Clermont
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①

Gpe de travail Zhu

Recall:

① (§9.2) For any prestack X over Λ we have a presentable, stable, Λ-linear ∞-category $QCoh(X)$ equipped with a t-structure.

For any morphism $f: X \to Y$ we have a (continuous) Λ-linear functor $f^*: QCoh(Y) \to QCoh(X)$.

We define f_* as its (potentially not continuous) right adjoint.

These define a sheaf theory

$$QCoh : Corr(PreStk_\Lambda)_{HR;\,All} \longrightarrow LinCat_\Lambda$$

HR: allows $*$-pushforwards for a class of morphisms which is not very explicit. One necessary condition is that f_* is continuous. This class contains all morphisms which are representable by qcqs alg. spaces, or more generally by concentrated stacks.

All: allows all pullbacks ($= *$-pullback)

Assume $\Lambda \supset \mathbb{Q}$ regular noetherian ring

② (§9.3.1) For an Artin stack X almost of finite presentation over Λ we have a presentable, stable, Λ-linear ∞-category $IndCoh(X)$ and equipped with a t-structure. ~~For any morphism $f: X \to Y$ we have a continuous functor $f_*: IndCoh(X) \to IndCoh(Y)$.~~

For any X we have a (continuous) Λ-linear t-exact functor

$$\Psi_X : IndCoh(X) \longrightarrow QCoh(X)$$

which restricts to an equivalence $IndCoh(X)^+ \xrightarrow{\sim} QCoh(X)^+$.

(afp = qcqs + lafp)

(§9.3.2) For any morphism $f: X \to Y$ we have a (continuous) Λ-linear functor $f_*^{IndCoh}: IndCoh(X) \longrightarrow IndCoh(Y)$.

If belongs to HR, then $\Psi_Y \circ f_*^{IndCoh} \simeq f_* \circ \Psi_X$.
(more generally if f_* is continuous)

For $f: X \to Y$ of finite tor-amplitude (i.e. f^* left t-exact up to a shift) we have a (continuous) Λ-linear functor $f^{IndCoh,*}: IndCoh(Y) \longrightarrow IndCoh(X)$, which is left adjoint to f_*^{IndCoh}.

These define a sheaf theory

$$IndCoh^*: Corr(IndArStk_\Lambda^{afp})_{All;\, ftor}$$

extended by left Kan extension (i.e. taking colimits)

morphisms representable by alg stacks and of finite tor-dimension

(§9.3.3) $\boxed{\text{Next goal}}$: define functors $f^{IndCoh,!}$ for all $f: X \to Y$.

Thm (9.39): There exists a sheaf theory

$$IndCoh^!: Corr(ArStk_\Lambda^{afp})_{Indafp;\, All} \longrightarrow LinCat_\Lambda$$

sending X to $IndCoh(X)$ and $X \xleftarrow{g} Z \xrightarrow{f} Y$ to

$$f_*^{IndCoh} \circ g^{IndCoh,!}: IndCoh(X) \to IndCoh(Y)$$

s.th.: (1) If g is an open embedding, $g^{IndCoh,!} = g^{IndCoh,*}$
(more generally: étale)

(2) If g is ind-proper, $g^{IndCoh,!}$ is right adjoint to g_*^{IndCoh}

Here Indafp: morphisms in $PreStk^{lafp}$ that are representable in $IndAlgSp_\Lambda^{afp}$

Rmk: From the existence of the theory IndCoh^*, we have functors f_*^{IndCoh} for all f. But compatibilities with the functors $f^{\mathrm{IndCoh},!}$ holds a priori only for morphisms in Indafp.

Important ingredients:

rather afp?

(1) For a separated morphism $f: X \to Y$ in $\mathrm{AlgSp}_\Lambda^{\mathrm{qcqs}}$, there exists a factorization $X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} Y$ with j qc open embedding and $\overline{f}$ proper (Lemma 9.38)

(2) If $f: X \to Y$ is (representable by alg. spaces and) proper, then f_*^{IndCoh} sends $\mathrm{Coh}(X)$ to $\mathrm{Coh}(Y)$, so its right adjoint $f^{\mathrm{IndCoh},!}: \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ is continuous and sends $\mathrm{IndCoh}(Y)^+$ to $\mathrm{IndCoh}(X)^+$ (cf. p. 316).

(Here $X, Y \in \mathrm{AlgStk}_\Lambda^{\mathrm{afp}}$.)

Other important property: If $f: X \to Y$ is smooth of relative dimension d, then $f^{\mathrm{IndCoh},!} \simeq \mathrm{Sym}^d(\mathbb{L}_{X/Y}[1]) \otimes f^{\mathrm{IndCoh},*}$.
In particular, if f is étale then $f^{\mathrm{IndCoh},!} \simeq f^{\mathrm{IndCoh},*}$.

Idea of proof:

(1) One restricts IndCoh^* to $\mathrm{AlgSp}_\Lambda^{\mathrm{afp}}$ (with only $*$-pushforwards). It extends to allow j^* for open embeddings (using left adjoints) and to all $f^{\mathrm{IndCoh},!}$ for proper morphisms (using right adjoints). The the factorization in (1) above allows to apply a general result to extend this to

$$\mathrm{IndCoh}^! : \mathrm{Corr}(\mathrm{AlgSp}_\Lambda^{\mathrm{afp}})_{\mathrm{All};\mathrm{sep}} \longrightarrow \mathrm{LinCat}_\Lambda$$

allow only separated morphisms for $(-)^{\mathrm{IndCoh},!}$

(2) We remove the separation requirement using the fact that for any morphism $f : X \to Y$ one can choose a cover $U \to X$ with U affine. Then $U \to X$ and $U \to Y$ are separated, and we have

$$\mathrm{IndCoh}(X) \xrightarrow{\sim} \mathrm{IndCoh}(U_\bullet)$$

with transition morphisms given by $(-)^* = (-)^!$ because étale maps

$\rightsquigarrow$ allows to define $f^!$ only in terms of separated morphisms.

(3) We restrict the theory to IndCoh^+, and then use right Kan extension to obtain

$$\mathrm{IndCoh}^{+,!} : \mathrm{Corr}(\mathrm{AlgStk}^{afp}_\Lambda)_{rp;All} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

morphisms that are representable by alg. spaces

One checks that this is the same (for categories and $*$-pushforward) as what was defined before.

(4) One can define $f^!$ for all morphisms using the restriction to Coh and the universal property of Ind-extension.

$\rightsquigarrow$ allows to get

$$\mathrm{IndCoh}^! : \mathrm{Corr}(\mathrm{AlgStk}^{afp}_\Lambda)_{rp;All} \to \mathrm{LinCat}_\Lambda$$

which coincides with IndCoh as before for categories & $*$-pushforwards.

(5) One extends to $\mathrm{Corr}(\mathrm{Ind}\cancel{\mathrm{AlgSt}}\mathrm{ArStk}^{afp}_\Lambda)_{\mathrm{Indafp};All}$ using right Kan extension. For a presentation $X = \mathrm{colim}_i\, X_i$ we have

$$\mathrm{IndCoh}^!(X) = \underset{\substack{i \\ \text{for } (-)^!}}{\lim}\, \mathrm{IndCoh}^!(X_i) = \underset{\substack{i \\ \text{for } (-)_*}}{\mathrm{colim}}\, \mathrm{IndCoh}^!(X_i)$$

So this the same as before. $\square$

Rmk: IndCoh and $(-)^!$ are also constructed by Gaitsgory (-Rozenblyum) when k is a field of char. 0. But this is not the same as the theory above. (Constructed for separated schemes, and then extended to all left prestacks by right Kan extension using $(-)^!$.)

$\boxed{\text{Grothendieck - Serre duality}}$ (§9.3.4)

Recall (Ex. 7.38): $\mathcal{A}$ commutative alg. in LinCat ($\mathcal{A} = \mathrm{Mod}_\Lambda$ in practice)

$\mathcal{C}$ an $\mathcal{A}$-algebra

A Frobenius structure on $\mathcal{C}$ is an $\mathcal{A}$-linear functor $\lambda : \mathcal{C} \longrightarrow \mathcal{A}$ such that

$$\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \xrightarrow{m} \mathcal{C} \xrightarrow{\lambda} \mathcal{A}$$

is the counit of a duality datum of $\mathcal{C}$.

Then we have an $\mathcal{A}$-module equivalence

$$\mathbb{D}^\lambda : \mathcal{C}^\vee \xrightarrow{\sim} \mathcal{C} \quad \text{such that} \quad \underset{\substack{\uparrow \\ \text{counit of duality}}}{e_{\mathcal{C}}}(c \boxtimes d) = \lambda(c \otimes \mathbb{D}^\lambda(d))$$

In case $\mathcal{C} = \mathrm{Ind}(\mathcal{M})$, we have $\mathcal{C}^\vee = \mathrm{Ind}(\mathcal{M}^{op})$, so $\mathbb{D}^\lambda$ must restrict to an equivalence $\mathcal{M}^{op} \xrightarrow{\sim} \mathcal{M}$

For $X \in \mathrm{IndArStk}^{afp}_\Lambda$, $\mathrm{IndCoh}(X)$ is symmetric monoidal with binary product

$$\mathcal{F} \otimes^! \mathcal{G} = \Delta_X^{\mathrm{IndCoh},!}(\mathcal{F} \boxtimes_\Lambda \mathcal{G})$$

for $\Delta_X : X \longrightarrow X \times_\Lambda X$ diagonal map.

The unit object is $\omega_X = \pi_X^{\mathrm{IndCoh},!}\Lambda$ for $\pi_X : X \to \mathrm{Spec}(\Lambda)$

$\boxed{\text{Thm (9.42): } (\pi_X)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{Mod}_\Lambda \text{ is a Frobenius structure}}$ 6

Consequence: We get $\mathbb{D}_X^{\mathrm{IndCoh}} : \mathrm{Coh}(X)^{op} \xrightarrow{\sim} \mathrm{Coh}(X)$: Grothendieck - Serre duality

Rmk: In case $\mathrm{IndCoh}(X \times X) \xleftarrow{\sim} \mathrm{IndCoh}(X) \otimes_\Lambda \mathrm{IndCoh}(X)$ this follows from the general formalism.
The unit of the duality datum is given by
$$(\Delta_X)_*^{\mathrm{IndCoh}}(\omega_X) \in \mathrm{IndCoh}(X) \otimes_\Lambda \mathrm{IndCoh}(X).$$

To ~~deal~~ prove the general case, Zhu uses a ~~de~~ construction of $\mathbb{D}_X^{\mathrm{IndCoh}}$ due to Gaitsgory in the setting of schemes (over k of char. 0 : in this case we do have $\mathrm{IndCoh}(X \times X) \simeq \mathrm{IndCoh}(X) \otimes_\Lambda \mathrm{IndCoh}(X)$, so we already know the thm, cf. Rmk 9.36).

First one assumes $X \in \mathrm{AlgSp}_\Lambda^{afp}$. From
$$- \otimes \mathcal{F} : \underset{\substack{\shortparallel \\ \mathrm{QCoh}(X)}}{\mathrm{IndPerf}(X)} \to \mathrm{IndCoh}(X)$$
by passing to the right adjoint we get
$$\mathcal{H}om(\mathcal{F}, -) : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$$
One checks that for $\mathcal{F} \in \mathrm{Coh}(X)$ we have
$$\mathcal{H}om(\mathcal{F}, \omega_X) \in \mathrm{Coh}(X)$$
$\rightsquigarrow$ allows to define $\underset{\substack{\shortparallel \\ \mathcal{H}om(-,\omega_X)}}{\mathbb{D}_X^{\mathrm{Coh}}} : \mathrm{Coh}(X)^{op} \to \mathrm{Coh}(X)$

One checks that this is involutive.

Then by descent one deduces a construction of $\mathbb{D}_X^{Coh}$ for any $X \in \mathrm{IndArStk}_\Lambda^{afp}$.

One checks that

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong (\pi_X)_* \left(\mathbb{D}_X^{Coh}(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}\right)$$

for $\mathcal{F}, \mathcal{G} \in Coh(X)$, ~~first for X eventually coconnective separated alg. space, then for X eventually coconnective alg. space~~.

This reduces to the case X is eventually coconnective ($\Rightarrow \mathcal{O}_X \in Coh(X)$). Then one treats separated alg. spaces, then general alg. spaces, then ~~general~~ alg. stacks, ~~then~~ ind-algebraic stacks.

implies that Δ_X is proper