4) **Horizontal traces** (7.2.5). Let $\phi: \mathcal{C} \to \mathcal{C}$ an A-lin endofunctor of $\mathcal{C}$. Let $u_\phi := (id_{\mathcal{C}} \otimes \phi)(u_c) \in \mathcal{C}^{\vee, A} \underset{A}{\otimes} \mathcal{C}$.

Recall:

$$u_{c/A} : A \to \mathcal{C}^{\vee,A} \underset{A}{\otimes} \mathcal{C}$$

$$e_{c/A} : \mathcal{C} \otimes \mathcal{C}^{\vee,A} \to A$$

$$sw : \mathcal{C}^{\vee,A} \underset{A}{\otimes} \mathcal{C} \simeq \mathcal{C} \underset{A}{\otimes} \mathcal{C}^{\vee,A}.$$

trace:

(7.35) $\boxed{tr(\mathcal{C}/A, \phi) := e_{\mathcal{C}/A}(sw(u_\phi)) \in A}$.

$tr(\mathcal{C}/A, id) =: tr(\mathcal{C}/A)$.

center

(7.36) $\boxed{Z(\mathcal{C}/A, \phi) := Hom_{\mathcal{C}^\vee \underset{A}{\otimes} \mathcal{C}^\vee}(u_c, u_\phi) \in A}$.

$Z(\mathcal{C}/A, id) =: Z(\mathcal{C}/A)$

Functoriality **Rmk 7.44** Suppose $A = Mod_\Lambda$ & $i: \mathcal{C} \hookrightarrow \mathcal{D}$ fully faithful. (A-dualizable)

then there exists a natural restriction map

$$Z(\mathcal{D}/A) \to Z(\mathcal{C}/A).$$

obtained by the following way.

$$\mathcal{D}^\vee \underset{\Lambda}{\otimes} \mathcal{D} \xrightarrow{i^\vee \otimes id} \mathcal{C}^\vee \otimes \mathcal{D} \xleftarrow{id \otimes i} \mathcal{C}^\vee \otimes \mathcal{C}.$$

$$u_{\mathcal{D}} \longmapsto u_i. \longleftarrow u_{\mathcal{C}}$$

$Z(\mathcal{D}/A) = End(u_{\mathcal{D}})$ & $Z(\mathcal{C}/A) = End(u_{\mathcal{C}})$.

but $id \otimes i$ is fully faithfull so the map is invertible,

$$Z(\mathcal{D}/A) \to End(u_i) \leftrightarrows Z(\mathcal{C}/A)$$

Chern character of compact objects

Proposition 7.47: Let $F : \mathcal{C} \to \mathcal{D}$ a 1-morphism in $\mathrm{Linear}_A^{\mathrm{dual}}$ and a transformation

$$\eta : F \circ \phi_{\mathcal{C}} \Rightarrow \phi_{\mathcal{D}} \circ F.$$

where $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ & $\phi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ are A lin fund.

$F^{\circ} := (F^R)^{\vee}$ (conjugate) then there is ~~suppose~~ $\alpha_F : (F^{\circ} \otimes F) \circ \mu_{\mathcal{C}} \Rightarrow \mu_D$ $\beta_F : e_{\mathcal{C}} \Rightarrow e_D \circ (F \otimes F^{\circ})$

then we have

$$\boxed{tr(\mathcal{C}, \phi_{\mathcal{C}}) \xrightarrow{tr(F,\eta)} tr(\mathcal{D}, \phi_{\mathcal{D}}) \in A}$$

$$= tr(\mathcal{C}, \phi_{\mathcal{C}}) = e_{\mathcal{C}}((id_{\mathcal{C}^\vee} \otimes \phi_{\mathcal{C}})(\mu_{\mathcal{C}})) \xrightarrow{\beta_F} e_D((F\phi_{\mathcal{C}} \otimes F^{\circ})(\mu_{\mathcal{C}})) \xrightarrow{\eta} e_D((\phi_D F \otimes F^{\circ})(\mu_{\mathcal{C}})) \xrightarrow{\alpha_F} e_D((\phi_D \otimes id_{D^\vee})(\mu_D))$$

(7.41) assume $c \in \mathcal{C}$ is a A-compact object apply the proposition w/ "$\mathcal{C} = A$" "$\mathcal{D} = \mathcal{C}$" "$F_c : A \to \mathcal{C}$" ($1_A \longmapsto c$) "$\eta = id$".

then one has.

$$\boxed{tr(F_c, id) : 1_A = tr(A) \to tr(C).}$$

$tr(F_c, id) = ch(c)$. = class Chern of c.

Apply again proposition 7.47 but with $\eta = id$.

(7.42)

$$\boxed{\begin{tikzcd} tr(C) \arrow[rr, "tr(F, id)"] \arrow[dr, phantom, "\circlearrowright"] & & tr(D) \\ & tr(A) \arrow[ul, "ch(c)"] \arrow[ur, "ch(F(c))"'] & \end{tikzcd}}$$

2) If $c \in \mathcal{C}$ is A-admissible, F_c admits a A-lin left adjoint. $(F_c : \mathcal{C} \to A)$.

Apply prop 7.47 again:

(7.42) $tr(F_c^L, id) : tr(\mathcal{C}) \to 1_A$.

we denote this functor by Θ_c

Twisted Chern character Suppose $\phi : \mathcal{C} \to \mathcal{C}$ is A-lin. $c \in \mathcal{C}$ is A-compact w/ a morphism $\phi_c : c \to \phi(c)$. then define $ch(c, \phi_c)$ as:

$$\mathcal{C} \underset{A}{\otimes} \mathcal{C}^{\vee,A}.$$

$$\boxed{\begin{array}{l} tr(c, \phi_c) : 1_A \to Hom_{\mathcal{C}/A}(c,c) \xrightarrow{\phi_c} Hom_{\mathcal{C}/A}(c, \phi(c)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow e_{\mathcal{C}/A} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad tr(\mathcal{C}/A, \phi) \end{array}}$$

Functorality of Twisted Chern character:

- $F : \mathcal{C} \to \mathcal{D}$ in $Lincat_A^{dual}$. & $\left\{ \begin{array}{l} \eta : F \circ \phi_{\mathcal{C}} \Rightarrow \phi_{\mathcal{D}} \circ F. \\ \phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \ \& \ \phi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \text{ A-line} \end{array} \right.$

- choose $c \in \mathcal{C}^{\omega}$ compact, $\underset{=:d}{\underline{F(c)}}$ is compact and $\phi_c : c$-

$\phi_d = \eta \circ F(\phi_c) : d \to F(\phi_{\mathcal{C}}(c)) \to \phi_{\mathcal{D}}(d)$ then.

$$\boxed{tr(F, \eta)(ch(c, \phi_c)) = ch(d, \phi_d).}$$

Trace and localization sequence

Let $M, N, \mathcal{C}$ dualizable A-cat (A = commutative algebra) and $M \xrightarrow{F} \mathcal{C} \xrightarrow{G} N$ a localization seq. ↳ (Def 7.26)

$\cancel{F^\circ = (F^R)^\vee : M^\vee \to \mathcal{C}^\vee}$

$\cancel{G^\circ = (G^R)^\vee : \mathcal{C}^\vee \to N^\vee}$

1) F^R, G^R A-lin exist, & $\begin{cases} \mathrm{id}_M \to F^R F \\ G G^R \to \mathrm{id}_N \end{cases}$ are equiv.

2) $G \circ F = 0$ and the seq. $F F^R c \to c \to G^R G c$ is a fiber sequence.

if G^R admits a A-lin right adjoint we say $(F(M), G^R(N))$ form a semi-orthogonal

Proposition 7.51. Let $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ A-lin

let $\underset{M \to M}{\phi_M := F^R \circ \phi_{\mathcal{C}} \circ F}$ $\quad \underset{N \to N}{\phi_N := G \circ \phi_{\mathcal{C}} \circ G^R}$.

There is a canonical fiber sequence in A.

$$tr(M, \phi_M) \xrightarrow{tr(F,\eta)} tr(\mathcal{C}, \phi_{\mathcal{C}}) \xrightarrow{tr(G,\delta)} tr(N, \phi_N).$$

In addition if $(F(M), G^R(N))$ form a semi-orthogonal decomposition of $\mathcal{C}$ and the adjunction $\phi_{\mathcal{C}} \circ G^R \Rightarrow G^R \circ \phi_N$ is an isomorphism, then the above sequence canonically splits.

Chern character and Grothendieck Group.

Let $c \in \mathcal{C}^\omega$ then $ch(c) : 1_A \to tr(\mathcal{C})$ defines an object in $tr(\mathcal{C})$. So one ~~can~~ define a functor.

$$ch : \mathcal{C}^\omega \to H^\circ tr(\mathcal{C})$$

($tr(\mathcal{C})$ is a Λ-mod and $H^\circ tr(\mathcal{C})$ is its H° cohomology)

sition 7.57 (that factor throught the grothendieck group.

$$ch : K^\circ(\mathcal{C}^\omega) \longrightarrow H^\circ tr(\mathcal{C})$$

3) (2) The construction is functorial

$$\begin{tikzcd} K_0(\mathcal{C}^\omega) \arrow[r, "ch"] \arrow[d, "K_0(F)"'] \arrow[dr, phantom, "\circlearrowleft"] & H^0 tr(\mathcal{C}) \arrow[d, "tr(F;id)"] \\ K_0(\mathcal{D}^\omega) \arrow[r, "ch"] & H^0 tr(\mathcal{D}) \end{tikzcd}$$

(3) Suppose $\mathcal{M} \to \mathcal{C} \to \mathcal{N}$ is a localization sequence in $Lincat_\Lambda$ which in addition induces a semi-orth. decomp of $\mathcal{C}$. Then (F, G^R) induce

$$K_0(\mathcal{M}^\omega) \oplus K_0(\mathcal{N}^\omega) \cong K_0(\mathcal{C}^\omega).$$

and ch is compatible w/ this decomposition.

<u>Vertical trace</u> (7.3.1) R = symmetric mon cat.

A ~~B~~ one associative algebra in R. Let F be an (A, A)-bimod

$$\boxed{Tr(A, F) = A \underset{A \otimes A^{rev}}{\otimes} F \in R}$$

write $\boxed{[-]_F : F \to Tr(A, F)}$

<u>Relation with horizontal trace</u>

Let $Morita(R)$ = $(\infty, 2)$-cat

<u>objects</u> = Ass. algebra A/R.

<u>morphisms</u> $A \longrightarrow B$ = (B, A)-bimodul

$$\mathrm{Morita}(\mathrm{Mod}_\Lambda) \subset \mathrm{Limat}_\Lambda.$$
$$A \longmapsto \mathrm{LMod}_A.$$
$$M \longmapsto M \otimes_B (-)$$

Every object is dualizable in Morita(R).

one has $\;{}_A\mathrm{BMod}_A \cong {}_{1_R}\mathrm{BMod}_{A^{rev}\otimes A}.$
$$\cong {}_{A\otimes A^{rev}}\mathrm{BMod}_{1_R}.$$

so $A^{\vee,\mathrm{Morita}(R)} = A^{rev}$ and one has $A^{\epsilon} \in {}_{1_R}\mathrm{BMod}_{A^{rev}\otimes A}$

$$A^{u} \in {}_{A^{rev}\otimes A}\mathrm{BMod}_{1_R}.$$

then $\mathrm{Tr}(A,F)_{horiz} = \mathrm{Tr}(A, \underset{\mathrm{End}_{\mathrm{Morita}(R)}(A)}{F})_{vert}.$

<u>Example</u> 7.66 take $\begin{cases} F = M\otimes N \\ M = \text{left } A\text{-mod} \\ N = \text{right } A\text{-mod}\end{cases}$

then $\mathrm{Tr}(A,F) = A \otimes_{A\otimes A^{rev}} (M\otimes N) \cong N \otimes_A M.$

<u>Example</u> 7.67 take F a $(A\text{-}A)$-bimod and $\phi : A \to A$

${}^{\phi}F^{\phi}$ = F but w/ left action being $a \cdot^{\phi} f = \phi(a)\cdot f.$

$$\mathrm{Tr}(A,\phi) \simeq A \otimes_{A\otimes A^{rev}} {}^{\phi}A$$

4) <u>Functoriality</u> <u>of vertical traces</u>. (7.3.2).

$F_A \in {}_A\mathrm{BMod}_A$
$F_B \in {}_B\mathrm{BMod}_B$. and $M \in {}_A\mathrm{BMod}_B$ dualiz.

let $\alpha : M \otimes_B F_B \to F_A \otimes_A M$.

one has a morphism

$$\boxed{Tr(M,\alpha) : Tr(B,F_B) \to Tr(A,F_A)}$$

In particular, if $B = F_B = 1_R$, $\alpha : M \to F_A \otimes_A M$.

$$Tr(M,\alpha) : 1_R \xrightarrow{u_M} N \otimes_A M \xrightarrow{id \otimes \alpha} N \otimes_A F_A \otimes M \cong (M \otimes N) \otimes_{A \otimes A^{rev}} F_A \xrightarrow{e_M \otimes id} Tr(A,F_A).$$

$\odot = - \otimes_A -$ and $\langle - \rangle = Tr(-)$

<u>Lemma 7.79</u> let F be a A-bimod. with a left dual G, then $Tr(A,F)$ is dualizable in R with dual $Tr(A,G)$ with unit.

$$1_R \xrightarrow{\mu} T_A \otimes_{A \otimes A^{rev}} A \xrightarrow{id_{T_A} \otimes u_F} T_A \otimes_{A \otimes A^{rev}} (G \otimes_A F) \cong \langle F \odot T_A \odot G \rangle \xrightarrow{\nu} \langle F \rangle$$

and counit

$$\langle F\rangle \otimes \langle G\rangle \xleftarrow{\varepsilon} \langle G \odot S_A \odot F\rangle \cong (F \underset{A}{\otimes} G) \underset{A\otimes A^{rev}}{\otimes} S_A$$

$$\downarrow e_F \otimes id_{S_A}$$

$$A \underset{A\otimes A^{rev}}{\otimes} S_A \xrightarrow[\delta]{} 1_R.$$

S_A is the left dual of A^e

T_A is the left dual of A^u

Categorical trace (7.3.4)

R = rigid symmetric monoidal cat. R (Lincat_Λ).

A = an ~~R~~ algebra in Lincat_R.

Remark 7.83 Let $[-]_F : F \to Tr(A,F)$ the canonical ~~morphism~~ functor. one has

$$[a \otimes f]_F \cong [f \otimes a]_F \text{ in } Tr(A,F).$$

take $F = {}^{\phi}A$

then $[\phi(a) \otimes b]_\phi \cong [b \otimes a]_\phi$ so if $b = 1_A$.

$$[\phi(a)]_\phi \cong [a]_\phi.$$

so the auto-equiv $(A, {}^{\phi}A) \xrightarrow{\phi} (A, {}^{\phi}A)$ is the identity on $Tr(A,\phi)$.

5)

Proposition 7.8.4: Let $\beta : M_1 \to M_2$ be a morphism of A-B bimod. suppose:

- M_i admits a left dual N_i in A-B-bimod
- β admits a A-B-lin right adjoint βR.

Suppose there are ~~functors~~ A-B lin maps

$$\alpha_i : M_i \underset{B}{\otimes} F_B \to F_A \underset{A}{\otimes} M_i \text{ and}$$

$$\eta : (id_{F_A} \otimes \beta) \circ \alpha_1 \Rightarrow \alpha_2 \circ (\beta \otimes id_{F_B})$$

a natural transformation of functors.

Then there exists a natural transformation of functors

$$Tr(\beta, \eta) : [Tr(M_1, \alpha_1) \Rightarrow Tr(M_2, \alpha_2)]$$

$$Tr(B, F_B) \to Tr(A, F_A).$$