# Stack of Langlands parameters

$F$ as usual NA local res field $k_F$ of size $q = p^r$

$G/F$ connected reductive $\rightsquigarrow$

$\hat{G}$ split dual gr $/\mathbb{Z}$ with a pinning $(\hat{B}, \hat{T}, \hat{e})$

$\Gamma_F = \mathrm{Gal}(\overline{F}/F) \xrightarrow{\xi} \mathrm{Aut}(\hat{G}, \hat{B}, \hat{T}, \hat{e})$

let $\Gamma_{\tilde{F}/F} = \Gamma_F / \ker \xi$ finite quot of $\Gamma_F$ (trivial if $G$ is split)

$\rho$ half-sum of positive roots of $G$

$\rightsquigarrow$ cocharacter $\rho_{ad} : \mathbb{G}_m \to \hat{G}_{ad} \hookrightarrow \mathrm{Aut}\,\hat{G}$

so $\mathbb{G}_m \circlearrowright \hat{G}$

<u>Def</u> ${}^L G := \hat{G} \rtimes \Gamma_{\tilde{F}/F} \xrightarrow{\mu_2} \Gamma_{\tilde{F}/F}$

$${}^c G = \hat{G} \rtimes (\mathbb{G}_m \times \Gamma_{\tilde{F}/F}) \xrightarrow[\mu_2]{} \mathbb{G}_m \times \Gamma_{\tilde{F}/F}$$

$\exists$ canonical hom

$$\text{Weil gr } W_F \xrightarrow{\mathrm{can}} \mathbb{Z}[\tfrac{1}{p}]^\times \times \Gamma_{\tilde{F}/F} \xrightarrow{\mu_2} \Gamma_{\tilde{F}/F}$$

$\Gamma_F \supset W_F$   $x \longmapsto (\|x\|^{-1}, \bar{x})$ image of $x$ in $\Gamma_{\tilde{F}/F}$

11.11: $\underbrace{W_F/I_F}_{\text{inertia of } F} \longrightarrow q^{\mathbb{Z}} \subset \mathbb{Z}[\tfrac{1}{p}]^\times$

lift of arith Frob $\longrightarrow q$

<u>Def</u> A $\mathbb{Z}[\frac{1}{p}]$-algebra. An abstract L-par / c-par over A is a hom

$$\varphi : W_F \longrightarrow {}^LG(A) \quad (\text{resp } \varphi : W_F \to {}^cG(A))$$

st

$$\begin{tikzcd} W_F \arrow[r, "\varphi"] \arrow[dr, "\mu_2 \circ \mathrm{can}"'] & {}^LG(A) \arrow[d, "\mu_2"] \\ & \Gamma_{\tilde{F}/F} \end{tikzcd} \qquad \left( \text{resp } \begin{tikzcd} W_F \arrow[r, "\varphi"] \arrow[dr, "\mathrm{can}"'] & {}^cG(A) \arrow[d, "\mu"] \\ & A^\times \times \Gamma_{\tilde{F}/F} \end{tikzcd} \right)$$

<u>RK</u> to give an abstract L-par

$\Leftrightarrow$ give 1-cocycle $c : \underline{W_F} \longrightarrow \underline{\hat{G}(A)}$

$\left( c(xy) = c(x)\,(x \cdot c(y)) \right)$

<u>Goal</u> construct & study moduli stacks of

L-par / c-par $\leftarrow$ abstract L-par / c-par with extra continuity condition

whatever $C^{\mathrm{cont}}$ means, it must include triviality on an open subgroup of $P_F$ =

wild inertia $= \Gamma_{F^t} \leftarrow$ max tame ext of $F$
$\quad\quad\quad\quad\quad\quad \| \; Gal(\overline{F}/F^t)$

Structure of tame inertia / tame Weil / tame Gal gp

---

$I_F^t := I_F / P_F$ is procyclic $\xrightarrow{\sim} \hat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$

$P_F$: p-Sylow of $I_F$

$\hat{\mathbb{Z}}^p$ = profinite cpl of $\mathbb{Z}[\frac{1}{p}]$

$\Gamma_F^t = \Gamma_F / P_F$

Th (Iwasawa) $\Gamma_F \longrightarrow \Gamma_F^t$ has a continuous splitting so $\Gamma_F \cong P_F \rtimes \Gamma_F^t$

Also $\exists$

$$i : \Gamma_q = \langle \sigma, \tau \mid \sigma \tau \sigma^{-1} = \tau^q \rangle \hookrightarrow \Gamma_F^t$$

st $i(\sigma)$ is a lift of arith Frob

$\quad i(\tau)$ is a top gener. of $I_F^t$

this induces $\hat{\Gamma}_q \xrightarrow{\sim} \Gamma_F^t$ ($\hat{\Gamma}_q$: profcpl)

$$1 \to \tau^{\mathbb{Z}[\frac{1}{p}]} \longrightarrow \Gamma_q \longrightarrow \langle \sigma \rangle \simeq \mathbb{Z} \to 1$$

$\tau^{\mathbb{Z}[\frac{1}{p}]} \cong \mathbb{Z}[\frac{1}{p}] = \mathbb{Z}[\frac{1}{q}]$

$$\frac{1}{q^n} \longmapsto \sigma^{-n} \tau \sigma^n = \tau_n$$

<u>Fix one such $i$</u> $\rightsquigarrow$ stacks $L_{G,i} = L_{{}^cG,i}$

$L_{{}^LG,i}$

over $\mathbb{Z}[\frac{1}{p}]$, whose BC to $\mathbb{Z}_\ell$ $(\ell \neq p)$
will be canonically indep of $i$

Define $\Gamma_F^i \hookrightarrow W_F$ by

$$\begin{tikzcd}
1 \arrow[r] & P_F \arrow[r] \arrow[d, equal] & \Gamma_F^i \arrow[r] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\square"] & \Gamma_q \arrow[r] \arrow[d, hook, "i"] & 1 \\
1 \arrow[r] & P_F \arrow[r] & \Gamma_F \arrow[r] & \Gamma_F^t \arrow[r] & 1
\end{tikzcd}$$

---

If $Q = \mathrm{Gal}(L/F^t)$ is a finite quotient of $P_F$ st $\ker(P_F \to Q)$ acts trivially on $\hat{G}$ (a $p$-group)

get $1 \to Q \to \Gamma_Q^i \to \Gamma_q \to 1$ by pushout of the top sequence along $P_F \twoheadrightarrow Q$

$\Gamma_Q^i \xrightarrow{\sim} Q \rtimes \Gamma_q$

$$\begin{tikzcd}
& W_F \arrow[r, "\mathrm{can}"] & \mathbb{Z}[\frac{1}{p}]^\times \times \Gamma_{\tilde{F}/F} \text{ induces} \\
\Gamma_F^i \arrow[ur, hook, "i"] \arrow[r, two heads] & \Gamma_Q^i \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

a map can: $\Gamma^i_{\mathbb{Q}} \to \mathbb{Z}[\frac{1}{p}]^\times \times \Gamma_{\tilde{F}/F}$

<u>Key pt</u> $\Gamma^i_{\mathbb{Q}}$ is a discrete fin gen group

<u>Rk/interlude</u> $\Gamma$ abstract group (eg $\Gamma^i_{\mathbb{Q}}$)

$H/\Lambda$ affine sm gr sch over a ring

$\rightsquigarrow \exists$ natural affine scheme $R^{cl}_{\Gamma, H}$ with action of $H$ (induced by $H \underset{conj}{\circlearrowleft} H$) st $\forall$ classical $\Lambda$-alg $A$

$$R^{cl}_{\Gamma, H}(A) = \mathrm{Hom}_{gr}(\Gamma, H(A))$$

This defines a derived affine scheme $R_{\Gamma, H}$ whose classical scheme is $R^{cl}_{\Gamma, H}$

$$R_{\Gamma, H} = \varprojlim_{(FM(I) \to \Gamma)^{op}} H^I$$

$FM(I)$: free monoid on a set $I$ (finite)

not filtered but sifted!

limit in derived affine schemes

point really want $R_{FM(I),\mathbf{H}} = H^{I}$

<u>Back to our stacks</u> $\quad P_F \twoheadrightarrow Q$ as before

define $L^{\square}_{G,Q,i}$ by

$$\begin{tikzcd}
L^{\square}_{G,Q,i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R_{\Gamma^{i}_{Q},\, {}^{c}G} \arrow[d, "pr"] \\
\operatorname{Spec} \mathbb{Z}[\tfrac{1}{p}] \arrow[r, "con"] & R_{\Gamma^{i}_{Q},\, \mathbb{G}_m \times \Gamma_{\tilde{F}/F}}
\end{tikzcd}$$

via $\Gamma^{i}_{Q} \xrightarrow{con} \mathbb{Z}[\tfrac{1}{p}]^{\times} \times \Gamma_{\tilde{F}/F}$

Set $L^{\square}_{G,i} = \varinjlim_{Q} L^{\square}_{G,Q,i}$

$L_{G,i} = L^{\square}_{G,i}/\hat{G}$ ← acts by conjugation

$L_{G,Q,i} = L^{\square}_{G,Q,i}/\hat{G}$

<u>Tame version</u> if $G$ is tamely ramified

then $\tilde{F}/F$ is tame & $P_F$ acts trivially on $\hat{G}$

can take $Q=1$ above

$L^{t,\square}_{G,i} := L^{\square}_{G,\{1\},i}$ $\qquad L^{t}_{G,i} := L^{t,\square}_{G,i}/\hat{G}$

Replacing ${}^{c}G$ by ${}^{L}G$ produces similar objects $L_{{}^{L}G,i}$, $L^{t}_{G,i}$, ...

<u>**Th**</u> (Dat - Helm - Kurinczuk - Moss, Zhu, Fargues - Scholze)

1) $L^{\square}_{G,\varphi,i}$ is a classical affine scheme of fin type & flat / $\mathbb{Z}[\frac{1}{p}]$, reduced, local complete intersection, equidim of dim $= \dim \hat{G}$ $(= 1 + \dim G)$

transition maps in the presentation of $L^{\square}_{G,i} = \varinjlim_{\varphi} L^{\square}_{G,\varphi,i}$ are clopen immersions

so $L^{\square}_{G,i}$ is a classical scheme, disj union of classical aff sch ft / $\mathbb{Z}[\frac{1}{p}]$

See $L_{G,i} = \varinjlim_{\varphi} \underbrace{L_{G,\varphi,i}}_{\substack{\text{clas alg stack} \\ \text{rel dim } 0}}$ as ind-alg stack

2) $\forall \ell \neq p$ $\quad L^{\square}_{G} := L^{\square}_{G,i} \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathbb{Z}_\ell$

is canonically indep of $i$, more precisely
$\forall$ classical $\mathbb{Z}_\ell$-alg $A$

$$L_G^{\square}(A) \overset{1-1}{\longleftrightarrow} \{ \underset{\text{abstract}}{c}\text{-param } \varphi : W_F \to {}^cG(A)$$

strongly continuous ie $\forall\ \Gamma \underset{\text{cp open}}{\leq} W_F$

$$\forall\ {}^cG_{\mathbb{Z}_\ell} \hookrightarrow \underset{\substack{\wr\wr \\ GL_{d,\mathbb{Z}_\ell}}}{GL}(\underset{\substack{\wr\wr \\ \mathbb{Z}_\ell^d}}{M}) \qquad \forall\ v \in M \otimes_{\mathbb{Z}_\ell} A$$

$\Gamma v$ is contained in a fin gen $\mathbb{Z}_\ell$-submodule of $M \otimes_{\mathbb{Z}_\ell} A$ on which $\Gamma$ acts cont for the $\ell$-adic top

RK 1) essentially by construction

$$L_{G,i}^{t,\square} = \{ (g_\tau, g_\sigma) \in \underset{\substack{\cap \\ {}^cG \times {}^cG}}{\hat{G}\bar{\tau} \times \hat{G}\bar{\sigma}}$$

st $g_\sigma g_\tau g_\sigma^{-1} = g_\tau^q \}$

here $\bar{\tau}, \bar{\sigma}$ are images of $\tau, \sigma$ via

$$\Gamma_q \xhookrightarrow{i} W_F^t \xrightarrow{\text{can}} G_m(\mathbb{Z}[\tfrac{1}{p}]) \times \Gamma_{\tilde{F}/F}$$

$$\cap$$

$$G(\mathbb{Z}[\tfrac{1}{p}])$$

In particular $\exists$

$$L^{\square,t}_{G,i} \xrightarrow[\mu_1]{} \hat{G}\,\bar{\tau} \quad \rightsquigarrow \quad L^t_{G,i} \longrightarrow \hat{G}\bar{\tau}/\hat{G}$$

$$(g_\tau . g_\sigma) \longmapsto g_\tau$$

$$\downarrow$$

$$\hat{G}\,\bar{\tau}/\!/\hat{G}$$

If $\bar{\tau} = 1$ (eg if $\tilde{F}/F$ is nr no OK if $G$ is unram) get $L^t_{G,i} \longrightarrow \hat{G}/\!/\hat{G}$

can then define

$$L^u_{G,i} = L^t_{G,i} \times_{\hat{G}/\!/\hat{G}} \{1\}$$

$$L^{\hat{u}}_{G,i} := L^t_{G,i} \times_{\hat{G}/\!/\hat{G}} \widehat{\{1\}} \swarrow \text{formal completion}$$

$$L^u_{G,i} \not\hookrightarrow L^{\hat{u}}_{G,i} \longleftarrow \text{ind-alg st}$$

2) over $\mathbb{Z}_\ell[\sqrt{q}^{\pm}]$ $\exists$

$$L_{{}^LG} \xrightarrow{\sim} L_G := L_{G,i} \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathbb{Z}_\ell$$

$$\ddot{\shortparallel}$$

$$\overset{\|}{L_{{}^LG, i} \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathbb{Z}_\ell}$$ (indep of $i$)

$\varphi$ L-par corr to a 1-cocycle $\varphi_0$ $\longmapsto$ $\tilde{\varphi} = (\varphi_0 \cdot (2\rho) \circ \|-\|^{\frac{1}{2}}, \mathrm{can})$

$\underbrace{C_{\hat{G}}(\varphi)}_{\text{centralizer of } \varphi \text{ in } \hat{G}}$ & $C_{\hat{G}}(\tilde{\varphi})$ are conj by $(2\rho)(\sqrt{q})$

3) (relation to WD reps)

$\exists$ $\underset{\text{classical}}{\wedge}$ ind-scheme $WD^{\square}_{G} = \varinjlim_{Q \text{ fin quot of } I_F \text{ acting on } \hat{G}} WD^{\square}_{G, Q}$

classifying $A$ $\mathbb{Q}$-alg $\mapsto$ pairs $(r, X)$

$r : W_F \longrightarrow {}^cG(A)$ abstract c-param $C^0$ for $\underline{\underline{\text{disc}}}$ top on ${}^cG(A)$ ($\ker r|_{I_F}$ is open)

$X \in \hat{\mathcal{N}}_{\mathbb{Q}}(A)$ st $\mathrm{Ad}_{r(\gamma)} X = q^{\|\gamma\|} X$ $\forall \gamma$

$\hat{\mathcal{N}}_{\mathbb{Q}} \subset \mathrm{Lie}\,\hat{G}_{\mathbb{Q}}$ nilp cone

$WD^{\square}_{G,\mathcal{Q}}$ param such pairs st $r$ is trivial on $\underbrace{W_F/W_L}$ where $\mathcal{Q} = \mathrm{Gal}(L/F^{nr}) \twoheadleftarrow I_F$

fin gen group $\qquad \qquad \| \; \mathrm{Gal}(\bar{F}/F^{nr})$

$$1 \to \mathrm{Gal}(L/F^{nr}) \to W_F/W_L \to \mathbb{Z} \to 1$$

so $WD^{\square}_{G,\square}$ is an aff sch ft / $\mathbb{D}$.

$\exists$ map of $\underset{\text{ind}}{\wedge}$ schemes / $\mathbb{D}$ $\qquad WD_G := WD^{\square}_G / \hat{G}$

$$WD^{\square}_G \xrightarrow{\phi_i} L^{\square}_{G,i} \underset{\mathbb{Z}[\frac{1}{p}]}{\otimes} \mathbb{D}$$

$W_F$

$\cup \; (r, X) \longrightarrow \varphi$

$\varphi : \Gamma^i_F \longrightarrow {}^cG(A)$

$\gamma \longrightarrow r(i(\gamma))e^{dX}$ $\qquad e: \hat{N}_{\mathbb{D}} \to \hat{U}_{\mathbb{D}}$

$\qquad \| \qquad\qquad \|$

$\qquad \exp \qquad$ unip variety in $\hat{G}_{\mathbb{D}}$

$d \in \mathbb{Z}[\frac{1}{p}]$ st im of $\gamma$ in $T_q$ is $\sigma^{\|\gamma\|} \cdot \tau^d$

<u>Th</u> (Thm, version of Grothendieck's quasi-unip th) $\phi_i$ is an isom, so

$$WD_G \xrightarrow{\sim} L_{G,i} \underset{\mathbb{Z}[\frac{1}{p}]}{\otimes} D$$

this $\Rightarrow$ $L^{\text{coarse}}_{G,i} := \operatorname{Spf} H^0(L_{G,i}, \mathcal{O})$

$$\cong \varinjlim_{Q} \underbrace{\operatorname{Spec} H^0(L_{G,Q,i}, \mathcal{O})}_{L^{\square}_{G,Q,i} /\!/ \hat{G}}$$

is indep of $i$