

6-functor formalisms

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HM stands for the paper of Heyer and Mann on 6-functor formalisms and smooth representations, LZ for the paper of Liu-Zheng, S for Scholze's lecture notes on 6-functor formalisms.

0.1 The category of correspondences

Let C be a category (i.e. ∞ -category) having finite limits, with final object $*$, and endow C with the Cartesian monoidal structure, for which it becomes a symmetric monoidal category. We start by introducing a bunch of dry definitions, but it is convenient to think of weakly stable (see below) classes as classes of morphisms f for which f^* (resp. $f_!$) are defined. More precisely in the context of Zhu's work H (for horizontal) is the class of morphisms f for which f^* is defined, and V (for vertical) is the class of morphisms for which $f_!$ is defined. So for a geometric setup (C, E) , E is the class of morphisms f for which $f_!$ is defined, also called $!$ -able morphisms. In practically all situations of Zhu's paper we will deal with geometric setups, but in the IndCoh^* theory this is not the case. In the ℓ -adic setting we will however always deal with geometric setups.

Definition 0.1. a) A (homotopy) class of morphisms E in C is called *weakly stable* if E contains all isomorphisms and is stable under compositions and arbitrary base change by morphisms in C . If E is weakly stable we let C_E be the wide subcategory of C with morphisms E .

b) A weakly stable class E is called *strongly stable* if moreover whenever $g, g \circ f \in E$ we also have $f \in E$ (in other words if $X \rightarrow S, Y \rightarrow S$ are in E , then any map $X \rightarrow Y$ over S is in E as well).

c) A *weak setup* is a triple (C, V, H) where C is a category with finite limits and V, H weakly stable classes in C .

d) A *geometric setup* is a weak setup in which V is strongly stable and H consists of all morphisms in C . We simply denote it (C, V) , and we often write E instead of V . A morphism of geometric setups $(C, E) \rightarrow (C', E')$ is a functor $F : C \rightarrow C'$ inducing a functor $C_E \rightarrow C'_{E'}$ and such that $F(X \times_S Y) \simeq F(X) \times_{F(S)} F(Y)$ whenever $X \rightarrow S \in E$.

Remark 0.2. 1. A weakly stable class E is strongly stable if and only if E is closed under relative diagonals (i.e. whenever $f : X \rightarrow Y \in E$ we also have $\Delta_f : X \rightarrow X \times_Y X \in E$), if and only if C_E has fibre products and the inclusion $C_E \subset C$ preserves fibre products. See lemma 2.1.5 in HM.

Fix now two weakly stable classes of morphisms V, H in C . A classical construction going back to Barwick associates to this a new category

$$\text{Corr} = \text{Corr}_{V;H}$$

having the same objects as C and 1-morphisms from X to Y are given by correspondences $X \leftarrow Z \rightarrow Y$, with $f : Z \rightarrow X \in H$ and $g : Z \rightarrow Y \in V$. Think of this correspondence as encoding the functor $g_! f^*$. Composition is defined as one expects, namely

$$(X_1 \leftarrow Y \rightarrow X_2) \circ (X_2 \leftarrow Z \rightarrow X_3) = X_1 \leftarrow Y \times_{X_2} Z \rightarrow X_3,$$

where the maps from the fiber product to X_1, X_3 are the projections to Y, Z composed with $Y \rightarrow X_1$ and $Z \rightarrow X_3$ (they are still in H and V).

The category Corr inherits a symmetric monoidal structure from C , given on objects by $X \otimes Y = X \times Y$ (product computed in C) and has the key property that it comes with symmetric monoidal (non-full) embeddings

$$C_H^{\text{op}} \rightarrow \text{Corr} \leftarrow C_V.$$

On objects both functors send X to X . The functor $C_H^{\text{op}} \rightarrow \text{Corr}$ sends $f : X \rightarrow Y$ to $Y \leftarrow X \rightarrow X$ ($X \rightarrow X$ being the identity of X), and the functor $\text{Corr} \rightarrow C_V$ sends $f : X \rightarrow Y$ in C_V to $X \leftarrow X \rightarrow Y$.

Here are some other key features of Corr . Call $X \in C$ H -nice if the canonical map $\pi_X : X \rightarrow *$ is in H , and so does the diagonal $\Delta_X : X \rightarrow X \times X$. Thus for a geometric setup all X 's are H -nice. If X is H -nice then π_X and Δ_X turn X into a commutative algebra object in C_H^{op} . Moreover any map $f : X \rightarrow Y$ between H -nice objects is in H and is a map $Y \rightarrow X$ of commutative algebra objects in C_H^{op} , making X an Y -module. If moreover $f \in V$ then f becomes a morphism of Y -modules in Corr (via the embedding of C_H^{op} in Corr).

Remark 0.3. 1. (HM lemma 2.2.11) Mapping (C, E) to $\text{Corr}(C, E) := \text{Corr}(C)_{\text{All}, E}$ gives a functor from the category of geometric setups to symmetric monoidal categories with lax symmetric monoidal functors.

2. Suppose that E is strongly stable and that $X \rightarrow * \in E$, then also $\Delta : X \rightarrow X \times X \in E$ and one checks (HM 2.3.9) that the correspondences $X \times X \leftarrow X \rightarrow *$ and $* \leftarrow X \rightarrow X \times X$ are the evaluation and unit exhibiting a self-duality of X in $\text{Corr}(C, E)$. In particular, if $E = \text{All}$ then $\text{Corr}(C)$ is closed symmetric monoidal, since every object is self-dual, moreover we have $\underline{\text{Hom}}(X, Y) = Y \times X$ in $\text{Corr}(C)$. There are versions for arbitrary weak setups (C, H, V) : if $\pi_X, \Delta_X \in V \cap H$, then X is dualizable in $\text{Corr}(C)_{V, H}$ (again, the unit and counit of the duality datum are given by the maps π_X and Δ_X), and more precisely it is self-dual.

0.2 Target categories

We will need to work with several categories as targets of our 3 (or 6)-functor formalisms (that Zhu calls sheaf theories).

- the category $\widehat{\text{Cat}}$ of not necessarily small categories, endowed with the Cartesian symmetric monoidal structure.
- the subcategory Pr^L of presentable categories with continuous (i.e. colimit-preserving) functors, endowed with Lurie's symmetric monoidal structure.
- the full subcategory $\text{LC} = \text{LinCat}$ of Pr^L consisting of presentable stable (or linear) categories. It is stable under the Lurie tensor product so becomes a symmetric monoidal category, and the functor $\text{Pr}^L \rightarrow \widehat{\text{Cat}}$ is lax symmetric monoidal.
- the sub-category LC^{cg} of LC consisting of compactly generated linear categories, with continuous functors that preserve compact objects. It is also stable under the Lurie tensor product, thus a symmetric monoidal category.
- the subcategory LC^{perf} of $\widehat{\text{Cat}}$ of small stable idempotent complete categories, with exact functors. The ind-completion functor

$$\text{Ind} : \text{LC}^{\text{perf}} \rightarrow \text{LC}^{\text{cg}}$$

is an equivalence, with quasi-inverse given by passage to the full subcategory of compact objects. We turn LC^{perf} into a symmetric monoidal category by declaring Ind symmetric monoidal.

Let Λ be a commutative ring. The derived category Mod_Λ of Λ -modules is a commutative algebra object in LC and we let LC_Λ be the category of Mod_Λ -modules in LC . It is a symmetric monoidal category itself, with unit Mod_Λ , and we denote by \otimes_Λ its monoidal structure. The objects of LC_Λ are called Λ -linear categories and the morphisms in LC_Λ are called Λ -linear functors. We similarly obtain $\text{LC}_\Lambda^{\text{cg}}$, etc.

0.3 Unwinding the definition of a 3-functor formalism

Definition 0.4. Let (C, V, H) be a weakly stable setup (for instance a geometric setup), with associated category of correspondences Corr . A 3-functor formalism on (C, V, H) is a lax symmetric monoidal functor

$$D : \text{Corr} \rightarrow \widehat{\text{Cat}}.$$

Such a formalism is called presentable (resp. Λ -linear, resp. Λ -linear compactly generated, etc) if it factors through a lax symmetric monoidal functor (still denoted) $D : \text{Corr} \rightarrow \text{Pr}^L$ (resp. $D : \text{Corr} \rightarrow \text{LC}_\Lambda$, resp. $D : \text{Corr} \rightarrow \text{LC}_\Lambda^{\text{cg}}$).

Let D be a 3-functor formalism on (C, V, H) . Composing with the (non-full) lax symmetric monoidal embeddings $C_H^{\text{op}} \rightarrow \text{Corr}$ and $C_V \rightarrow \text{Corr}$ yields lax symmetric monoidal functors

$$D^* : C_H^{\text{op}} \rightarrow \widehat{\text{Cat}}, \quad D_! : C_V \rightarrow \widehat{\text{Cat}}.$$

For $f : X \rightarrow Y \in C_H$ let $f^* = D^*(f) : D(Y) \rightarrow D(X)$ and for $f : X \rightarrow Y \in C_V$ let $f_! = D_!(f) : D(X) \rightarrow D(Y)$ the resulting functors, so that

$$f^* = D(Y \leftarrow X \rightarrow X), \quad f_! = D(X \leftarrow X \rightarrow Y).$$

By definition, for any $X \in \mathcal{C}$ we are given a canonical functor

$$\boxtimes : D(X) \times D(X) \rightarrow D(X \times X),$$

called external tensor product. If D is Λ -linear, it factors through a functor

$$\boxtimes_\Lambda : D(X) \otimes_\Lambda D(X) \rightarrow D(X \times X).$$

Remark 0.5. D is presentable if and only if all $D(X)$ are presentable and all $f_!, g^*$ are continuous. D is Λ -linear if and only if all $D(X)$ are Λ -linear categories and all $f_!, g^*$ are Λ -linear functors.

Recall that $X \in \mathcal{C}$ is H -nice if $\pi_X : X \rightarrow *$, $\Delta_X : X \rightarrow X \times X$ are in H , and then X is a commutative algebra object in C_H^{op} , thus also in Corr . Since D is lax symmetric monoidal, it follows that $D(X)$ is naturally a symmetric monoidal category, whose unit is denoted 1_X . Concretely, $1_X = \pi_X^* 1_*$ and

$$F \otimes G = \Delta_X^*(F \boxtimes G), \quad F, G \in D(X).$$

Now take $f : X \rightarrow Y$ a map between H -nice objects. As f is a map between commutative algebra objects in C_H^{op} , f^* is symmetric monoidal, and endows $D(X)$ with a structure of $D(Y)$ -module. If moreover $f \in V$ then f is a map of Y -modules in Corr , thus $f_!$ is a $D(Y)$ -linear functor, which concretely translates into canonical isomorphisms (called projection formula)

$$f_!(F \otimes f^*G) \simeq f_!F \otimes G.$$

Definition 0.6. A 3-functor formalism D is called a 6-functor formalism if all functors $f_!$ (with $f \in V$), f^* (with $f \in H$) admit right adjoints, and if all symmetric monoidal categories $D(X)$ (with X H -nice) are closed, i.e. have internal Hom 's.

Recall that any continuous functor between presentable categories has a right adjoint. Using this, one gets (see HM lemma 3.2.5 and Zhu 8.22 (2)):

Proposition 0.7. *A 3-functor formalism D is presentable (resp. Λ -linear) if and only if all $D(X)$ are in Pr^L (resp. LC_Λ) and all functors $f^*, g_!$ (for $f \in H, g \in V$) are continuous (resp. are Λ -linear), in which case D is automatically a 6-functor formalism.*

In particular for any Λ -linear D we can talk about the right adjoint f_* of f^* for all $f \in H$ and about the right adjoint $f^!$ of $f_!$ for all $f \in V$. Note however that these are not necessarily continuous, but they do interact well with inner Hom 's (when these exist):

Proposition 0.8. *If $f : X \rightarrow Y$ is a map between H -nice objects then there are canonical isomorphisms*

$$\underline{\text{Hom}}(F, f_*G) \simeq f_*\underline{\text{Hom}}(f^*F, G).$$

If moreover $f \in V$ there are also natural isomorphisms

$$f^!\underline{\text{Hom}}(F, G) \simeq \underline{\text{Hom}}(f^*F, f^!G), \quad f_*\underline{\text{Hom}}(F, f^!G) \simeq \underline{\text{Hom}}(f_!F, G).$$

Proof. Let's do the second one, the others are done in a similar way. For arbitrary $A \in D(X)$ we compute, using adjunctions and the projection formula

$$\begin{aligned} \mathrm{Hom}(A, f^! \underline{\mathrm{Hom}}(F, G)) &\simeq \mathrm{Hom}(f_! A, \underline{\mathrm{Hom}}(F, G)) \simeq \mathrm{Hom}(f_! A \otimes F, G) \\ &\simeq \mathrm{Hom}(f_!(A \otimes f^* F), G) \simeq \mathrm{Hom}(A \otimes f^* F, f^! G) \simeq \mathrm{Hom}(A, \underline{\mathrm{Hom}}(f^* F, f^! G)) \end{aligned}$$

and we are done using Yoneda. \square

0.4 Liu-Zheng and Mann's construction of 3-functor formalisms "for free"

Essentially all 3-functor formalisms constructed in real-life come from the following data:

- we are given a geometric setup (C, E) and a functor

$$D_0 : C^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}),$$

encoding the functors $(-)^*$ and \otimes . We write $f^* = D(f)$ for a morphism f in C .

- we are also given two **strongly** stable classes¹ of morphisms I and P such that all $f \in I \cap P$ are n -truncated for some $n \geq -2$ (depending on f) and of the form $f = pj$ with $p \in P, j \in I$ (we could also ask that any $f \in E$ is of the form $f = jp$). The critical assumptions we make are:

a) if $j : X \rightarrow Y \in I$ then j^* has a left adjoint $j_!$ compatible with base change and satisfying the projection formula.²

b) if $p : X \rightarrow Y \in P$ then p^* has a right adjoint p_* compatible with base change and satisfying the projection formula.³

c) the functors $j_!$ and p_* are compatible in cartesian diagrams, i.e. if $p : X \rightarrow Y \in P$ and $j : Y' \rightarrow Y \in I$ with corresponding base-changes $p' : X' \rightarrow Y'$ and $j' : X' \rightarrow X$, the Beck-Chevalley map $j_! p'_* \rightarrow p_* j'_!$ is an isomorphism.

In practice a) and c) are easy (and very often once a) and b) are proved part c) is automatic, for instance this is the case if all $j \in I$ are monomorphisms), and the whole work is in part b), i.e. proving proper base change. See HM prop. 3.3.3 for the proof of the following deep result.

Theorem 0.9. *(Liu-Zheng, Mann) Under the above assumptions there is a unique 3-functor formalism D on (C, E) such that $p_! = p_*$ for $p \in P$, $j_! = j_!$ for $j \in J$ and the proper base change isomorphism encoded by D is given by the Beck-Chevalley map.*

¹We don't need to ask that I and P contain all isomorphisms and are stable under equivalences, but this happens in practice.

²This means that if $f : Y' \rightarrow Y$ is any map in C , with base change $j' : X' = Y' \times_Y X \rightarrow Y'$ and projection $f' : X' \rightarrow X$, then the Beck-Chevalley map $j'_!(f')^* \rightarrow f^* j_!$ is an isomorphism of functors, and moreover that the natural map $j_!(j^* F \otimes G) \rightarrow F \otimes j_! G$ is an isomorphism.

³i.e. for any $f : Y' \rightarrow Y$ in C with base change $p' : X' = Y' \times_Y X \rightarrow Y'$ and projection $f' : X' \rightarrow X$ the Beck-Chevalley map $f^* p_* \rightarrow (p')_*(f')^*$ is an isomorphism and the natural map $F \otimes p_* G \rightarrow p_*(f^* F \otimes G)$ is an isomorphism.

0.5 An example: étale sheaves

0.5.1 Étale (complexes of) sheaves

Take C the category of qcqs schemes, and E the class of morphisms $f : X \rightarrow Y$ of finite expansion in C , which concretely means that f factors $X \rightarrow X' \rightarrow Y$ with $X \rightarrow X'$ integral and $X' \rightarrow Y$ separated finitely presented, in particular E contains all separated finitely generated maps, but also perfection of such maps, or pro-(finite étale) maps of qcqs schemes. We take for I the class of open immersions and for P the class of "proper" maps in E , i.e. those which are universally closed. For D_0 we fix a *torsion* ring Λ and we let $D_0(X) = D(X_{\text{et}}, \Lambda)$ be the derived ∞ -category of sheaves of Λ -modules on X_{et} , which identifies with the category of $\text{Mod}_\Lambda = D(\Lambda)$ -valued hypersheaves on X_{et} .

Theorem 0.10. *The assumptions in the LZM machinery (theorem 0.9) are satisfied, and so there is a 3-functor formalism*

$$D : \text{Corr}(\text{Sch}^{\text{qcqs}}, E) \rightarrow \text{LC}_\Lambda$$

such that $j_!$ is the left adjoint of j^* for $j \in I$ and $p_!$ is the right adjoint p_* of p^* for $p \in P$.

Proof. This is Proposition 7.16 in Scholze's appendix to lecture VII. After proving an analogue of Nagata compactification in this setting (due to Hamacher), the tricky part is showing that for any $p : X \rightarrow Y \in P$ the functor p_* is continuous, satisfies the projection formula and is compatible with arbitrary pullbacks by morphisms of schemes. It turns out that p is automatically the composition of an integral map and a finitely presented proper map. By limit arguments we are reduced to the case when p is proper finitely presented. In this case writing Y as cofiltered inverse limit with affine transitions of schemes of finite type over \mathbf{Z} the map p is the base-change of a proper map between schemes of finite type over \mathbf{Z} . In this case p_* has finite cohomological dimension, so we can reduce to complexes in D^+ , and then to sheaves of Λ -modules, i.e. to the classical setting in SGA. \square

Remark 0.11. Instead of taking a torsion ring Λ one could work as in Scholze's lecture with $D_{\text{pf}}(\mathbf{Z})$ -valued sheaves on X_{et} , where $D_{\text{pf}}(\mathbf{Z})$ is the full sub-category of $D(\mathbf{Z})$ consisting of profinitely complete complexes of abelian groups, i.e. for which $A \simeq \varprojlim_n A/Ln$. We could also work with $D_{\text{pf}}(\mathbf{Z})$ -valued hypersheaves on X_{et} , resulting in a different functor formalism. Yet another option is to work with the left-completion of the category of $D_{\text{pf}}(\mathbf{Z})$ -valued sheaves (which is the same as the left-completion of the category of $D_{\text{pf}}(\mathbf{Z})$ -valued hypersheaves), resulting in yet another formalism. There are natural maps between these formalisms, and for schemes of finite type over \mathbf{Z} or over algebraically closed fields there is no difference between sheaves and hypersheaves, and the category is already left-complete for the canonical t -structure.

0.6 Extending 3-functor formalisms

Let D be a 3-functor formalism on a geometric setup (C, E) .

Definition 0.12. Let $f : X \rightarrow Y$ be a morphism in C , with Čech nerve X_\bullet , and maps $f_n : X_n \rightarrow Y$.

- f is a D^* -cover if the functor $(f_n^*)_n : D(Y) \rightarrow \varprojlim_{n \in \Delta} D(X_n)$ is an equivalence.
- Suppose that D is a 6-functor formalism and $f \in E$. We say that f is a $D^!$ -cover if the functor $(f_n^!)_n : D(Y) \rightarrow \varprojlim_{n \in \Delta} D(X_n)$ is an equivalence.

We say that f is a universal $D^?$ -cover if any base change of f along a map $Y' \rightarrow Y$ in C is a $D^?$ -cover.

Proposition 0.13. (HM 3.4.8) Let (C, E) and (C, E') be geometric setups with $E \subset E'$ and let D be a 3-functor formalism on (C, E) . Under any of the following two assumptions D extends uniquely to a 3-functor formalism D' on (C, E') (which is presentable in the second case):

- any $f \in E'$ has a pullback by a universal D^* -cover that belongs to E .
- D is presentable and for any $f \in E'$ there is a universal $D^!$ -cover g in E such that $f \circ g \in E$.

Caution: the first condition is actually subtle, the most naive candidate for E' would be that of maps which belong to E after pullback by a suitable universal D^* -cover, but this does not seem to form a strongly stable class (I don't see any reason for stability by composition).

How to define $f_!$ for $f : X \rightarrow Y \in E'$? In the first case take a universal D^* -cover $g : Y' \rightarrow Y$ such that the pullback $f' : X' \rightarrow Y'$ is in E . Let $g_\bullet : Y'_\bullet \rightarrow Y$ and $g'_\bullet : X'_\bullet \rightarrow X'$ be the Čech nerves of g and of its base-change g' . The functor $f_!$ is defined by

$$D(X) \simeq \varprojlim_{n \in \Delta} D(X'_n) \rightarrow \varprojlim_{n \in \Delta} D(Y'_n) \simeq D(Y),$$

where the first map is induced by the $(g'_n)^*$'s (note that g' is a D^* -cover), the second by $!$ -pushforward along $f'_n : X'_n \rightarrow Y'_n$ (which is in E by assumption) and the last is the inverse of the isomorphism induced by the g_n^* 's. By proper base change the $(f'_n)_!$ behave well with respect to transition maps, so this makes sense (it is of course not obvious how to make all this homotopy coherent, compatible with compositions etc).

In the second case take $f : X \rightarrow Y \in E$ and let $g : X' \rightarrow X$ be a universal $D^!$ -cover such that $f \circ g \in E$. Let $g_\bullet : X'_\bullet \rightarrow X$ be the Čech nerve of g . Using the equivalence $(g_n^!)_n : D(X) \simeq \varprojlim_{n \in \Delta} D(X'_n)$ and adjunctions one deduces that for any $F \in D(X)$ the natural map

$$F \rightarrow \varinjlim_{n \in \Delta^{\text{op}}} g_n, !g_n^! F$$

is an equivalence (the colimit exists since $D(X)$ is presentable). One then sets

$$f_! F := \varinjlim_{n \in \Delta^{\text{op}}} (f \circ g_n)_! g_n^! F.$$

For the next technical but crucial result the context is as follows. Let τ be a **sub-canonical** Grothendieck topology on C and let D be a 3-functor formalism on (C, E)

which is sheafy, i.e. the functor $D^* : C^{\text{op}} \rightarrow \widehat{\text{Cat}}$ satisfies τ -descent. Let $\text{Sh}(C)$ be the category of τ -sheaves of anima on C and let E_r be the class of maps $f : X \rightarrow Y$ in $\text{Sh}(C)$ which are representable in E , i.e. for any map $Z \rightarrow Y$ in $\text{Sh}(C)$ with $Z \in C$ we have $X \times_Y Z \rightarrow Z \in E$. See HM 3.4.2 for the proof.

Proposition 0.14. *The obvious map $(C, E) \rightarrow (\text{Sh}(C), E_r)$ is a morphism of geometric setups and D extends uniquely to a sheafy 3-functor formalism D' on $(\text{Sh}(C), E_r)$, via right Kan extension along $\text{Corr}(C, E) \rightarrow \text{Corr}(\text{Sh}(C), E_r)$. Moreover, the functor $(D')^* : \text{Sh}(C)^{\text{op}} \rightarrow \widehat{\text{Cat}}$ is obtained by right Kan extension along $C^{\text{op}} \rightarrow \text{Sh}(C)^{\text{op}}$, and if D is presentable (resp. Λ -linear) so is D' .*

Note that we can always choose the trivial topology on C , then $\text{Sh}(C)$ is the category of prestaks (i.e. (accessible) Ani-valued presheaves) on C .

Again, intuitively this is clear and making this precise is a pain. Namely for any $X \in \text{Sh}(C)$ one defines (and is forced to do so)

$$D'(X) = \varprojlim_{Z \in C^{\text{op}}/X} D(Z),$$

transitions being induced by pullbacks and the $f_!$ (for $f \in E_r$) are defined "component-wise".

0.7 Left and right adjoints in 2-categories

See the appendix D to HM for proofs of the results stated here. A 2-category (i.e. $(\infty, 2)$ -category) is by definition a Cat -enriched category (this makes sense since Cat is symmetric monoidal for its Cartesian monoidal structure). Fix a 2-category K , for any objects $X, Y \in K$ we thus have a category $\text{Fun}_K(X, Y)$, whose objects are called morphisms from X to Y (these are 1-morphisms) and whose morphisms are called 2-morphisms. Say a morphism $f : X \rightarrow Y$ is a left adjoint if there is a morphism $g : Y \rightarrow X$ (called a right adjoint of f) and 2-morphisms $\eta : \text{id}_X \rightarrow gf$, $\varepsilon : fg \rightarrow \text{id}_Y$ such that the composite $f \rightarrow ffg \rightarrow f$ (first map induced by η , second one by ε) is id in $\text{Fun}_K(X, Y)$ and similarly $g \rightarrow gfg \rightarrow g$ is id in $\text{Fun}_K(Y, X)$. For instance, a functor $F : C \rightarrow D$ of categories is a left adjoint (in usual sense) if and only if it's a left adjoint seen as 1-morphism in the 2-category of categories.

Remark 0.15. • If f has a right adjoint g , then g is essentially unique: for any two right adjoints g_1, g_2 of f the map $g_1 \rightarrow g_2fg_1 \rightarrow g_2$ is an isomorphism (HM D.2.6). So we will speak about *the* right adjoint of f .

• Any 2-functor between 2-categories sends left adjoint morphisms to left adjoint morphisms.

Let $f : X \rightarrow Y$ be a morphism in K . For any object Z of K we get an induced functor $f_{*,Z} : \text{Fun}_K(Z, X) \rightarrow \text{Fun}_K(Z, Y)$ and for any morphism $h : Z \rightarrow T$ we get functors $h_X^* : \text{Fun}_K(T, X) \rightarrow \text{Fun}_K(Z, X)$ and $h_Y^* : \text{Fun}_K(T, Y) \rightarrow \text{Fun}_K(Z, Y)$. One then has the very practical criterion:

Proposition 0.16. (HM D.2.8) $f : X \rightarrow Y$ is a left adjoint if and only if for any object Z of K the functor $f_{*,Z}$ has a right adjoint G_Z and for any morphism $h : Z \rightarrow T$ the natural transformation $h_X^* G_T \rightarrow G_Z h_Y^*$ is an isomorphism, and then the right adjoint of f is $G_Y(\text{id}_Y) : Y \rightarrow X$.

Moreover, it suffices to check the existence of G_Z only for $Z = X, Y$, and the fact that the natural map $G_Y(\text{id}_Y) \circ f \rightarrow G_X(f)$ is an isomorphism (in $\text{Fun}_K(X, X)$) after applying $\text{Hom}(\text{id}_Y, -)$.

Let I be a category. We say that K is compatible with I -limits (resp. I -colimits) if for all objects X, Y the category $\text{Fun}_K(X, Y)$ has all limits indexed by I (resp. all colimits indexed by I) and for any morphism $h : Z \rightarrow T$ and any object X the functor $h_X^* : \text{Fun}_K(T, X) \rightarrow \text{Fun}_K(Z, X)$ preserves such limits (resp. colimits).

Proposition 0.17. (HM D.4.13, D.4.9) a) If K is compatible with I -colimits and I^{op} -limits, then the full subcategory $\text{Fun}_L(X, Y) \subset \text{Fun}(X, Y)$ spanned by morphisms that are left adjoints has all I -colimits and the inclusion preserves them.

b) If K is compatible with I -limits and $Y = \varprojlim_{i \in I} Y_i$ is a limit in C , then a morphism $f : X \rightarrow Y$ is a left adjoint as soon as all $X \rightarrow Y_i$ are left adjoints.

0.8 Suave duality and Verdier duality

Let D be a 3-functor formalism on a geometric setup (C, E) . It turns out that one can associate to each object S of C a 2-category K_S of kernels, with objects the maps $f : X \rightarrow S$ in E and such that

$$\text{Fun}_{K_S}(X, Y) = D(Y \times_S X).$$

The composition of $A : X \rightarrow Y$ and $B : Y \rightarrow Z$ (thus $A \in D(Y \times_S X)$ and $B \in D(Z \times_S Y)$) is

$$B \circ A = p_{13,!}(p_{23}^* A \otimes p_{12}^* B) \in D(Z \times_S X),$$

the p_{ij} being the projections from $Z \times_S Y \times_S X$. In particular $\text{Fun}_{K_S}(X, S)$ and $\text{Fun}_{K_S}(S, X)$ both identify with $D(X)$, i.e. we can see any $A \in D(X)$ as a morphism $X \rightarrow S$ and as a morphism $S \rightarrow X$ in K_S .

Remark 0.18. 1. (HM 4.1.5) The 2-category K_S comes equipped with canonical 2-functors, whose composition is D

$$\text{Corr}((C_E)/S, \text{All}) \rightarrow K_S \rightarrow \text{Cat}_2,$$

the first functor sending X/S to X/S (on objects) and a correspondence $X \leftarrow Z \rightarrow Y$ to $u_!(1) \in D(Y \times_S X)$, where $u : Z \rightarrow Y \times_S X$ is the induced map. The second functor sends X/S to $D(X)$ and sends $A \in \text{Fun}_S(X, Y) = D(Y \times_S X)$ to $p_{1,!}(A \otimes p_2^*(-)) : D(X) \rightarrow D(Y)$, where p_i are the projections from $Y \times_S X$. This explains the name category of kernels: any morphism $A \in \text{Fun}_S(X, Y)$ gives rise to a kernel functor.

2. (HM 4.2.4) Any morphism $f : T \rightarrow S$ induces a 2-functor $f^* : K_S \rightarrow K_T$ sending X/S to X_T/T , where $X_T = X \times_S T$, and which on functor categories $D(Y \times_S X) \rightarrow D(Y_T \times_T X_T)$ is the obvious $*$ -pullback functor. Similarly, any $g : T \rightarrow S$ in E induces a 2-functor $g_! : K_T \rightarrow K_S$ which on objects is the forgetful functor and on functor categories is given by $u_! : D(Y \times_T X) \rightarrow D(Y \times_S X)$, where $u : Y \times_T X \rightarrow Y \times_S X$ is the obvious map.
3. (HM 4.3.1) If $X' \rightarrow X$ is a universal D^* -cover with Cech nerve X'_\bullet , the natural 2-functor $K_X \rightarrow \varprojlim_{n \in \Delta} K_{X'_n}$ is fully faithful.

Definition 0.19. Let $f : X \rightarrow S$ be a map in E . We say that $A \in D(X)$ is f -suave if seen as a morphism $X \rightarrow S$ in K_S , A is a left adjoint. We call $D_f(A) \in D(X) = \text{Fun}_{K_S}(S, X)$ the corresponding right adjoint. We say that f is suave (or D -suave) if the monoidal unit $1_X \in D(X)$ is f -suave, and we say that f is smooth (or D -smooth) if f is suave and $\omega_f := D_f(1_X)$ is invertible in $D(X)$.

The next result resumes the key stability properties of this notion:

Theorem 0.20. *Let D be a 3-functor formalism on (C, E) , $f : X \rightarrow S \in E$ and $A \in D(X)$.*

1. (*Verdier duality*) *If A is f -suave then so is $D_f(A)$ and the natural map $A \rightarrow D_f(D_f(A))$ is an isomorphism.*
2. (*$*$ -locality on the target*) *Let $g : S' \rightarrow S$ be a map in C with base changes $g' : X' = X \times_S S' \rightarrow X$ and $f' : X' \rightarrow S'$. If A is f -suave then $(g')^*A$ is f' -suave. The converse holds if g is a universal D^* -cover.*
3. (*limits/colimits stability*) *Let I be a category such that all $D(X)$ have colimits indexed by I and limits indexed by I^{op} , and these are preserved by \otimes separately in each variable, and by the f^* and $f^!$ for $f \in E$. Then f -suave objects are stable under I -colimits and I^{op} -limits. In particular, for a stable formalism f -suave objects are stable under taking fibers, cofibers and retracts.*

Proof. 1) is formal, see HM 4.4.4. The key point is that $D(X \times_S Y) \simeq D(Y \times_S X)$ induces an identification $K_S \simeq K_S^{\text{op}}$.

For 2), stability by pullback follows since the 2-functor $g^* : K_S \rightarrow K_{S'}$ sends A to $(g')^*A$ and preserves adjunctions, as does any 2-functor. For the second part, note that $K_S \rightarrow \varprojlim_{n \in \Delta} K_{X'_n}$ is a fully faithful 2-functor (where X'_\bullet is the Cech nerve of g), since g is of universal D^* -descent (remark 0.18). By assumption the image of $A : X \rightarrow S$ in any $K_{X'_n}$ is a left adjoint, and then it follows formally that A itself is a left adjoint.

Part 3) is a simple consequence of propositions 0.17 and 0.16. \square

The next result is incredibly powerful and underlies most non formal properties of suave maps:

Theorem 0.21. Consider a 3-functor formalism D on a geometric setup (C, E) and let $h : X \rightarrow S, g : Y \rightarrow S$ be in E and let $f : X \rightarrow Y$ be a map over S . If $A \in D(Y)$ is g -suave and $B \in D(X)$ is f -suave then $f^*A \otimes B$ is h -suave and

$$D_h(f^*A \otimes B) \simeq f^*D_g(A) \otimes D_f(B).$$

Proof. There is a 2-functor $K_Y \rightarrow K_S$ sending $B \in D(X) = \text{Fun}_{K_Y}(X, Y)$ to $u_!B \in D(Y \times_S X) = \text{Fun}_{K_S}(X, Y)$, where $u : X \rightarrow Y \times_S X$ is (f, id) . Thus $u_!B$ is a left adjoint, with right adjoint $u^!D_f(B)$, and so $A \circ u_!B$ is also a right adjoint, with the obvious right adjoint. Next one computes in K_S

$$A \circ u_!B = f^*A \otimes B.$$

□

Corollary 0.22. Let D be a 3-functor formalism on (C, E) .

1. *Suave maps are stable under composition and base change. If $f : X \rightarrow Y$ is a morphism over S (with $X \rightarrow S, Y \rightarrow S \in E$) and $X \rightarrow S$ is suave and $Y \rightarrow S$ has suave diagonal, then f is suave.*
2. *Let $f : X \rightarrow Y$ be a morphism over S such that $g : Y \rightarrow S \in E$ and $h : X \rightarrow S \in E$.*
 - *If g is suave then all f -suave objects A are h -suave and there are natural isomorphisms $D_h(A) \simeq D_f(A) \otimes f^*\omega_g$.*
 - *if f is suave then any dualizable object $A \in D(X)$ is f -suave with $D_f(A) \simeq A^\vee \otimes \omega_f$.*
 - *if Δ_f is suave, then all h -suave objects A are f -suave and there are natural isomorphisms $D_f(A) \simeq D_h(A) \otimes \omega_{(f, \text{id}_Y)}$, where*
 - *if Δ_f is suave then any f -suave object A is dualizable in $D(X)$ and $A^\vee \simeq D_f(A) \otimes \omega_{\Delta_f}$, and if h is suave then so is g .*
3. *(*-locality for suave pullback) Suppose that f is suave and let $A \in D(Y)$. If A is g -suave then f^*A is h -suave and $D_h(f^*A) \simeq f^*D_g(A) \otimes \omega_f$. Conversely, if f^*A is h -suave, f is a universal D^* -cover and D is compatible with colimits then A is g -suave and $D_g(A) \simeq f_!D_h(f^*A)$.*

Proof. These are all immediate consequences of the above two theorems. □

Let $f : X \rightarrow S \in E$ and $A \in D(X)$, seen as a morphism $A : X \rightarrow S$ in K_S . Then the corresponding functor $A_* : \text{Fun}_{K_S}(Z, X) = D(X \times_S Z) \rightarrow \text{Fun}_{K_S}(Z, S) = D(Z)$ is simply $p_{2,!}(p_1^*A \otimes (-))$, where p_i are the projections from $X \times_S Z$. If D is a 6-functor formalism, a simple computation shows that this has a right adjoint $G_Z = \underline{\text{Hom}}(p_1^*A, p_2^!(-))$ and we have $G_S(\text{id}_S) = \underline{\text{Hom}}(A, f^!1)$. Since $\text{Hom}(\text{id}_X, -) = \text{Hom}(1, \Delta^!(-))$ (where $\Delta : X \rightarrow X \times_S X$ is the diagonal of f), proposition 0.16 yields:

Theorem 0.23. Let D be a 6-functor formalism and let $f : X \rightarrow S \in E$ and $A \in D(X)$.

- if A is f -suave there is a canonical isomorphism

$$D_f(A) \simeq \underline{\mathrm{Hom}}(A, f^!1),$$

inducing isomorphisms of functors

$$D_f(A) \otimes f^*(-) \simeq \underline{\mathrm{Hom}}(A, f^!(-)), \quad D_f(A) \boxtimes (-) \simeq \underline{\mathrm{Hom}}(p_1^*A, p_2^!(-))$$

for any $Z \rightarrow S \in E$, with projections $p_1 : X \times_S Z \rightarrow X, p_2 : X \times_S Z \rightarrow Z$.

- Conversely, if for $Z = X$ the natural map

$$\underline{\mathrm{Hom}}(A, f^!1) \boxtimes A \rightarrow \underline{\mathrm{Hom}}(p_1^*A, p_2^!A)$$

is an isomorphism (even after applying $\mathrm{Hom}(1, \Delta^!(-))$), then A is f -suave.

Corollary 0.24. Let D be a 6-functor formalism on (C, E) and let $f : X \rightarrow S \in E$.

1. If f is suave then there is a natural isomorphism $\omega_f \simeq f^!1$ inducing natural isomorphisms

$$\omega_f \otimes f^* \simeq f^!, \quad f^* \simeq \underline{\mathrm{Hom}}(\omega_f, f^!).$$

Moreover the formation of $f^!$ commutes with arbitrary $*$ -pullback and for any map $g : S' \rightarrow S$ in C with base changes $g' : X' = X \times_S S' \rightarrow X$ and $f' : X' \rightarrow S'$ there are natural Beck-Chevalley isomorphisms

$$f^*g_* \simeq g'_*(f')^*, \quad (g')^*f^! \simeq (f')^!g^*,$$

If moreover $g \in E$ then there are natural Beck-Chevalley isomorphisms

$$f^!g^! \simeq f'_!(g')^!, \quad (f')^*g^! \simeq (g')^!f^*.$$

2. Conversely, if the formation of $f^!1$ commutes with base change along f , i.e. the natural map $p_1^*f^!1 \rightarrow p_2^!1$ is an isomorphism (where p_1, p_2 are the projections from $X \times_S X$), then f is suave.

Proof. Most of these results follow by taking $A = 1$ in the above theorem. To see that $f^*g_* \simeq g'_*(f')^*$, use that $f^* \simeq \underline{\mathrm{Hom}}(\omega_f, f^!)$ and similarly with f' , combined with prop. 0.8. □

Theorem 0.25. Let $f : X \rightarrow S \in E$ be suave, where D is a 6-functor formalism. If $D(X)$ has all countable limits and colimits and if f^* is conservative, then f is a universal D^* and $D^!$ -cover.

Proof. The key is Lurie's Beck-Chevalley conditions for descent, combined with suave base-change (part 2 of the above corollary). This reduces us to checking only the following things: $f^!$ is conservative (OK since f^* is so and $f^* = \underline{\mathrm{Hom}}(\omega_f, f^!)$) and $f^!$ preserves colimits over $f^!$ -split simplicial objects in $D(X)$, which is the case simply because $f^! \simeq \omega_f \otimes f^*$ preserves all small colimits. This proves $D^!$ -descent, hence also universal $!$ -descent as the hypotheses are preserved under base-change. For D^* -descent one needs to check that f^* preserves f^* -split totalizations of cosimplicial objects in $D(X)$, which holds since $f^* \simeq \underline{\mathrm{Hom}}(\omega_f, f^!)$. □

1 Ind-constructible sheaves and cosheaves, placidity

Setup: k is an algebraically closed field in which a prime ℓ is invertible, Λ is a \mathbf{Z}_ℓ -algebra of coefficients, by which we mean either a finite (as a set!) \mathbf{Z}_ℓ -algebra, a localization of a noetherian I -adically complete ring A with A/I finite (for some ideal I of A), or more generally a filtered colimit of such localizations.

1.1 Spaces vs pfp spaces

Let Sp be the category of perfect algebraic spaces over k and call a morphism $f : X \rightarrow Y$ pfp (perfectly finitely presented) if f is the perfection of a finitely presented morphism $f' : X' \rightarrow Y'$ of (usual) algebraic spaces over k . If f' can be chosen proper we call f pfp proper. Let $* = \mathrm{Spec}(k)$ and for $X \in \mathrm{Sp}$ let $\pi_X : X \rightarrow *$ be the structure map. We say X is pfp if π_X is so and we let $\mathrm{Sp}^{\mathrm{pfp}}$ be the full subcategory of Sp consisting of pfp spaces.

Definition 1.1. A *presentation* of a morphism $f : X \rightarrow Y$ in Sp is an isomorphism $X \simeq \varprojlim_{i \in I} X_i$ in Sp/Y $X \simeq \varprojlim_{i \in I} X_i$, where $(f_i : X_i \rightarrow Y)_i$ is a cofiltered inverse system of pfp maps with affine transition maps $X_i \rightarrow X_j$. A presentation of X is by definition a presentation of π_X .

We will use the following application of standard "noetherian approximation" results:

Proposition 1.2. (Zhu 10.5) *a) Any morphism in Sp has a presentation.*

b) If $X \rightarrow Y$ is a pfp morphism in Sp and $Y = \varprojlim_i Y_i$ is a presentation of Y , then f is the base-change of some pfp morphism $f_i : X_i \rightarrow Y_i$ along the projection $Y \rightarrow Y_i$ for some i . If f is separated, étale, pfp proper then we can choose f_i of the same form.

It easily follows from this that pfp morphisms and separated pfp proper morphisms are strongly stable classes in Sp .

1.2 Constructible sheaves

1.2.1 As a plain functor

We first define a lax symmetric monoidal functor

$$\mathrm{Sh}_c^*(-, \Lambda) : \mathrm{Sp}^{\mathrm{op}} \rightarrow \mathrm{LC}_\Lambda^{\mathrm{Perf}}, X \mapsto \mathrm{Sh}_c^*(X, \Lambda), f : X \mapsto Y \mapsto f^* : \mathrm{Sh}_c^*(Y, \Lambda) \rightarrow \mathrm{Sh}_c^*(X, \Lambda)$$

as the left Kan extension of a similarly defined (see below) functor on $\mathrm{Sp}^{\mathrm{pfp}}$, in other words for any presentation $X = \varprojlim_i X_i$ of X we have

$$\mathrm{Sh}_c^*(X, \Lambda) = \varinjlim_i \mathrm{Sh}_c^*(X_i, \Lambda).$$

This is independent of the presentation, as it is also the colimit of the $\mathrm{Sh}_c^*(S, \Lambda)$ over all maps $X \rightarrow S$ with $S \in \mathrm{Sp}^{\mathrm{pfp}}$.

So it remains to explain what is $\mathrm{Sh}_c^*(X, \Lambda)$ when X is pfp (i.e. π_X is pfp).

- If Λ is finite, then mapping X to $D(X_{\mathrm{et}}, \Lambda)$ (the derived ∞ -category of the abelian category of étale Λ -sheaves on X) gives a lax symmetric monoidal functor

$$\mathrm{Sp}^{\mathrm{op}} \rightarrow \mathrm{LC}_\Lambda, X \mapsto D(X_{\mathrm{et}}, \Lambda), f \mapsto f^*,$$

each $D(X_{\mathrm{et}}, \Lambda)$ being closed symmetric monoidal. In this case we let $\mathrm{Sh}_c^*(X, \Lambda)$ be the smallest Λ -linear idempotent complete stable full subcategory of $D(X_{\mathrm{et}}, \Lambda)$ containing all $j_! \Lambda$ with $j : U \rightarrow X$ étale map in Sp (where $j_!$ is the left adjoint of j^*). A key property of this functor is that it is *finitary*, i.e. turns cofiltered limits with affine transition maps into colimits (this is a standard argument, see Zhu 10.7).

- If A is a noetherian I -adic ring set

$$\mathrm{Sh}_c^*(X, A) = \varprojlim_n \mathrm{Sh}_c^*(X, A/I^n).$$

If $\Lambda = S^{-1}A$ with A as above and S a multiplicative subset define

$$\mathrm{Sh}_c^*(X, \Lambda) = \mathrm{Sh}_c^*(X, A) \otimes_{\mathrm{Perf}_A} \mathrm{Perf}_\Lambda,$$

relative tensor product in $\mathrm{LC}_\Lambda^{\mathrm{Perf}}$ (in other words formally localize with respect to S , then take idempotent-completion).

- Finally if $\Lambda = \varinjlim \Lambda_i$ is a filtered colimit of such localizations set

$$\mathrm{Sh}_c^*(X, \Lambda) = \varinjlim_i \mathrm{Sh}_c^*(X, \Lambda_i)$$

with transition maps given by extension of scalars.

Since Λ is fixed, we shorten notation and set

$$\mathrm{Sh}_c^* = \mathrm{Sh}_c^*(-, \Lambda).$$

The next result is proved in various degrees of generality in the work of Gabber, Bhatt-Mathew, Hansen-Scholze:

Theorem 1.3. *Sh_c^* is a finitary v -sheaf of ∞ -categories.*

1.2.2 As a 3-functor formalism

Theorem 1.4. *There is a 3-functor formalism*

$$\text{Corr}(\text{Sp}, \text{pfp}) \rightarrow \text{LC}_\Lambda^{\text{perf}}$$

such that for all j étale $j_!$ is the left adjoint of $j^!$ and for all morphisms p that are pfp proper $p_!$ is the right adjoint of p^* . Restricted to the geometric setup $(\text{Sp}^{\text{pfp}}, \text{all})$ this is a 6-functor formalism, i.e. f^* has a right adjoint f_* and $f_!$ has a right adjoint $f^!$ for any map f between pfp spaces.

Proof. Take for E the strongly stable class of separated pfp morphisms in Sp , for I the class of étale morphisms and for P that of separated pfp proper morphisms. Using Nagata compactification (in particular the version for algebraic spaces) and passing to suitable limits in the classical proper base-change theorem (Zhu 10.14) shows that the Liu-Zheng axioms are satisfied, so we get a 3-functor formalism Sh_c^* on (Sp, E) . One easily checks that finite qcqs Zariski covers are of universal D^* -descent. Since for any pfp map $f : X \rightarrow Y$ there is a finite qcqs Zariski cover $g : X' \rightarrow X$ such that $X' \rightarrow Y \in E$, it follows that we can extend Sh_c^* to (Sp, pfp) . The second part follows similarly using noetherian approximation to reduce to the standard finiteness theorems in SGA.

As an example of arguments based on limits, let us prove that for $p : X \rightarrow Y \in P$ a pfp proper morphism the functor p^* has a right adjoint p_* whose formation commutes with base-change by $g : Y' \rightarrow Y$ and satisfies projection formula. By using dévissage in Λ we reduce to the case when Λ is finite. If Y is pfp, the existence of a right adjoint p_* commuting with base-change by g with Y' pfp is a consequence of the proper base-change theorem in SGA (and passage to perfections). Keep Y pfp, and no longer suppose that Y' is pfp, and let us check that $g^*p_*F \simeq (p')_*(g')^*F$ still holds. Take a presentation $Y' = \varprojlim_i Y_i$ of Y' and let $X_i = Y_i \times_Y X$, with induced maps $p_i : X_i \rightarrow Y_i$ and $g_i : Y_i \rightarrow Y$. Let $a_i : Y' \rightarrow Y_i, b_i : X' \rightarrow X_i$ be the natural projections, so that $g' = g'_i \circ b_i$ and $g = g_i \circ a_i$, where $g'_i : X_i \rightarrow X$ is the base-change of g_i along P . Take any $A \in \text{Sh}_c^*(Y')$, WLOG $A = a_0^*A_0$ with $A_0 \in \text{Sh}_c^*(Y_0)$. Then letting A_i be the pullback of A_0 to Y_i we have

$$\text{Hom}(A, g^*p_*F) = \text{Hom}(a_0^*A_0, a_0^*g_0^*p_*F) = \varinjlim_i \text{Hom}(A_i, g_i^*p_*F).$$

Applying the previous case, we can identify this to

$$\varinjlim_i \text{Hom}(A_i, p_{i,*}(g'_i)^*F) = \varinjlim_i \text{Hom}(p_i^*A_i, (g'_i)^*F).$$

A similar computation shows that this identifies in turn with $\text{Hom}((p')^*A, (g')^*F) = \text{Hom}(A, p'_*(g')^*F)$ and we conclude using Yoneda.

Thus we know the result when Y is pfp. In general take a presentation $Y = \varprojlim Y_i$ of Y and use that p is the base-change of some pfp proper $p_i : X_i \rightarrow Y_i$, which allows one to reduce the general case to the one previously treated. \square

1.3 Ind-constructible sheaves

Define a 3-functor formalism

$$\mathrm{Sh}^* : \mathrm{Corr}(\mathrm{Sp}, \mathrm{pfp}) \rightarrow \mathrm{LC}_\Lambda$$

by composing Sh_c^* with the ind-completion functor. The next result is not obvious, in particular the sheaf property uses that k is algebraically closed and would fail for general fields k (Zhu 10.24). It holds whenever k has finite ℓ -cohomological dimension. The key point here is that under our assumptions on k , for any $X \in \mathrm{Sp}^{\mathrm{pfp}}$ and any *finite* Λ taking inductive limits yields an equivalence

$$\mathrm{Sh}^*(X) \simeq D(X_{\mathrm{et}}, \Lambda).$$

Theorem 1.5. (Zhu 10.21, 10.25) *For the 3-functor formalism Sh^* each $f_!$ (with f pfp) has a continuous right adjoint $f^!$ and each f^* (with f arbitrary) has a continuous right adjoint f_* . Moreover Sh^* is a sheaf for the étale topology, and for any étale surjective map $f : X \rightarrow Y$ with Čech nerve f_\bullet and any $F \in \mathrm{Sh}_c^*(Y)$*

$$|(f_\bullet)_!(f_\bullet)^*F| \simeq F,$$

in particular $f^* = f^! : \mathrm{Sh}^*(Y) \rightarrow \mathrm{Sh}^*(X)$ is conservative.

Proof. For the second part, one first proves the isomorphism

$$|(f_\bullet)_!(f_\bullet)^*F| \simeq F,$$

by reducing to the case when $X, Y \in \mathrm{Sp}^{\mathrm{pfp}}$ (use a presentation of Y and base-change and limit arguments), in which case we can use the equivalences $\mathrm{Sh}^*(Z) \simeq D(Z_{\mathrm{et}}, \Lambda)$ for $Z = X, Y$ and the fact that $X \mapsto D(X_{\mathrm{et}}, \Lambda)$ is an étale sheaf. This isomorphism obviously implies the conservativity of f^* , and using Lurie's Beck-Chevalley criterion for descent we see that this is enough to conclude concerning étale descent. For the existence of continuous right adjoints see Zhu 10.21 (or use lemma 10.17 in the appendix VII of Scholze's 6 functor formalism course). \square

Remark 1.6. These right adjoints f_* and $f^!$ are a bit complicated to work with. For instance, say $f : X \rightarrow Y$ is pfp and $Y = \varprojlim Y_i$ is a presentation of Y . We may assume that f is the base-change of a pfp map $f_0 : X_0 \rightarrow Y_0$. To describe $f^!$ it suffices to give its values on constructible sheaves, each coming from some finite layer Y_i , and by replacing Y_0 by Y_i it suffices to describe $f^! p_0^* F_0$ for a constructible sheaf F_0 on Y_0 , where $p_i : Y \rightarrow Y_i$ is the projection map. If F_i is the pullback of F to Y_i , then

$$f^! F = \varinjlim_i q_i^* f_i^! F_i,$$

where $f_i : X_i \rightarrow Y_i$ is the base-change of f_0 and $q_i : X \rightarrow X_i$ is the projection.

Similarly, if $f : X \rightarrow Y$ is an arbitrary morphism with a presentation $f = \varprojlim (f_i : X_i \rightarrow Y)$ and projections $p_i : X \rightarrow X_i$, we have

$$f_* p_0^* F_0 = \varinjlim_i f_{i,*} F_i,$$

where again F_i is the pullback of the constructible sheaf F_0 on X_0 to X_i . This works as long as the $f_{i,*}$ are well-defined, in particular it works if Y is pfp. Thus to define f_* in general it remains to explain how to define it for f pfp. But then taking a presentation $Y = \varprojlim Y_i$ we obtain a compatible family of maps $f_i : X \rightarrow Y_i$ and we have

$$f_* F = \varinjlim p_i^* f_{i,*} F$$

where $p_i : Y \rightarrow Y_i$ are the projections.

1.4 ULA, smooth, pro-smooth, unipotent,... maps

Definition 1.7. A (necessarily pfp) map f in Sp is called ULA/smooth/unipotent) if it is suave/smooth/unipotent) for the 6-functor formalism $\mathrm{Sh}^*(-, \mathbf{F}_\ell)$.

Remark 1.8. 1. An ULA (resp. smooth) map f is then automatically suave (resp. smooth) for $\mathrm{Sh}^*(-, \Lambda)$. One can prove that a pfp morphism $f : X \rightarrow Y$ is ULA if and only if for any geometric point $\bar{x} \rightarrow X$ and any generalization $\bar{y} \rightarrow f(\bar{x})$ the natural map

$$\Lambda \rightarrow \mathrm{R}\Gamma(X_{(\bar{x})} \times_{Y_{(f(\bar{x}))}} \bar{y}, \Lambda)$$

is an isomorphism. Moreover, ULA maps are universally open.

2. One can check that a smooth map f is unipotent if and only if its geometric fibers have \mathbf{F}_ℓ -acyclic cohomology, if and only if $f^!$ is fully faithful.
3. By the general formalism, ULA maps and smooth maps form weakly stable classes (and then so do unipotent maps), and for $f : X \rightarrow Y$ ULA the formation of $f^!$ commutes with arbitrary $*$ -base change. Moreover, for $g : Y' \rightarrow Y$ with base-changes $g' : X' \rightarrow X$, $f' : X' \rightarrow Y'$ there are natural Beck-Chevalley isomorphisms

$$(g')_!(f')^! \simeq f^! g!, \quad (g')^* f^! \simeq (f')^! g^*, \quad (f')^* g^! \simeq (g')^! f^*, \quad f^* g_* \simeq g'_*(f')^*$$

when everything makes sense (thus we ask that g is pfg if the formula involves $g_!$ or $g^!$). Moreover, the last isomorphism holds if f is a cofiltered limit of ULA maps $f_i : X_i \rightarrow Y$ (Zhu 10.32), by passage to limits.

Theorem 1.9. *If f is ULA then $f^!$ preserves constructibility.*

Proof. Say $f : X \rightarrow Y$ is ULA and let $\Delta : X \rightarrow X \times_Y X$ be the relative diagonal, then Δ is pfp, so $\Delta_!$ preserves constructibility and so $\Delta_! 1_X$ is constructible, in particular compact. On the other hand $f^! \simeq f^* \otimes A$, with $A = f^! 1$ being f -suave (as it is $D_f(1_X)$).

It suffices to see that A is compact, and for this we compute for arbitrary constructible B , using the projections $p_1, p_2 : X \times_Y X \rightarrow X$

$$\begin{aligned} \mathrm{Hom}(A, B) &= \mathrm{Hom}(p_{1,!} \dagger \Delta_! A, B) = \mathrm{Hom}(\Delta_! A, p_1^! B) = \mathrm{Hom}(\Delta_! 1_X \otimes p_2^* A, p_1^! B) = \\ &= \mathrm{Hom}(\Delta_! 1_X, \underline{\mathrm{Hom}}(p_2^* A, p_1^! B)) = \mathrm{Hom}(\Delta_! 1_X, B \boxtimes_{\Lambda} D_f(A)). \end{aligned}$$

As the last functor commutes with filtered colimits in B , we are done. \square

Definition 1.10. Let E be a class of morphisms in Sp and let $f : X \rightarrow Y$ be a morphism in Sp .

1. f is called *pseudo-pro- E* if f is a cofiltered limit of morphisms $f_i : X_i \rightarrow Y$ in E , the transitions $X_i \rightarrow X_j$ being affine.
2. f is called *pro- E* if we can moreover ensure that the transitions $X_i \rightarrow X_j$ are also in E . If moreover we can ensure that the transitions are surjective, we call f *strongly pro- E* .
3. f is called *weakly E* if there is a surjective morphism $X' \rightarrow X$ in E such that $X' \rightarrow X \rightarrow Y$ is also in E .

We can iterate these things and obtain the notions of morphisms which are weakly pseudo-pro- E , etc.

Convention We will write *ps* for pro-smooth and *pu* for pro-unipotent.

1.5 Constructible and Ind-constructible cosheaves

We define now lax-monoidal functors

$$\mathrm{Sh}_c^! : \mathrm{Sp}^{\mathrm{op}} \rightarrow \mathrm{LC}_{\Lambda}^{\mathrm{Perf}}, \quad \mathrm{Sh}^! : \mathrm{Sp}^{\mathrm{op}} \rightarrow \mathrm{LC}_{\Lambda}.$$

by asking that

$$\mathrm{Sh}_c^!(X) = \mathrm{Sh}_c^*(X)^{\mathrm{op}}, \quad \mathrm{Sh}^!(X) = \mathrm{Ind}(\mathrm{Sh}_c^!(X)).$$

For a map $f : X \rightarrow Y$ the corresponding "pullback" functor is denoted

$$f^! : \mathrm{Sh}_c^!(Y) \rightarrow \mathrm{Sh}_c^!(X), \quad f^! = (f^*)^{\mathrm{op}}$$

and

$$f^! : \mathrm{Sh}^!(Y) \rightarrow \mathrm{Sh}^!(X), \quad f^! := \mathrm{Ind}((f^*)^{\mathrm{op}}).$$

Note that what is usually called f^* in 3-functor formalisms is now called $f^!$. Also note that $\mathrm{Sh}^!$ is simply obtained by composing $\mathrm{Sh}_c^* : \mathrm{Sp}^{\mathrm{op}} \rightarrow \mathrm{LC}_{\Lambda}^{\mathrm{cg}}$ with the natural self-duality of $\mathrm{Lambda}^{\mathrm{cg}}$, sending C to $C^{\vee} = \mathrm{Ind}((C^{\omega})^{\mathrm{op}})$. In particular the above functors extend to 3-functor formalisms

$$\mathrm{Sh}_c^! : \mathrm{Corr}(\mathrm{Sp}^{\mathrm{op}}, \mathrm{pfp}) \rightarrow \mathrm{LC}_{\Lambda}^{\mathrm{Perf}}, \quad \mathrm{Sh}^! : \mathrm{Corr}(\mathrm{Sp}^{\mathrm{op}}, \mathrm{pfp}) \rightarrow \mathrm{LC}_{\Lambda}.$$

The functor $\mathrm{Sh}_c^!$ becomes lax-symmetric monoidal for the same external tensor product, each $\mathrm{Sh}_c^!(X)$ becoming symmetric monoidal for the tensor product

$$F \otimes^! G := \Delta_X^!(F \boxtimes_{\Lambda} G).$$

We call ω_X the monoidal unit of $\mathrm{Sh}_c^!(X)$.

For $A \in \mathrm{Sh}_c^*(Y)$ I will let $\bar{A} \in \mathrm{Sh}_c^!(Y)$ be the corresponding object, so that for a map $f : X \rightarrow Y$ in Sp we have

$$f^!(\bar{A}) = \overline{f^*(A)}.$$

Here is the key motivation for the above mumbo-jumbo, which looks extremely weird: suppose that X is pfp and let $\omega_X := \pi_X^!(1_X) \in \mathrm{Sh}_c^*(X)$ (here we use the $()^!$ in the Sh_c^* -theory). This is the classical dualizing sheaf of X . Verdier duality gives an involution

$$D_X : \mathrm{Sh}_c^*(X)^{\mathrm{op}} \simeq \mathrm{Sh}_c^*(X), \quad F \mapsto D_X(F) := \underline{\mathrm{Hom}}(F, \omega_X).$$

Moreover for any map $f : X \rightarrow Y$ in $\mathrm{Sp}^{\mathrm{pfp}}$ there are canonical isomorphisms

$$D_X \circ f^* \circ D_Y \simeq f^!, \quad D_Y \circ f_* \simeq D_X \simeq f_!$$

It follows that for X pfp there is a canonical equivalence

$$u_X : \mathrm{Sh}_c^!(X) \simeq \mathrm{Sh}_c^*(X), \quad u_X(\bar{A}) := D_X(A)$$

such that for any f we have $u_Y \circ f^! = f^! \circ u_Y$, $u_Y \circ f_* = f_* \circ u_Y$. Moreover, one can check (using that Verdier duality commutes with external tensor products) that u_X is symmetric monoidal.

We will now extend the class of $!$ -able maps in the theory $\mathrm{Sh}^!$. The next result is the key input.

Proposition 1.11. *Consider the formalism $\mathrm{Sh}^!$ on $\mathrm{Corr}(\mathrm{Sp}^{\mathrm{op}}, \mathrm{pfp})$. Let $f : X \rightarrow Y$, $g : Y' \rightarrow Y$ with base changes $g' : X' \rightarrow X$, $f' : X' \rightarrow Y'$.*

a) *If f is ULA then f_* has a left adjoint f^* preserving constructibility and endowed with canonical Beck-Chevalley isomorphisms*

$$(f')^* g^! \simeq (g')^! f^*.$$

b) *If f is pseudo ps then f is HR, i.e. (after any base change of f) the right adjoint f_{\bullet} of $f^!$ satisfies the projection formula and for any $g : Y' \rightarrow Y$ with corresponding base-changes f', g' we have*

$$g^! f_{\bullet} \simeq (f')_b (g')^!,$$

and if g is pfp

$$g_*(f')_b \simeq f_b(g')_*.$$

Proof. a) This follows formally from the fact that $f^!$ preserves constructibility and the corresponding base-change properties satisfied by ULA morphisms in the formalism Sh^* , by passing to opposite categories and ind-completion.

b) One can reduce to the case of f smooth by a careful consideration of limits. Suppose that f is smooth.

Lemma 1.12. *We have a natural isomorphism*

$$f_b = f_*(f^*(\omega_Y) \otimes^! (-)).$$

Proof. We may reduce by the standard argument to the case when Y is pfp, in which case so is X and we can use Verdier duality on X and Y (which I will simply denote by D): we start with the isomorphism $f^! \simeq f^* \otimes f^! 1_Y$, then apply D and get $f^* D \simeq f^! D \otimes^! f^* D(1_Y) = f^! D \otimes^! f^* \omega_Y$, hence $f^* \simeq f^! \omega_Y \otimes^! f^!$. The lemma follows then easily. \square

The formula above makes it clear that f_b preserves constructibility and is continuous. For the projection formula, letting $L_f = f^*(\omega_Y)$ we have

$$f_b F \otimes^! G \simeq f_b(L_f \otimes^! F) \otimes^! G \simeq f_b(L_f \otimes^! F \otimes^! f^! G) \simeq f_b(F \otimes^! f^! G).$$

Next, let's prove the isomorphism $g^! f_b \simeq f'_b(g')^!$. We have (using the lemma and proper base change for the formalism $\text{Sh}^!$)

$$g^! f_b F = g^! f_*(L_f \otimes^! F) = f'_*(g')^!(L_f \otimes^! F) = f'_*((g')^! L_f \otimes^! (g')^! F)$$

and it suffices to check that $(g')^! L_f \simeq L_{f'}$, but using a) we can write

$$(g')^! L_f = (g')^! f^* \omega_Y = (f')^* g^! \omega_Y = (f')^* \omega_{Y'}.$$

Similarly, if g is pfp we have

$$g_* f'_b F = g_* f'_*(L_{f'} \otimes^! F) = f_* g'_*(L_{f'} \otimes^! F)$$

and we are reduced to checking that

$$g'_*(L_{f'} \otimes^! F) = L_f \otimes^! g'_* F,$$

which follows from the projection formula and the isomorphism $(g')^! L_f = L_{f'}$, proved again using base-change results in a). \square

Theorem 1.13. *(Zhu 10.81) The functor $\text{Sh}^! : \text{Sp}^{\text{op}} \rightarrow \text{LC}_\Lambda$ extends to a 3-functor formalism $\text{Sh}^!$ on the geometric setup (Sp, Epet) , where Epet is the class of essentially pro-étale maps, such that*

1. for f pfp proper the corresponding lower $!$ -functor, denoted f_* is the left adjoint of $f^!$ and for f pro-étale f_* is the right adjoint of $f^!$.
2. for f ULA f_* has a left adjoint f^* preserving constructibility and satisfying base-change $(f')^* g^! \simeq (g')^! f^*$.
3. pseudo pro-smooth morphisms f are HR.

Proof. This follows from the above proposition, for instance by applying the Liu-Zheng-Mann machinery to the class E of separated essentially pro-étale maps, with I the class of pro-étale maps and P the class of pfp proper maps. \square

We can now further extend (by right Kan extension) to a 3-functor formalism $\mathrm{Sh}^!$ on $\mathrm{Corr}(\mathrm{PSt}, \mathrm{Epet}_r)$ and a 3-functor formalism $\mathrm{Sh}_c^!$ on $\mathrm{Corr}(\mathrm{PSt}, \mathrm{pfp}_r)$ (recall that the subscript r means representable). Similarly we obtain 3-functor formalisms Sh_c^* and Sh^* on $\mathrm{Corr}(\mathrm{PSt}, \mathrm{pfp}_r)$. By construction we have a fully faithful embedding

$$\mathrm{Sh}_c^!(X) \subset \mathrm{Sh}^!(X).$$

Moreover, since $(-)^{\mathrm{op}}$ commutes with limits we have a canonical equivalence

$$\mathrm{Sh}_c^!(X) \simeq \mathrm{Sh}_c^*(X)^{\mathrm{op}},$$

but for general prestacks X it can happen that $\mathrm{Sh}^!(X)$ is not compactly generated, and even when it is compactly generated its compact objects may not be $\mathrm{Sh}_c^!(X)$.

Remark 1.14. 1. It follows formally that for f representable étale f_* is the right adjoint of $f^!$ and for f representable pfp proper f_* is the left adjoint of $f^!$. For f representable ULA the functor f_* has a left adjoint f^* preserving constructibility and satisfying base-change $(f')^*g^! \simeq (g')^!f^*$. Also, representable ps morphisms are HR.

2. By reduction to the case of algebraic spaces (which can be done thanks to the above established base-change results) one proves that $\mathrm{Sh}^!$ satisfies a suitable form of excision: if $j : U \rightarrow X$ is a qc open embedding of prestacks with closed complement $i : Z \rightarrow X$ a pfp closed immersion, then i_* induces an equivalence between $\mathrm{Sh}^!(Z)$ and the full subcategory of $A \in \mathrm{Sh}^!(X)$ for which $j^!A = 0$, j_* induces an equivalence between $\mathrm{Sh}^!(U)$ and the full subcategory of $A \in \mathrm{Sh}^!(X)$ for which $i^!A = 0$, and for any $A \in \mathrm{Sh}^!(X)$ there is a canonical fibre sequence

$$i_*i^!F \rightarrow F \rightarrow j_*j^!F.$$

Proposition 1.15. *If X, Y are prestacks such that $\mathrm{Sh}^!(X)$ is dualizable, then the external product*

$$\mathrm{Sh}^!(X) \otimes_{\Lambda} \mathrm{Sh}^!(Y) \rightarrow \mathrm{Sh}^!(X \times Y)$$

is fully faithful.

Proof. One first deals with the case when $X, Y \in \mathrm{Sp}$, and using limits arguments reduces this to the case when X, Y are pfp, when this is classical. Next, suppose that X is a space, then $\mathrm{Sh}^!(X)$ is dualizable (since compactly generated) and so tensoring with it commutes with limits. Then

$$\mathrm{Sh}^!(X) \otimes_{\Lambda} \mathrm{Sh}^!(Y) = \varprojlim_{S \rightarrow \mathrm{Sp}/Y} (\mathrm{Sh}^!(X) \otimes_{\Lambda} \mathrm{Sh}^!(S)) \rightarrow \varprojlim_{S \in \mathrm{Sp}/Y} \mathrm{Sh}^!(X \times S)$$

is fully faithful and a cofinality argument shows that

$$\varprojlim_{S \in \mathbb{S}p/Y} \mathrm{Sh}^!(X \times S) = \mathrm{Sh}^!(X \times Y).$$

Finally, since we know that the result holds when one of X, Y is an algebraic space, we can redo the same argument as before under the only hypothesis that $\mathrm{Sh}^!(X)$ is dualizable. □

1.6 Placidity

Recall that ps, pu means pro-smooth, pro-unipotent. Write r? for representable ?.

Definition 1.16. 1. A space $X \in \mathbb{S}p$ is called placid⁴ if the map $X \rightarrow *$ is essentially pro-smooth.

2. Consider a qc stack X and a map $f : S \rightarrow X$ from a placid space S .

- we say that f is a quasi-placid atlas if f is rps surjective.
- f is called a placid atlas if it's a quasi-placid atlas, of universal homological descent⁵ and strongly ps.
- f is called a very placid atlas if it factors $S \rightarrow X' \rightarrow X$ with $S \rightarrow X'$ essentially pu and $X' \rightarrow X$ an open embedding.

We say that X is quasi-placid, placid, very placid if it has a quasi-placid, placid, or very placid atlas.

Remark 1.17. 1. One can define a more general notion of quasi-placid/placid/very placid stacks without imposing quasi-compactness. By convention whenever I use quasi-placid/placid/very placid I implicitly assume that the stack is qc.

Placid spaces behave very much like pfp spaces:

Theorem 1.18. *Let $f : X \rightarrow Y$ be a map of placid spaces.*

- a) *If $F, G \in \mathrm{Sh}_c^*(X)$ then $\underline{\mathrm{Hom}}(F, G) \in \mathrm{Sh}_c^*(X)$.*
- b) *If f is pfp then f_* and $f^!$ preserve the categories of constructible sheaves.*
- c) *If f is pseudo pu then f_* preserves constructibility.*

Proof. I only give the proof of a), for which the key input is

Lemma 1.19. *If f is (weakly) pseudo ps then f^* commutes with internal Hom on Sh_c^* .*

⁴Zhu calls it standard placid

⁵Which means that the map induced by !-pullbacks $\mathrm{Sh}(X) \rightarrow \varprojlim_{n \in \Delta} \mathrm{Sh}(S_n)$ is an equivalence and stays so after any base-change of f .

Proof. One can use v -descent to reduce to the case of pseudo ps morphisms (which is enough to prove the theorem, anyway). If f is smooth this follows from the fact that $f^! \underline{\mathrm{Hom}}(F, G) = \underline{\mathrm{Hom}}(f^* F, f^! G)$, since $f^!$ differs from f^* by tensoring with an invertible sheaf. In general write $f = \varprojlim_i f_i$ with $f_i : X_i \rightarrow Y$ smooth and $X_i \rightarrow X_j$ affine. We want to show that for A constructible

$$\mathrm{Hom}(A \otimes f^* F, f^* G) = \mathrm{Hom}(A, f^* \underline{\mathrm{Hom}}(F, G)).$$

We may assume that A comes from some A_0 on X_0 , with pullback A_i to X_i . Then

$$\mathrm{Hom}(A \otimes f^* F, f^* G) = \varinjlim_i \mathrm{Hom}(A_i \otimes f_i^* F, f_i^* G)$$

and by the case of smooth maps already discussed this is the same as

$$\varinjlim_i \mathrm{Hom}(A_i, f_i^* \underline{\mathrm{Hom}}(F, G)) = \mathrm{Hom}(A, f^* \underline{\mathrm{Hom}}(F, G)).$$

□

□

Corollary 1.20. *Let $f : X \rightarrow Y$ be a map of placid spaces and work with the theory $\mathrm{Sh}^!$.*

a) *If f is pfp then $f^!, f_*$ have left adjoints $f_!, f^*$ preserving $\mathrm{Sh}_c^!$ and such that for any g weakly pseudo ps there are canonical Beck-Chevalley isomorphisms*

$$f_!(g')^! \simeq g^! f_!, \quad (f')^* g^! \simeq (g')^! f^*.$$

b) *If f is pu, then $f^!$ has a left adjoint $f_!$ preserving constructibility and for g as in a) there is a canonical Beck-Chevalley isomorphism*

$$f_!(g')^! \simeq g^! f_!.$$

Proof. The existence of left adjoints follows formally from the above theorem. The base-change isomorphisms can be checked on constructible sheaves as everything is continuous and preserves constructibility. By v -descent we can reduce to the case when g is pseudo ps. The case when f is pu can be reduced again by limit arguments to the case f pfp. When f is pfp passage to opposite categories reduces us to:

Lemma 1.21. a) *if f is a cofiltered limit of ULA maps $X_i \rightarrow Y$ then $f^* g_* \simeq g'_*(f')^*$ on Sh^* (already seen).*

b) *if f is pseudo ps and g is pfp then $(f')^* g^! \simeq (g')^! f^*$ on Sh^* (again by limit arguments this reduces to the case f smooth, already seen).*

□

If $f : S \rightarrow X$ is a quasi-placid atlas then f is a v -cover and since $\mathrm{Sh}_c^!$ satisfies v -descent it follows that $F \in \mathrm{Sh}^!(X)$ is constructible (i.e. belongs to the essential image of $\mathrm{Sh}_c^!(X)$) if and only if $f^!F$ is constructible in $\mathrm{Sh}^!(S)$, or equivalently if F is compact. Since f_b is continuous (this always happens for representable (pseudo) ps morphisms 10.87 (3)), by a standard argument $f^!$ preserves compactness, and we conclude that for any quasi-placid stack X we have an inclusion

$$\mathrm{Sh}^!(X)^\omega \subset \mathrm{Sh}_c^!(X).$$

Under extra hypotheses we can refine this result:

Proposition 1.22. *(Zhu 10.144) Suppose that X has a placid atlas $f : S \rightarrow X$ representable essentially pu. Then $\mathrm{Sh}^!(X)$ is compactly generated, with compact object the smallest full idempotent complete stable sub-category containing all $f_!F$ with $F \in \mathrm{Sh}_c^!(S)$. If f is representable pu then the compact object are exactly the constructible cosheaves.*

Proof. Use that $f^!$ is conservative, that we can describe $\mathrm{Sh}^!(X)$ using the cosheaves on the Cech nerve of f and that $f^!$ has a left adjoint $f_!$. When f is representable pu the functor $f^!$ is fully faithful, so $f_!f^! = \mathrm{id}$ and so \square

Proposition 1.23. *Let $f : X \rightarrow Y$ be a representable morphism between quasi-placid staks, and let $g : Y' \rightarrow Y$ be a weakly ps map of prestacks, with base changes $g' : X' \rightarrow X$ and $f' : X' \rightarrow Y'$.*

1. *if f is essentially pu then $f^!$ on constructible subcategories has a left adjoint $f_!$ and $(f')_!(g')^! \simeq g^!f_!$.*
2. *if f is pfp then f_* preserves the constructible subcategories and has a left adjoint f^* on them such that $(f')^*g^! = (g')^!f^*$.*
3. *if f is smooth then f_b preserves the constructible subcategories and $(f')_b(g')^! \simeq g^!f_b$.*

Proof. This is a formal consequence of similar results for placid spaces. \square