$\boxed{\text{Affine Hecke categories and affine Deligne-Lusztig theory}}$

(§§4.2–4.3–4.4)

Recall notation from chap 3 (cf. talk by Benjamin)

(Not clear why this is necessary)

F non archimedian local field

$\mathcal{O}_F$ ring of integers. $\varpi \in \mathcal{O}_F$ uniformizer

k_F residue field

k alg. closure of k_F ← all stacks are defined over k.

G connected quasi-split reductive gp over F. ~~For simplicity we assume~~ which splits over a tamely ramified extension ~~that G is split over F, so we have a canonical $\mathcal{G}$ model $\mathcal{G}$ over $\mathcal{O}_F$~~. We choose a pinning (B, T, e) ⇝ gives us $\mathcal{K}$ (special max'l parahoric subgp) and $\mathcal{I}$ (Iwahori gp scheme)

Loop group: $LG : \begin{cases} k\text{-algebras} \longrightarrow \text{Gps} \\ R \longmapsto G\left(W_0(R)\left[\frac{1}{\varpi}\right]\right) \end{cases}$

~~we fix a (split) maximal torus $T \subset G$ and a Borel subgp~~

~~...defined over $\mathcal{O}_F$ ... get ... Iwahori group scheme~~

~~$\mathcal{I}$~~. We set $Iw = L^+\mathcal{I} : \begin{cases} k\text{-algebras} \longrightarrow \text{Gps} \\ R \longmapsto \mathcal{I}(W_0(R)) \end{cases}$

(torus over k)

We have $Iw = \mathcal{S}_k \ltimes Iw^u$ (pro-unipotent)

here $S \subset G$ maximally $\breve{F}$-split torus, $\mathcal{S}$ Néron model over $\mathcal{O}_F$, $\mathcal{S}_k$ base-change to k.

$\boxed{\text{Affine Hecke categories}}$ (§4.2~~...~~)

(1) We consider $Iw\backslash LG/Iw = \mathbb{B}(Iw) \times_{\mathbb{B}(LG)} \mathbb{B}(Iw)$

Here $\mathbb{B}(Iw) \to \mathbb{B}(LG)$ is an ind-pfp morph. ~~of ... affine ... ind-placid stacks~~ (because LG/Iw is an ind-scheme ind-pfp over k). So $\boxed{\mathrm{Shv}(Iw\backslash LG/Iw)}$ has a monoidal structure (by general convolution formalism). Called unipotent affine Hecke category.

Prop. (4.41)(1)

(1) $Shv({}_{Iw}\backslash LG/{}_{Iw})$ is compactly generated, with compact objects (those) which satisfy one of the following equiv. cond.:

- $\forall w \in \tilde{W}$, $i_w^* \mathcal{F} \in Shv({}_{Iw}\backslash Fl_w) \simeq Shv(\mathbb{B}\mathcal{I}_k)$ is compact, and $i_w^* \mathcal{F} = 0$ for all but finitely many w's

  seems false: need to assume $\mathcal{F}$ is supported on a finite union of Schubert var.

- Same with $i_w^!$

(2) $Shv({}_{Iw}\backslash LG/{}_{Iw})$ is semi-rigid Λ-linear.

(3) For every X prestack over k,

$$Shv({}_{Iw}\backslash LG/{}_{Iw}) \otimes_\Lambda Shv(X) \xrightarrow{\sim} Shv({}_{Iw}\backslash LG/{}_{Iw} \times X).$$

Rmk: $Shv(\mathbb{B}\mathcal{I}_k)$ is compactly generated. But in $Shv(\mathbb{B}\mathcal{I}_k)$, $\underline{\Lambda}$ is NOT compact. A typical compact object is $(pt \to pt/\mathcal{I}_k)_! \Lambda$. In particular the unit of $Shv({}_{Iw}\backslash LG/{}_{Iw})$ is not compact, so this is NOT rigid!

Idea of proof

(1) ~~Compact generation follows from a general statement~~ This follows from the fact that

$$Shv({}_{Iw}\backslash LG/{}_{Iw}) = \operatorname{colim}_w Shv({}_{Iw}\backslash LG_{\leq w}/{}_{Iw})$$

(very placid stack $\downarrow$ cf. Ex. 10.117)

[Prop. 10.144: $Shv(X)$ comp. gen'd if X very placid stack with suitable atlas]

(2) The category is compactly generated, hence dualizable. Linearity of the right adjoint to multiplication follows from Lemma 1 in Quan's notes. [Prop. 8.67]

Characterization of compact objects is similar to the corresp. claim for $Isoc_G$

(3) Using localization sequences, the proof reduces to the similar statement for $Shv({}_{Iw}\backslash LG_w/{}_{Iw}) \otimes_\Lambda Shv(X)$, which reduces to a statement for $Shv(BH_w) \otimes_\Lambda Shv(X)$ for appropriate H_w, proved as Prop. 10.109. □

(2) We also consider

(maybe not in equal char. ?)

$$\boxed{\mathrm{Shv}_{f.g.}(I_w\backslash LG/I_w)} \subset \mathrm{Shv}(I_w\backslash LG/I_w)$$

(small unipotent affine Hecke category)

Concrete description:

For $w \in \widetilde{W}$ (affine Weyl gp) we have $\boxed{i_w :}$ $I_w\backslash LG_w/I_w \hookrightarrow I_w\backslash LG/I_w$.

LG_w/I_w: Schubert ~~variety~~ cell (I_w-orbit, isom. to $\mathbb{A}_k^{\ell(w)}$).

Set

$$\boxed{\begin{aligned} \Delta_w &:= (i_w)_!\, \omega_{I_w\backslash LG_w/I_w}[-\ell(w)] \\ \nabla_w &:= (i_w)_*\, \omega_{I_w\backslash LG_w/I_w}[-\ell(w)] \end{aligned}}$$

Fact: $\mathrm{Shv}_{f.g.}(I_w\backslash LG/I_w)$ = smallest idempotent complete stable subcategory of $\mathrm{Shv}(I_w\backslash LG/I_w)$ containing the objects $(\Delta_w : w \in \widetilde{W})$

= ______________________________

______________________________ $(\nabla_w : w \in \widetilde{W})$

We will also study $\boxed{\mathrm{IndShv}_{f.g}(I_w\backslash LG/I_w)}$

(big unipotent affine Hecke category).

Convolution preserves Shv_{fg} so we get a monoidal structure

[More formally, $\mathbb{B}(I_w)$ is a very placid stack, cf. Ex 10.117. $\mathbb{B}(LG)$ is a sifted very placid stack because $\mathbb{B}(I_w) \to \mathbb{B}(LG)$ is ind-pfp proper. So we can consider IndShv_{fg} on them. Moreover since $\mathbb{B}(I_w)$ is ind-very placid and $\mathbb{B}(I_w) \to \mathbb{B}(LG)$ is ind-pfp we are ok, cf. §10.6.5, §10.5.4, Ex. 10.136]

Prop. (4.41(2))

(1) $\mathrm{IndShv}_{f.g.}(I_W\backslash LG/I_W)$ is compactly generated, with compact objects those which satisfy one of the following equiv. conditions:

- $\forall w \in \tilde{W}$, $i_w^* \mathcal{F} \in \mathrm{IndShv}_{f.g.}(I_W\backslash LG_w/I_W) \simeq \mathrm{IndShv}_{f.g.}(\mathbb{B}\mathcal{J}_k)$ is compact, and $i_w^* \mathcal{F} = 0$ for all but finitely many w's
- same with $i_w^!$

(2) $\mathrm{IndShv}_{f.g.}(I_W\backslash LG/I_W)$ is rigid Λ-linear.

(3) For every quasi-compact placid stack X,
$$\mathrm{IndShv}_{f.g.}(I_W\backslash LG/I_W) \otimes_\Lambda \mathrm{IndShv}_{f.g.}(X) \xrightarrow{\sim} \mathrm{IndShv}_{f.g.}(I_W\backslash LG/I_W \times X)$$

Proof is similar to the previous one (using Cor. 10.143 for (3)). Here the unit is compact so the category is rigid.

③ Recall that $I_W = I_W^u \rtimes \mathcal{J}_k$ (uses monodromic sheaves from talk by Alexandre)

We consider $I_W^u\backslash LG/I_W^u \circlearrowleft \mathcal{J}_k \times \mathcal{J}_k$

$$\boxed{\mathrm{Shv}_{mon}(I_W^u\backslash LG/I_W^u)} = \mathrm{Shv}_{(\mathcal{J}_k\times\mathcal{J}_k,\,mon)}(I_W^u\backslash LG/I_W^u)$$
$$= \mathrm{Shv}_{(\mathcal{J}_k\text{-mon})}(I_W^u\backslash LG/I_W^u)$$

(monodromic affine Hecke category)

either for right or for left action.

For $w \in \tilde{W}$, after choosing a lift $\dot{w} \in N_G(S)(\breve{F})$ we have $pr_{\dot{w}} : I_W^u\backslash LG_w/I_W^u \longrightarrow \mathcal{J}_k$, which induces

$$(\mathrm{pr}_{\dot w})^! : \mathrm{Shv}_{\mathrm{mon}}(\mathcal{J}_k) \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{mon}}(I_W^u \backslash LG_w / I_W^u)$$

This allows to define

$$\Delta^{\mathrm{mon}}_{\dot w},\ \nabla^{\mathrm{mon}}_{\dot w} : \mathrm{Shv}_{\mathrm{mon}}(\mathcal{J}_k) \to \mathrm{Shv}_{\mathrm{mon}}(I_W^u \backslash LG / I_W^u)$$

In particular we have

$$\tilde{\Delta}^{\mathrm{mon}}_{\dot w} = \Delta^{\mathrm{mon}}_{\dot w}(\underbrace{\mathrm{Ch}_{\mathrm{mon}}}_{\left(\text{unit of } \mathrm{Shv}_{\mathrm{mon}}(\mathcal{J}_k)\right)}), \quad \tilde{\nabla}^{\mathrm{mon}}_{\dot w} = \nabla^{\mathrm{mon}}_{\dot w}(\mathrm{Ch}_{\mathrm{mon}})$$

*

Prop (4.44)

(1) $\mathrm{Shv}_{\mathrm{mon}}(I_W^u \backslash LG / I_W^u)$ is compactly generated, with compact objects $\mathrm{Shv}_{\mathrm{mon}}(I_W^u \backslash LG / I_W^u) \cap \mathrm{Shv}(I_W^u \backslash LG / I_W^u)^{\omega}$

(2) This category is semi-rigid
Λ-linear

(3) For every prestack X with an action of H we have

$$\mathrm{Shv}_{\mathrm{mon}}(I_W^u \backslash LG / I_W^u) \otimes_{\Lambda} \mathrm{Shv}_{\mathrm{mon}}(X) \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{mon}}(I_W^u \backslash LG / I_W^u \times X).$$

(${\Lambda}$ ~~$\mathcal{J}$~~ a torus)

Proof is similar to those of the previous propositions (using Lemma 4.~~30~~ 31 for (3)).

By the same construction as for $I_W \backslash LG / I_W$, we have a monoidal structure on $\mathrm{Shv}(I_W^u \backslash LG / I_W^u)$ defined via $(-)_* \circ (-)^!$ for the diagram $I_W^u \backslash LG / I_W^u \leftarrow I_W^u \backslash LG \times^{I_W^u} LG / I_W^u \to I_W^u \backslash LG / I_W^u \times I_W^u \backslash LG / I_W^u$

The full subcategory $\mathrm{Shv}_{\mathrm{mon}}(I_W^u \backslash LG / I_W^u)$ is a (non-unital!) monoidal subcategory. Then one argues that $\tilde{\Delta}^{\mathrm{mon}}_e = \tilde{\nabla}^{\mathrm{mon}}_e$ provides a unit, so we get a monoidal structure $\longrightarrow$

Alternatively one can ~~obt~~ obtain this structure from the sheaf theory Shv_{mon}, ~~with~~ since $\left(I_W^u \backslash LG / I_W^u , \, \mathcal{J}_k \times \mathcal{J}_k \right) =$ ~~Cech~~ $\left(\text{Cech nerve of } \left(B(I_W^u), \mathcal{J}_k \right) \to \left(B(LG), \{1\} \right) \right)$ is an algebra object.

~~Below~~ Later we will also consider the monoidal category

$$\mathrm{Shv}\left((I_W, \hat{u}) \backslash LG / (I_W, \hat{u}) \right) := \mathrm{Shv}\left(\underbrace{\left(I_W^u \backslash LG / I_W^u \right)}_{(\mathcal{J}_k \times \mathcal{J}_k, \, u\text{-mon})} \right)$$

Here the unit is

$$\Delta^{mon}_{e,\hat{u}} = \Delta^{mon}_{e}(Ch_{\hat{u}}) \quad \left(= \nabla^{mon}_{e,\hat{u}} \right).$$

(called unipotent monodromic affine Hecke category)

Construction: (cf. Ex. 4.36) (with dual torus $\hat{H}$ over Λ)

Recall that for a torus H we have $R_{I_F^t, \hat{H}}$: ind-scheme ind-finite over Λ. (If $\Lambda = \bar{\mathbb{F}}_\ell$ or $\bar{\mathbb{Q}}_\ell$, $R_{I_F^t,\hat{H}} = \bigsqcup_{x \in \hat{H}(\Lambda)} \hat{x}$.)

[moduli stack over $\mathbb{Z}_\ell$ of strongly continuous $\hat{H}$-valued rep^{ns} of I_F^t, base-changed to Λ cf §2.1.2]

for $\varphi : \mathbb{G}_m \to H$ cocharacter we have

$$\hat{\varphi} : R_{I_F^t, \hat{H}} \longrightarrow R_{I_F^t, \mathbb{G}_m}$$

We set $x_\varphi = u \times_{R_{I_F^t, \mathbb{G}_m}} R_{I_F^t, \hat{H}}$

We have $\boxed{Ch : \mathrm{IndCoh}(R_{I_F^t,\hat{H}}) \simeq \mathrm{Shv}_{mon}(H)}$

We have an exact sequence

$$1 \to Ch(\omega_{x_\varphi}) \longrightarrow Ch_{mon} \longrightarrow Ch_{mon} \longrightarrow 1$$

in $\mathrm{Shv}_{mon}(H)^\heartsuit$.

For $s \in \tilde{W}$ simple reflection we have an associated ~~coot~~ coroot $\hat{\alpha}_s : \mathbb{G}_m \to \mathcal{T}_k$. We set $\widetilde{Ch}_s = Ch(\omega_{x_{\hat{\alpha}_s}}) \in \mathrm{Shv}_{mon}(\mathcal{T}_k)$.

Proposition (4.47)

Let $\mathcal{L}, \mathcal{L}' \in \mathrm{Shv}_{mon}(\mathcal{T}_k)$ we have

(1) for $w, y \in \tilde{W}$ s.th. $\ell(wy) = \ell(w) + \ell(y)$

$$\Delta^{mon}_{\dot{w}}(\mathcal{L}) *^u \Delta^{mon}_{\dot{y}}(\mathcal{L}') \simeq \Delta^{mon}_{\dot{w}\dot{y}}(\mathcal{L} * w(\mathcal{L}'))$$

$$\nabla^{mon}_{\dot{w}}(\mathcal{L}) *^u \nabla^{mon}_{\dot{y}}(\mathcal{L}') \simeq \nabla_{\dot{w}\dot{y}}(\mathcal{L} * w(\mathcal{L}')).$$

(2) For $w \in \tilde{W}$ we have canonical isomorphisms

$$\nabla^{mon}_{\dot{w}}(\mathcal{L}) *^u \Delta^{mon}_{\dot{w}^{-1}}(\mathcal{L}') \simeq \Delta^{mon}_{e}(\mathcal{L} * w(\mathcal{L}')) \simeq \Delta^{mon}_{\dot{w}}(\mathcal{L}) *^u \nabla^{mon}_{\dot{w}^{-1}}(\mathcal{L}')$$

(3) For $s \in \tilde{W}$ simple reflection we have fiber sequences

$$\nabla^{mon}_{e}(\mathcal{L} * s(\mathcal{L}')) \to \nabla^{mon}_{\dot{s}}(\mathcal{L}) *^u \nabla^{mon}_{\dot{s}}(\mathcal{L}') \to \nabla^{mon}_{s}(\mathcal{L} * \widetilde{Ch}_s * s(\mathcal{L}'))[1]$$

$$\Delta^{mon}_{\dot{s}}(\mathcal{L} * \widetilde{Ch}_s * s(\mathcal{L}')) \to \Delta^{mon}_{\dot{s}}(\mathcal{L}) *^u \Delta^{mon}_{\dot{s}}(\mathcal{L}') \to \Delta^{mon}_{e}(\mathcal{L} * s(\mathcal{L}')).$$

Idea of proof

This follows from standard arguments

(1) The main ingredients are:

- $LG_w \times^{I_w} LG_y \xrightarrow{\sim} LG_{wy}$ if ~~[illegible]~~ $\ell(wy) = \ell(w) + \ell(y)$

- Convolution can be defined either using $*$-pushforward or with $!$-pushforward (cf. Lemma 4.27)

(2) This reduces to the case ~~s~~ w is a simple reflection, hence to an SL_2-computation. In the end one uses that $H^*(\mathbb{A}^1, j_! \Delta_{\mathbb{G}_m}) = 0$ to show that the complex is supported on ${}_{I_w^u}\backslash I_w / I_w^u$

(3) Again this an SL_2-computation, done in $\mathrm{IndCoh}(R_{I_F^t}, \hat{\mathfrak{g}}_k)$. □

$\boxed{\text{Affine Deligne–Lusztig theory}}$ (§4.3)

[recall subtlety: induces $LG_w/\mathrm{Ad}_\sigma(I_w) \to I_w \backslash LG_{w^{-1}} / I_w$]

Recall the diagram

$$\begin{tikzcd} I_w \backslash LG / I_w & \underbrace{LG/\mathrm{Ad}_\sigma(I_w)}_{\substack{\| \\ \mathrm{Sht}^{loc}_{\mathcal{I}}}} \arrow[l, "\delta"', "\text{repr. coh. pro-smooth}"] \arrow[r, "Nt", "\text{ind-pfp proper}"'] & \underbrace{LG/\mathrm{Ad}_\sigma(LG)}_{\substack{\| \\ \mathrm{Isoc}_G}} \end{tikzcd}$$

We set

$$(4.43) \quad \boxed{\mathrm{Ch}^{unip}_{LG,\phi} := (Nt)^{\mathrm{Indfg}}_* \circ \delta^{\mathrm{Indfg},!} : \mathrm{IndShv}_{fg}({}_{I_w}\backslash LG / I_w) \longrightarrow \mathrm{IndShv}_{fg}(\mathrm{Isoc}_G)}$$

Note: $(Nt)^{\mathrm{Indfg}}_*$ is well defined because Nt is ind-pfp ~~[illegible]~~ (cf. comments above (10.62))

cf §10.65 ~~ind stacks placid stacks, cf Ex 10.136~~

$\delta^{\mathrm{Indfg},!}$ is well defined because δ is representable in ind-~~[illegible]~~placid stacks.

Since δ is repr. ~~[illegible]~~ in qc placid stacks $\delta^{\mathrm{Ind\,fg},!}$ preserves Shv_{fg} - since Nt is ind-pfp, $(Nt)_*^{\mathrm{Ind\,fg}}$ preserves Shv_{fg} (cf Prop. 10.178) - so $\mathrm{Ch}^{unip}_{LG,\phi}$ restricts to

$$\mathrm{Shv}_{f.g.}(I_w\backslash LG/I_w) \longrightarrow \mathrm{Shv}_{fg}(\mathrm{Isoc}_G)$$

In fact we have

$$\begin{tikzcd}
\mathrm{Shv}_{f.g.}(I_w\backslash LG/I_w) \arrow[r, "\mathrm{Ch}^{unip}_{LG,\phi}"] \arrow[d, hook, "\text{(because ind-very placid stack)}"'] & \mathrm{Shv}_{fg}(\mathrm{Isoc}_G) \arrow[d, hook, "\text{(Prop. 3.91)}"] \\
\mathrm{Shv}(I_w\backslash LG/I_w) \arrow[r, "(Nt)_*\,\delta^!"'] & \mathrm{Shv}(\mathrm{Isoc}_G)
\end{tikzcd}$$

<u>Variant:</u> We consider

$$I_w^u\backslash LG/I_w^u \xleftarrow{\delta^u} LG/\mathrm{Ad}_\sigma(I_w^u) \xrightarrow{Nt^u} LG/\mathrm{Ad}_\sigma(LG) = \mathrm{Isoc}_G$$

$$(4.46)\quad \boxed{\mathrm{Ch}^{mon}_{LG,\phi} := (Nt^u)_*(\delta^u)^! : \mathrm{Shv}_{mon}(I_w^u\backslash LG/I_w^u) \longrightarrow \mathrm{Shv}(\mathrm{Isoc}_G)}$$

We will consider the following objects:

$$\boxed{R^*_w = \mathrm{Ch}^{unip}_{LG,\phi}(\nabla_{w^{-1}})} = Nt_*\,\tilde{i}_{w*}\,\omega_{Sht^{loc}_w}[-l(w)]$$

$$\boxed{R^!_w = \mathrm{Ch}^{unip}_{LG,\phi}(\Delta_{w^{-1}})} = Nt_*\,\tilde{i}_{w!}\,\omega_{Sht^{loc}_w}[-l(w)]$$

standard base change thm for

$$\begin{tikzcd}
Sht^{loc}_w = LG_w/\mathrm{Ad}_\sigma I_w \arrow[r, hook, "\tilde{i}_w"] \arrow[d] \arrow[dr, phantom, "\square"] & Sht^{loc} = LG/\mathrm{Ad}_\sigma I_w \arrow[d, "\delta"] \\
I_w\backslash LG_{w^{-1}}/I_w \arrow[r, hook, "i_{w^{-1}}"] & I_w\backslash LG/I_w
\end{tikzcd}$$

base change thm with $(-)_!$ and $(-)^!$, which applies since δ is coh. ps (cf. Prop. 10.145)