/\mathrm{Ho}(\mathcal{A})$.

For details, see [BGT1, Proposition 5.15] and [HSS, §5.1]. Following [BGT1, Definition 5.18] (see also [HSS, Definition 5.2]), we will say that a sequence as in (1.7) is *split exact* if it is exact and f and g admit right adjoints. (If these adjoints are denoted i and j respectively, then we automatically have $i \circ f \cong \mathrm{id}$ and $g \circ j \cong \mathrm{id}$.)

Remark 1.6. As explained to us by S. Scherotzke, the existence of a right adjoint for an exact functor between ∞-categories can be checked at the level of homotopy categories. (In fact, the adjoint at the level of homotopy categories is automatically an $\mathcal{H}$-enriched adjoint, since in [Lu1, Definition 5.2.2.7], if the composition is a bijection between connected components then it is automatically an equivalence between mapping spaces. Then the claim follows from [Lu1, Proposition 5.2.2.12].) Hence the fact that a sequence as in (1.7) is a split exact sequence can also be checked at the level of infinity-categories.

We will say that a sequence as above, but now in $\mathrm{Cat}_k^{\mathrm{perf}}$, is exact, resp. split exact, if its image in $\mathrm{Cat}^{\mathrm{perf}}$ is exact, resp. split exact. In this case, the adjoints i and j considered above are automatically Vect_k^{ω}-linear, see [HSS, Proposition 4.9(3)]. See also [HSS, Proposition 5.4] for another characterization of this condition.

Following [HSS, Definitions 5.11 and 5.16], if $\mathcal{D}$ is an object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$, we will say that a functor $F : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathcal{D}$ is an *additive invariant*, resp. a *localizing invariant*, if the following conditions are satisfied:

- F preserves filtered colimits;
- F preserves zero objects;
- F sends split exact sequences, resp. exact sequences, in $\mathrm{Cat}_k^{\mathrm{perf}}$ to cofiber sequences in $\mathcal{D}$.

In fact, as noted in [HSS, Footnote 6 on p. 138], if F is an additive invariant, given a split exact sequence

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "f"] & \mathcal{B} \arrow[l, shift left, "i"] \arrow[r, shift left, "g"] & \mathcal{C}, \arrow[l, shift left, "j"] \end{tikzcd}$$

the functors i and g induce an equivalence

$$F(\mathcal{A}) \vee F(\mathcal{C}) \xrightarrow{\sim} F(\mathcal{B})$$

in $\mathcal{D}$. It is clear that a localizing invariant is a fortiori an additive invariant. These definitions are "linear variants" of concepts considered in [BGT1]; in particular we have a notion of additive or localizing invariant from $\mathrm{Cat}^{\mathrm{perf}}$ to $\mathcal{D}$..

The ∞-category of additive invariants with values in $\mathcal{D}$ will be denoted

$$\mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_k^{\mathrm{perf}}, \mathcal{D}).$$

Proposition 1.7. *The functor*

$$\mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Vect}_k$$

given by $\mathcal{A} \mapsto \mathrm{HH}(\mathrm{Ind}(\mathcal{A}))$ *is a localizing invariant.*

Proof. First, recall that in this context $\mathrm{Ind}(\mathcal{A})$ is always dualizable, see §1.3, so that $\mathrm{HH}(\mathrm{Ind}(\mathcal{A}))$ is well defined. We need to check that the 3 properties characterizing localizing invariants are satisfied. The commutation with colimits follows from the recipe to compute Hochschild homology explained in §1.5 (see [HSS, Corollary 4.25]). The second property is clear. Finally, the third property is proved in [HSS, Theorem 3.4 and Proposition 5.4]. □

Remark 1.8. In fact, [HSS, Theorem 3.4 and Proposition 5.4] give a version of the claim on images of exact sequences which allows endomorphisms of the categories.

2. K-theory as a universal additive invariant

2.1. **K-theory of ∞-categories.** Algebraic (connective) K-theory is a functor

$$\mathsf{K} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}, \tag{2.1}$$

see [BGT1, §7.1] and [HSS, §5.4]. See also [Ba] for another construction of algebraic K-theory, which hopefully is equivalent. This functor takes values in *connective* spectra, i.e. spectra whose π_n vanishes for $n < 0$.

What we will mainly consider is the composition of this functor with (1.2), which we will denote

$$\mathsf{K}_k : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Vect}_k, \tag{2.2}$$

or even its "restriction" to $\mathrm{Cat}_k^{\mathrm{perf}}$. This functor takes values in complexes concentrated in non-positive (cohomological) degrees.

As explained in [Lu2, Remark 11], the group $\mathsf{K}^0(\mathcal{A}) := \pi_0(\mathsf{K}(\mathcal{A}))$ identifies with the Grothendieck group $\mathsf{K}^0(\mathrm{Ho}(\mathcal{A}))$ of the triangulated category $\mathrm{Ho}(\mathcal{A})$. As a consequence we have $\mathsf{H}^0(\mathsf{K}_k(\mathcal{A})) = k \otimes_{\mathbb{Z}} \mathsf{K}^0(\mathrm{Ho}(\mathcal{A}))$.

Remark 2.1.
(1) The main result of [Ba] says that if $\mathcal{A}$ is equipped with a bounded t-structure with heart $\mathcal{A}^\heartsuit$ (an abelian category), then there is an equivalence of spectra $\mathsf{K}(\mathcal{A}) \cong \mathsf{K}(\mathcal{A}^\heartsuit)$ where the right-hand side is K-theory of abelian categories in the sense of Quillen. In particular, this provides a different proof, in this setting, that $\mathsf{K}^0(\mathcal{A})$ is the Grothendieck group $\mathsf{K}^0(\mathcal{A}^\heartsuit)$ of the abelian category $\mathcal{A}^\heartsuit$, or equivalently of the triangulated category $\mathrm{Ho}(\mathcal{A})$.
(2) It is important here to work with *small* stable ∞-categories, as opposed to presentable ∞-categories, as the Grothendieck group of a category admitting arbitrary colimits will often vanish.

2.2. **Universal additive invariant – absolute case.** There are various statements in the literature which make sense of the idea that "K-theory is the universal additive invariant" on $\mathrm{Cat}^{\mathrm{perf}}$ or $\mathrm{Cat}_k^{\mathrm{perf}}$. The following version was explained to us by V. Saunier (to whom all the proofs in this section are due).

We start with the "absolute" case, i.e. for $\mathrm{Cat}^{\mathrm{perf}}$. Consider the functor

$$\iota : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$$

which is the composition of the functor sending an ∞-category to the underlying ∞-groupoid (see e.g. [GR, Chap. 1, §1.1.3]) with the functor Σ^∞ considered in Remark 1.4.

Proposition 2.2. *There exists a morphism $\iota \to \mathsf{K}$ which is initial among the morphisms from $\imath$ to an additive invariant on $\mathrm{Cat}^{\mathrm{perf}}$ with values in Sp.*

Sketch of proof. In [BGT1] the authors construct a presentable stable ∞-category Mot of "noncommutative motives" and a functor

$$\mathcal{U}_{\mathrm{add}} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Mot}$$

which is the "universal additive invariant" in the sense that for any $\mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$, composition with $\mathcal{U}_{\mathrm{add}}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Mot}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}^{\mathrm{perf}}, \mathcal{D}). \tag{2.3}$$

For formal reasons, this implies that for $\mathcal{D}$ as above, if $F : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{D}$ is a (colimit-preserving) functor, there is an initial morphism from F to an additive invariant with values in $\mathcal{D}$, given by

$$F \to P_1(\mathcal{U}_{\mathrm{add},!}F) \circ \mathcal{U}_{\mathrm{add}}$$

where $\mathcal{U}_{\mathrm{add},!}$ is left Kan extension along $\mathcal{U}_{\mathrm{add}}$, and $P_1(-)$ is the target of the universal morphism to a colimit-preserving functor.

We apply this process to the morphism ι. The functor Ω^∞ commutes with limits, hence with Kan extension, and the first functor in the definition of ι is corepresented by Sp^ω. As a consequence, $\mathcal{U}_{\mathrm{add},!}\iota$ is corepresented by $\mathcal{U}_{\mathrm{add}}(\mathrm{Sp}^\omega)$. Now since Mot is a stable ∞-category, applying P_1 to the functor corepresented by $\mathcal{U}_{\mathrm{add}}(\mathrm{Sp}^\omega)$ produces the functor $\mathrm{Mot}(\mathcal{U}_{\mathrm{add}}(\mathrm{Sp}^\omega), -)$ where we use the notation of Remark 1.4.

Finally, one uses the following fact: algebraic K-theory has the property that for $\mathcal{A}, \mathcal{B}$ in $\mathrm{Cat}_k^{\mathrm{perf}}$ with $\mathcal{B}$ compact, there is a canonical equivalence

$$\mathrm{Hom}_{\mathrm{Mot}}(\mathcal{U}_{\mathrm{add}}(\mathcal{B}), \mathcal{U}_{\mathrm{add}}(\mathcal{A})) \cong \mathsf{K}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{B}, \mathcal{A}))$$

in Sp, see [BGT1, Theorem 7.13]. In particular, when $\mathcal{B} = \mathrm{Sp}^\omega$ we have $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{B}, \mathcal{A}) = \mathcal{A}$, hence an equivalence

$$\mathrm{Hom}_{\mathrm{Mot}}(\mathcal{U}_{\mathrm{add}}(\mathrm{Sp}^\omega), \mathcal{U}_{\mathrm{add}}(\mathcal{A})) \cong \mathsf{K}(\mathcal{A}),$$

which concludes the proof. $\square$

One also has a version of this property which takes monoidal structures into account. Namely, in [BGT2, Proposition 5.9], the authors show that the functor (2.1) is lax-monoidal. Arguments similar to those for the proof of Proposition 2.2, based on the results of [BGT2], show that this functor is initial among lax-monoidal additive invariants with values in Sp.

2.3. **Universal additive invariant – linear case.** Most of the main results of [BGT1, BGT2] are adapted to a relative case in [HSS] (see in particular [HSS, Theorem 5.12] and [HSS, Theorem 5.24]). Using these variants one shows that:

- the composition $\mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}} \xrightarrow{\mathsf{K}} \mathrm{Sp}$ admits a morphism from the composition $\mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}} \xrightarrow{\iota} \mathrm{Sp}$, and this morphism is initial among the morphisms from this composition to an additif invariant on $\mathrm{Cat}_k^{\mathrm{perf}}$ with values in Sp.
- the composition $\mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}} \xrightarrow{\mathsf{K}} \mathrm{Sp}$ is lax-monoidal, and is initial among lax-monoidal additive invariants on $\mathrm{Cat}_k^{\mathrm{perf}}$ with values in Sp.

2.4. **Application to Hochschild homology.** We have seen in Proposition 1.7 that the functor $\mathrm{HH}(\mathrm{Ind}(-))$ is a localizing invariant, hence in particular an additive invariant, on $\mathrm{Cat}_k^{\mathrm{perf}}$, with values in Vect_k. This functor is monoidal. Hence its composition with (1.3) is lax monoidal. (Here we use the fact that (1.2) is monoidal, so that its right adjoint is lax-monoidal, see [GR, Chap. 1, Lemma 3.2.4].) Using the results explained in §2.3, this composition receives a canonical morphism from K; by adjointness we deduce a canonical map

$$\mathsf{K}_k(\mathcal{A}) \to \mathrm{HH}(\mathrm{Ind}(\mathcal{A})) \tag{2.4}$$

in Vect_k for any $\mathcal{A}$ in $\mathrm{Cat}_k^{\mathrm{perf}}$, called the Chern character, see [BZCHN, §2.1.3]. This morphism can also be obtained using the linear analogue of Proposition 2.2, once one produces a morphism from ι to $\mathrm{HH}(\mathrm{Ind}(-))$, which can be done e.g. using the computation in terms of the complex of §1.5.

In the cases considered below, the Hochschild homology $\mathrm{HH}(\mathrm{Ind}(\mathcal{A}))$ will be concentrated in nonnegative degrees; in this case this map must factor through a map

$$k \otimes_{\mathbb{Z}} \mathsf{K}^0(\mathrm{Ho}(\mathcal{A})) \to \mathrm{HH}(\mathrm{Ind}(\mathcal{A})). \tag{2.5}$$

By functoriality, for any $a \in \mathcal{A}$ this map sends the image in the left-hand side of the class of a in $\mathsf{K}^0(\mathrm{Ho}(\mathcal{A}))$ to $[a] \in \mathrm{HH}^0(\mathrm{Ind}(\mathcal{A}))$ (see §1.4).

3. Hecke category and Bezrukavnikov's equivalence

3.1. **Hecke categories.** From now on we set $k = \overline{\mathbb{Q}}_\ell$. We fix a connected reductive algebraic group G over k, with a fixed choice of Borel subgroup $B \subset G$ and maximal torus $T \subset B$. We will also denote by W the Weyl group of (G,T), and by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ the Lie algebras of G, B, T. The choice of B determines in the usual way a system of generators $S \subset W$ (the "simple reflections") such that (W,S) is a Coxeter system.

The *Springer resolution* $\widetilde{\mathcal{N}}$ is the cotangent bundle $T^*(G/B)$ of the flag variety G/B. This is a smooth quasi-projective scheme over k, endowed with an action of G. After fixing a G-invariant bilinear form on $\mathfrak{g}$, which gives rises to a G-invariant identification $\mathfrak{g} \cong \mathfrak{g}^*$, we obtain an identification $\widetilde{\mathcal{N}} = G \times^B \mathfrak{u}$ where $\mathfrak{u}$ is the Lie algebra of the unipotent radical U of B. From this identification we see that there exists a canonical G-invariant morphism $\widetilde{\mathcal{N}} \to \mathfrak{g}$, and we set

$$\mathcal{Z} := \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}.$$

Here the fiber product is a *derived* fiber product, and $\mathcal{Z}$ is a *derived* scheme, called the (derived) Steinberg variety. The action of G on $\widetilde{\mathcal{N}}$ extends to an action of $G \times \mathbb{G}_{\mathrm{m},k}$, where $z \in \mathbb{G}_{\mathrm{m},k}$ acts by multiplication by z^{-1} on the fibers of the projection $\widetilde{\mathcal{N}} \to G/B$, and we obtain an induced action of $\mathcal{Z}$. We will also consider the *non-derived* fiber product of $\widetilde{\mathcal{N}}$ with itself over $\mathfrak{g}$, which we will denote $\mathcal{Z}^{\mathrm{cl}}$; this is a scheme, equipped with a canonical morphism $i : \mathcal{Z}^{\mathrm{cl}} \to \mathcal{Z}$.

It is sometimes useful to consider a variant of this construction in which $\widetilde{\mathcal{N}}$ is replaced by the Grothendieck resolution $\widetilde{\mathfrak{g}} = G \times^B \mathfrak{b}$, and consider

$$Z = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}.$$

Here, even if we take the derived fiber product, Z is an honest scheme ; in fact it is a local complete intersection in $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$. It is a standard fact that the irreducible components of Z are in bijection with W, and we will denote by Z_w the component associated with w. (It is the closure of the preimage of the G-orbit on $G/B \times G/B$ associated with w.) It is not difficult to check that the scheme-theoretic intersection of Z_w with $\widetilde{\mathcal{N}} \times \widetilde{\mathfrak{g}}$ is contained in $\mathcal{Z}^{\mathrm{cl}}$; this closed subscheme will be denoted Z'_w.

The main players in this section will be the Hecke category

$$\mathbf{H} = \mathfrak{Coh}(\mathcal{Z}/G)$$

and its "mixed variant"

$$\mathbf{H}^{\mathrm{m}} = \mathfrak{Coh}(\mathcal{Z}/G \times \mathbb{G}_{\mathrm{m},k}).$$

Here we consider the derived stacks $\mathcal{Z}/G$ and $\mathcal{Z}/G \times \mathbb{G}_{\mathrm{m},k}$, and the full ∞-subcategories of the (stable, presentable, k-linear) ∞-categories of quasi-coherent sheaves on these derived stacks consisting of coherent sheaves, i.e. complexes whose pullback to $\mathcal{Z}$ are bounded complexes of coherent sheaves. These are objects in $\mathrm{Cat}_k^{\mathrm{perf}}$. The homotopy categories $\mathrm{Ho}(\mathbf{H})$ and $\mathrm{Ho}(\mathbf{H}^{\mathrm{m}})$ can be described as the bounded derived categories of equivariant coherent sheaves on $\mathcal{Z}$ realized as a dg-scheme. As such, $\mathrm{Ho}(\mathbf{H})$ has appeared in [B3]. Using the general formalism of [BFN], one obtains that $\mathbf{H}$ and $\mathbf{H}^{\mathrm{m}}$ admit canonical structures of (nonsymmetric!) monoidal ∞-category, with monoidal product given by convolution.

3.2. **Braid objects.** The (extended) affine Weyl group W_{aff} is the semidirect product

$$W_{\mathrm{aff}} := W \ltimes X^*(T).$$

There exists a canonical subset S_{aff} containing S (depending on the initial choice of B) and an abelian subgroup $\Omega \subset W_{\mathrm{aff}}$ such that:

- the subgroup $\langle S_{\mathrm{aff}} \rangle \subset W_{\mathrm{aff}}$ generated by S_{aff} is normal in W_{aff};
- the pair $(\langle S_{\mathrm{aff}} \rangle, S_{\mathrm{aff}})$ is a Coxeter system, and the action of Ω by conjugation preserves S_{aff};
- multiplication induces an isomorphism $\Omega \ltimes \langle S_{\mathrm{aff}} \rangle \xrightarrow{\sim} W_{\mathrm{aff}}$.

In particular, the length function on $\langle S_{\mathrm{aff}} \rangle$ extends to W_{aff} by setting $\ell(\omega w) = \ell(w)$ for $\omega \in \Omega$ and $w \in \langle S_{\mathrm{aff}} \rangle$. The *affine braid group* is the group generated by symbols $(T_w : w \in W_{\mathrm{aff}})$, with relations

$$T_y T_w = T_{yw} \quad \text{if } y, w \in W_{\mathrm{aff}} \text{ satisfy } \ell(yw) = \ell(y) + \ell(w).$$

It turns out that this group admits another presentation, similar to the Bernstein presentation of the associated Hecke algebra, with generators $(T_w : w \in W) \cup (\theta_\lambda : \lambda \in X^*(T))$, and a series of relations which includes in particular:

$$T_y T_w = T_{yw} \quad \text{if } y, w \in W \text{ satisfy } \ell(yw) = \ell(y) + \ell(w);$$
$$\theta_\lambda \theta_\mu = \theta_{\lambda+\mu} \quad \text{if } \lambda, \mu \in X^*(T).$$

Using the results of [BR], one can check that there exists a group morphism from B_{aff} to the group of isomorphism classes of invertible objects in the category $\mathrm{Ho}(\mathbf{H})$, which is given as follows on generators:

$$\theta_\lambda \mapsto [i_* \mathscr{O}_{\Delta\widetilde{\mathcal{N}}}(\lambda)] \ (\lambda \in X^*(T)); \quad T_w \mapsto [i_* \mathscr{O}_{Z'_w}] \ (w \in W).$$

Here $\mathscr{O}_{\Delta\widetilde{\mathcal{N}}}(\lambda)$ is the pullback to the diagonal copy of $\widetilde{\mathcal{N}}$ in $\mathcal{Z}^{\mathrm{cl}}$ of the line bundle on G/B corresponding to λ, seen (via pushforward) as a coherent sheaf on $\mathcal{Z}^{\mathrm{cl}}$, and $\mathscr{O}_{Z'_w}$ is the structure sheaf of Z'_w, again seen as a coherent sheaf on $\mathcal{Z}^{\mathrm{cl}}$. The image of T_w^{-1} is the (pushforward of the) dualizing sheaf $\omega_{Z'_{w^{-1}}}$ of the scheme $Z'_{w^{-1}}$ (which is Cohen–Macaulay).

3.3. **Constructible side.** We now fix a prime power q prime with ℓ, and consider the connected reductive algebraic group $G^\vee$ over $\overline{\mathbb{F}}_q$ which is Langlands dual to G, and a choice of Borel subgroup $B^\vee \subset G^\vee$ and maximal torus $T^\vee \subset B^\vee$ such that $X_*(T^\vee) = X^*(T)$, and the choice of positive roots of (G,T) determined by B corresponds to the set of positive coroots of $(G^\vee, T^\vee)$ determined by $B^\vee$.

Associated with these data we have:

- the loop group $\mathrm{L}G^\vee$, defined as the group ind-scheme over $\overline{\mathbb{F}}_q$ representing the functor $R \mapsto G^\vee(R((z)))$ (where z is an indeterminate);
- the arc group $\mathrm{L}^+G^\vee$, defined as the group scheme over $\overline{\mathbb{F}}_q$ representing the functor $R \mapsto G^\vee(R[[z]])$;
- the Iwahori subgroup $I^\vee$, defined as the preimage of $B^\vee$ under the natural morphism $\mathrm{L}^+G^\vee \to G^\vee$ sending z to 0.

It is a classical fact that the étale sheafification of the quotient $\mathrm{L}G^\vee/I^\vee$ is represented by an ind-projective ind-scheme, called the *affine flag variety* and denoted $\mathsf{Fl}_{G^\vee}$. We will denote by $\mathrm{Sh}(I^\vee\backslash \mathrm{L}G^\vee/I^\vee)$ the bounded derived category of étale k-sheaves on the stack $I^\vee\backslash\mathsf{Fl}_{G^\vee}$, seen as a triangulated category. A standard construction provides a monoidal structure on this category, with product given by convolution.

Remark 3.1. In fact the quotient $I^\vee\backslash\mathsf{Fl}_{G^\vee}$ does not make sense as an algebraic stack, since $\mathsf{Fl}_{G^\vee}$ is an ind-scheme rather than a scheme, and $I^\vee$ is not of finite type. But there is a standard remedy for this: one can write $\mathsf{Fl}_{G^\vee}$ as a filtered colimit of schemes $\mathrm{colim}_n X_n$ where each X_n is projective, stable under the action of $I^\vee$, and such that the induced action factors through a quotient of finite type type along a pro-unipotent subgroup. Then one defines the category $\mathrm{Sh}(I^\vee\backslash X_n)$ by replacing $I^\vee$ by any such quotient, and $\mathrm{Sh}(I^\vee\backslash \mathrm{L}G^\vee/I^\vee)$ is defined as the colimit of such categories.

The (affine version of the) Bruhat decomposition provides a parametrization of the $I^\vee$-orbits on $\mathsf{Fl}_{G^\vee}$ by W_{aff} (identified here with $W \ltimes X_*(T^\vee)$[3]), in such a way that the orbit $\mathsf{Fl}_{G^\vee,w}$ labelled by w is isomorphic to an affine space of dimension $\ell(w)$. For $w \in W_{\mathrm{aff}}$ we denote by $j_w : \mathsf{Fl}_{G^\vee,w} \to \mathsf{Fl}_{G^\vee}$ the embedding, and set

$$\Delta_w = j_{w!}\underline{k}[\ell(w)], \quad \nabla_w = j_{w*}\underline{k}[\ell(w)].$$

These are $I^\vee$-equivariant perverse sheaves on $\mathsf{Fl}_{G^\vee}$, which we see as objects in $\mathrm{Sh}(I^\vee\backslash \mathrm{L}G^\vee/I^\vee)$.

[3] Note that W identifies canonically with the Weyl group of $(G^\vee, T^\vee)$.

It is a standard fact that there exists a group morphism from B_{aff} to the group of isomorphism classes of invertible objects in the monoidal category $\mathrm{Sh}(I^\vee\backslash \mathrm{L}G^\vee/I^\vee)$, given on generators by
$$T_w \mapsto [\nabla_w] \ (w \in W_{\mathrm{aff}}).$$
The image of T_w^{-1} is the class of $\Delta_{w^{-1}}$.

3.4. **Bezrukavnikov's equivalence.** The following statement is one of the main results of [B3].

Theorem 3.2. *There exists a canonical equivalence of monoidal triangulated categories*
$$\mathrm{Ho}(\mathbf{H}) \xrightarrow{\sim} \mathrm{Sh}(I^\vee\backslash \mathrm{L}G^\vee/I^\vee).$$

The following statement can easily be deduced from [BL, Proposition 5.8].

Proposition 3.3. *The equivalence of Theorem 3.2 intertwines the group morphisms from B_{aff} considered in §3.2 and §3.3.*

3.5. **Some consequences.** Let us now fix an enumeration $w_0, w_1, \cdots$ of W_{aff} such that if $w_i < w_j$ in the Bruhat order, then $i < j$. For any i, the object corresponding to T_w^{-1} under the group morphism considered in §3.2 can be "lifted" to an object $\mathscr{E}_i \in \mathbf{H}^{\mathrm{m}}$. For $i \geq 0$, we denote by $\mathbf{H}^{\mathrm{m}}_{\leq i}$ the full idempotent-complete stable ∞-subcategory of $\mathbf{H}^{\mathrm{m}}$ generated under the action of $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})^\omega$ by the objects $\mathscr{E}_0, \cdots, \mathscr{E}_i$. Then we have
$$\mathbf{H}^{\mathrm{m}} = \mathrm{colim}_i\, \mathbf{H}^{\mathrm{m}}_{\leq i}. \tag{3.1}$$
Below we will also make extensive use of the exact sequences
$$\mathbf{H}^{\mathrm{m}}_{\leq i-1} \to \mathbf{H}^{\mathrm{m}}_{\leq i} \to \mathbf{H}^{\mathrm{m}}_{\leq i}/\mathbf{H}^{\mathrm{m}}_{\leq i-1} \tag{3.2}$$
for $i \geq 1$.

We will denote by $\overline{\mathscr{E}}_i$ the image of $\mathscr{E}_i$ in $\mathbf{H}^{\mathrm{m}}_{\leq i}/\mathbf{H}^{\mathrm{m}}_{\leq i-1}$, and set
$$A_i = (\mathbf{H}^{\mathrm{m}}_{\leq i}/\mathbf{H}^{\mathrm{m}}_{\leq i-1})_{\mathbb{G}_{\mathrm{m}}}(\overline{\mathscr{E}}_i, \overline{\mathscr{E}}_i).$$
Then A_i is an algebra object in $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$. Moreover, denoting by A the symmetric algebra of $\mathfrak{t}$, considered as an algebra object in $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$ where $\mathfrak{t}$ is placed in degree 2 and has $\mathbb{G}_{\mathrm{m},k}$-weight 1, then we have
$$\mathsf{H}^\bullet(A_i) = A. \tag{3.3}$$
In fact the proof of this claim relies on Theorem 3.2. More specifically, if we denote by $\mathbf{H}_{\leq i}$ the full idempotent-complete stable ∞-subcategory of $\mathbf{H}^{\mathrm{m}}$ generated by the objects $\mathscr{E}_0, \cdots, \mathscr{E}_i$, then for any i the equivalence of Theorem 3.2 restricts to an equivalence
$$\mathrm{Ho}(\mathbf{H}_{\leq i}) \cong \mathrm{Sh}(I^\vee\backslash \mathsf{Fl}_{G,\leq i})$$
where $\mathsf{Fl}_{G,\leq i}$ is the closed subscheme of Fl_G which is the union of the orbits Fl_{G,w_j} with $j \leq i$. As a consequence, we have
$$\mathrm{Ho}(\mathbf{H}_{\leq i}/\mathbf{H}_{\leq i-1}) \cong \mathrm{Sh}(I^\vee\backslash \mathsf{Fl}_{G,w_i}),$$
so that $\mathsf{H}^\bullet(A_i)$ identifies with the $I^\vee$-equivariant cohomology of Fl_{G,w_i}, or equivalently of the point, which is A. To identify the action of $\mathbb{G}_{\mathrm{m},k}$, we also need to know that, under this equivalence, multiplication by q on the left-hand side corresponds to pullback under Frobenius on the right-hand side, which is proved in [B3].

Remark 3.4. Using standard claims for the corresponding subcategories of the triangulated category $\mathrm{Sh}(I^\vee\backslash \mathrm{L}G^\vee/I^\vee)$, one can check that for any $i\geq 0$, the composition

$$\langle \mathscr{E}_i\rangle_{\mathbb{G}_\mathrm{m}} \to \mathbf{H}^\mathrm{m}_{\leq i} \to \mathbf{H}^\mathrm{m}_{\leq i}/\mathbf{H}^\mathrm{m}_{\leq i-1}$$

is an equivalence, where the left-hand side is the full idempotent-complete stable ∞-subcategory of $\mathbf{H}^\mathrm{m}$ generated by the objects $V\otimes\mathscr{E}_i$ with V in $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})^\omega$, and the second arrow is the quotient functor.

In the homotopy category $\mathrm{Ho}(\mathbf{H}^\mathrm{m})$, the images of the elements $(T_w : w\in W_\mathrm{aff})$ form a graded quasi-exceptional set (for the Bruhat order on W_aff), with dual the images of the elements $(T^{-1}_{w^{-1}} : w\in W_\mathrm{aff})$. (See [B1] for quasi-exceptional sets, and [B2] for graded exceptional sets. The definition of a graded quasi-exceptional set should be easy to guess from these.) In particular, for any i the Grothendieck group $\mathsf{K}^0(\mathbf{H}^\mathrm{m}_{\leq i}/\mathbf{H}^\mathrm{m}_{\leq i-1}) = \mathsf{K}^0(\mathrm{Ho}(\mathbf{H}^\mathrm{m}_{\leq i}/\mathbf{H}^\mathrm{m}_{\leq i-1}))$ is free of rank 1 over $\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}] = \mathsf{K}^{\mathbb{G}_{\mathrm{m},k}}(\mathrm{pt})$, with generator the class of the image of $\mathscr{E}_i$, and we have a short exact sequence

$$\mathsf{K}^0(\mathbf{H}^\mathrm{m}_{\leq i-1}) \hookrightarrow \mathsf{K}^0(\mathbf{H}^\mathrm{m}_{\leq i}) \twoheadrightarrow \mathsf{K}^0(\mathbf{H}^\mathrm{m}_{\leq i}/\mathbf{H}^\mathrm{m}_{\leq i-1}). \tag{3.4}$$

We also have

$$\mathsf{K}^0(\mathbf{H}^\mathrm{m}) = \mathrm{colim}_i\, \mathsf{K}^0(\mathbf{H}^\mathrm{m}_{\leq i}). \tag{3.5}$$

4. Hochschild homology of $\mathrm{Ind}(\mathbf{H}^\mathrm{m})$

In this section we explain the description of the algebra $\mathrm{HH}(\mathrm{Ind}(\mathbf{H}^\mathrm{m}))$, which is the main result of [BZCHN, §2]. Note that by results of Drinfeld–Gaitsgory, the category $\mathrm{Ind}(\mathbf{H}^\mathrm{m})$ identifies with the ∞-category of ind-coherent sheaves on $\mathcal{Z}/G\times\mathbb{G}_{\mathrm{m},k}$, see [BZCHN, §1.6.2] for details and references. Similarly, $\mathrm{Ind}(\mathbf{H})$ identifies with the ∞-category of ind-coherent sheaves on $\mathcal{Z}/G$.

4.1. **A preliminary lemma.** Let $\mathcal{C}$ be an object of $\mathrm{Pr}^\mathrm{L}_{\mathrm{St},k}$ endowed with an action of $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$. As in §1.5 we then have the enriched morphism spaces $\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,b)\in\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$ for $a,b\in\mathcal{C}$. In particular, if $a\in\mathcal{C}$ we have the algebra object $\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,a)$ in $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$.

Lemma 4.1. *If $a\in\mathcal{C}$ is compact and generates $\mathcal{C}$ under the action of $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$, then we have equivalences of ∞-categories*

$$\mathcal{C}\xrightarrow{\sim}\mathrm{RMod}_{\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,a)}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})),\quad \mathcal{C}^\omega\xrightarrow{\sim}\mathrm{Perf}_{\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,a)}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}))$$

where

- $\mathrm{RMod}_{\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,a)}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}))$ *is the ∞-category of right $\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,a)$-modules in $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$;*
- $\mathrm{Perf}_{\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,a)}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}))$ *is the ∞-subcategory generated by objects of the form $V\otimes\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,a)$ with $V\in\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})^\omega$.*

Proof. The second equivalence follows from the first one by passing to compact objects. For the first equivalence we consider the adjunction

$$\mathcal{C}\underset{V\mapsto V\otimes a}{\overset{\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,-)}{\rightleftarrows}}\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}).$$

The associated monad on $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$ is given by tensoring by $\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,a)$. The functor $\mathcal{C}_{\mathbb{G}_\mathrm{m}}(a,-)$ preserves colimits because its left adjoint preserves compact objects, and it is conservative by our assumption that a generates $\mathcal{C}$ under the action of $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$. Then the equivalence follows from the Barr–Beck–Lurie theorem, see [GR, Chap. I, Proposition 3.7.7]. □

4.2. **Construction of a basis of Hochschild homology.** Recall the algebra objects A_i ($i \geq 0$) and A introduced in §3.5.

Lemma 4.2. *For any $i \geq 0$, there exists an equivalence of $\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})$-categories*

$$\mathrm{RMod}_{A_i}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})) \cong \mathrm{RMod}_{A}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}))$$

intertwining A_n in the left-hand side with A in the right-hand side.

I don't really know how to prove this lemma formally in the setting of ∞-categories. However this is an analogue of a standard claim for dg-algebras. Namely, let E be a dgg-algebra, i.e. a $\mathbb{Z}^2$-graded algebra

$$E = \bigoplus_{n,m \in \mathbb{Z}} E^{n,m}$$

endowed with a differential d (of square 0) of bidegree $(1,0)$, which satisfies the Leibniz rule. The cohomology $\mathsf{H}(E)$ is then naturally $\mathbb{Z}^2$-graded, and we assume that we have

$$\mathsf{H}^{n,m}(E) \neq 0 \quad \Rightarrow \quad n = 2m.$$

Then E is formal; more specifically, setting

$$E' = \left(\bigoplus_{\substack{n,m \in \mathbb{Z} \\ n<2m}} E^{i,j} \right) \oplus \left(\bigoplus_{m \in \mathbb{Z}} \ker(d^{2m,m}) \right),$$

E' is a sub-dgg-algebra of E and we have natural quasi-isomorphisms of dgg-algebras

$$E \hookleftarrow E' \twoheadrightarrow \mathsf{H}(E).$$

In particular we can consider the derived category $\mathsf{DGGMod}(E)$ of dgg-E-modules, i.e. $\mathbb{Z}^2$-graded E-modules endowed with a differential satisfying the Leibniz rule. We can also forget about the second grading of E, and consider the derived category $\mathsf{DGMod}(E)$ of dg-E-modules. We also have similar categories for $\mathsf{H}(E)$. Since quasi-isomorphisms induce equivalences on derived categories of dg(g)-modules, we deduce equivalences of categories

$$\mathsf{DGGMod}(E) \cong \mathsf{DGGMod}(\mathsf{H}(E)), \quad \mathsf{DGMod}(E) \cong \mathsf{DGMod}(\mathsf{H}(E)).$$

These considerations apply here thanks to (3.3).

Corollary 4.3. *Let $i \geq 0$. We have $\mathrm{HH}^n(\mathrm{RMod}_{A_i}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}))) = 0$ for any $n \neq 0$, and an isomorphism*

$$\mathrm{HH}^0(\mathrm{RMod}_{A_n}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}))) \cong k[\mathbf{q}, \mathbf{q}^{-1}],$$

where $\mathbf{q}$ is an indeterminate.

Proof. In view of Lemma 4.2, it suffices to prove that

$$\mathrm{HH}(\mathrm{RMod}_{A}(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k}))) \cong k[\mathbf{q}, \mathbf{q}^{-1}].$$

Now the left-hand side can be computed using the complex (1.6), with the set S consisting of $\{A\}$. Here, since A/k has positive weights for the action of $\mathbb{G}_{\mathrm{m},k}$, for any m we have

$$(A^{\otimes m})^{\mathbb{G}_{\mathrm{m},k}} = 0.$$

Hence the complex is the same as for the computation of $\mathrm{HH}(\mathrm{Vect}_k)$ (with S consisting of $\{k\}$), up to tensor product with $\mathscr{O}(\mathbb{G}_{\mathrm{m},k}) = k[\mathbf{q}, \mathbf{q}^{-1}]$. The desired claim follows. □

Since $\mathrm{HH}(\mathrm{Ind}(-))$ is a localizing invariant (see Proposition 1.7), from (3.1) we deduce that

$$\mathrm{HH}(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}})) = \mathrm{colim}_i\, \mathrm{HH}(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}_{\leq i})). \tag{4.1}$$

Moreover, for any $i \geq 1$, applying this functor to the exact sequence (3.2) we obtain a cofiber sequence

$$\mathrm{HH}(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}_{\leq i-1})) \to \mathrm{HH}(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}_{\leq i})) \to \mathrm{HH}(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}_{\leq i}/\mathbf{H}^{\mathrm{m}}_{\leq i-1})).$$

Using Corollary 4.3 and induction, we deduce that for any $i \geq 0$ we have

$$\mathrm{HH}^n(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}_{\leq i})) = 0 \quad \text{for any } n \neq 0,$$

and that for $i \geq 1$ we have canonical exact sequence

$$\mathrm{HH}^0(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}_{\leq i-1})) \hookrightarrow \mathrm{HH}^0(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}_{\leq i})) \twoheadrightarrow k[\mathbf{q}, \mathbf{q}^{-1}]. \tag{4.2}$$

In view of (4.1) we also have

$$\mathrm{HH}^n(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}})) = 0 \quad \text{for any } n \neq 0.$$

4.3. **Identification with the affine Hecke algebra.** Let $\mathscr{H}_{\mathrm{aff}}$ be the Hecke algebra over k associated with W_{aff}. In other words $\mathscr{H}_{\mathrm{aff}}$ is a $k[\mathbf{q}, \mathbf{q}^{-1}]$-algebra which is free as a $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-module, with a basis $(T_w : w \in W_{\mathrm{aff}})$, and multiplication determined by the following rules:

- $(T_s - \mathbf{q}) \cdot (T_s + 1) = 0$ for $s \in S_{\mathrm{aff}}$;
- $T_y T_w = T_{yw}$ if $y, w \in W_{\mathrm{aff}}$ are such that $\ell(yw) = \ell(y) + \ell(w)$.

By a classical result of Kazhdan–Lusztig and Ginzburg, there exists a canonical isomorphism of $k[\mathbf{q}, \mathbf{q}^{-1}]$-algebras

$$\mathscr{H}_{\mathrm{aff}} \xrightarrow{\sim} k \otimes_{\mathbb{Z}} \mathsf{K}^0(\mathbf{H}^{\mathrm{m}}) \tag{4.3}$$

intertwining multiplication by $\mathbf{q}$ with the automorphism given by tensoring with the tautological $\mathbb{G}_{\mathrm{m},k}$-module.

Remark 4.4. More precisely, Kazhdan–Lusztig and Ginzburg proved this result under the assumption that G has simply-connected derived subgroup. The general case follows.

We are now in a position to explain the first main result of [BZCHN, §2].

Theorem 4.5. *We have* $\mathrm{HH}^n(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}})) = 0$ *for any* $n \neq 0$*, and an isomorphism of* k*-algebras*

$$\mathscr{H}_{\mathrm{aff}} \xrightarrow{\sim} \mathrm{HH}^0(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}})).$$

Proof. The vanishing claim was already explained at the end of §4.2. Using this fact, the constructions of §2.4 provide an algebra morphism

$$k \otimes_{\mathbb{Z}} \mathsf{K}^0(\mathbf{H}^{\mathrm{m}}) \to \mathrm{HH}^0(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}))$$

This map is compatible in the obvious way with the filtrations on both sides (see (3.4) and (3.5) for the left-hand side, and (4.1) and (4.2) for the right-hand side), and one sees using Corollary 4.3 that it induces an isomorphism on each associated subquotient (see in particular (3.4)). It is therefore an isomorphism. We conclude using (4.3). □

4.4. **A q-specialization.** We fix $q \in k \smallsetminus \{0,1\}$, and still denote by $q : \mathcal{Z}/G \to \mathcal{Z}/G$ the automorphism given by the action of $q \in \mathbb{G}_{\mathrm{m},k}(k)$. The following statement is the second main result of [BZCHN, §2].

Theorem 4.6. *We have* $\mathrm{HH}^n(\mathrm{Ind}(\mathbf{H}), q_*) = 0$ *for any* $n \neq 0$*, and an isomorphism of* k*-algebras*

$$\mathscr{H}_{\mathrm{aff}\,|\mathbf{q}=q} \xrightarrow{\sim} \mathrm{HH}^0(\mathrm{Ind}(\mathbf{H}), q_*),$$

where the left-hand side denotes the specialization of $\mathscr{H}_{\mathrm{aff}}$ *at* $\mathbf{q} = q$.

Note that the forgetful (or pullback) functor $\mathbf{H}^{\mathrm{m}} \to \mathbf{H}$ induces a morphism of pairs

$$(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}), \mathrm{id}) \to (\mathrm{Ind}(\mathbf{H}), q_*).$$

We deduce a canonical algebra morphism

$$\mathrm{HH}(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}})) \to \mathrm{HH}(\mathrm{Ind}(\mathbf{H}))$$

and, in view of Theorem 4.5, to conclude it suffices to prove that this morphism induces an isomorphism

$$\mathrm{HH}(\mathrm{Ind}(\mathbf{H}^{\mathrm{m}}))_{|\mathbf{q}=q} \xrightarrow{\sim} \mathrm{HH}(\mathrm{Ind}(\mathbf{H})).$$

For this we use the filtration of $\mathbf{H}$ similar to that of $\mathbf{H}^{\mathrm{m}}$ considered in §3.5 (see Remark 1.8). For any i we have an automorphism of A_i induced by q_*, and a compatible automorphism of A (denoted φ_q) induced by multiplication by q on $\mathfrak{t}$. What we have to prove is therefore that the obvious forgetful functor induces an isomorphism

$$\mathrm{HH}(\mathrm{RMod}_A(\mathrm{Rep}(\mathbb{G}_{\mathrm{m},k})))_{|\mathbf{q}=q} \xrightarrow{\sim} \mathrm{HH}(\mathrm{RMod}_A(\mathrm{Vect}_k)).$$

Now the left-hand side has been computed in the proof of Corollary 4.3. For the right-hand side we can use Remark 1.5, and compute the derived tensor product using a Koszul resolution. This gives the claim.

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