Functoriality of IndCoh^{*}

I) Definition of $\mathrm{IndCoh}\,X$

Λ = regular noetherian ring

<u>Lemma</u> Let $X \in \mathrm{AlgStk}^{\mathrm{afp}}_{\Lambda}$ then

$$(\mathrm{QCoh}\,X)^{\heartsuit} = \underbrace{(\mathrm{QCoh}\,X_{cl})^{\heartsuit}}_{\text{usual abelian cat}}$$

Definition 9.19 We say $\mathcal{F} \in \mathrm{QCoh}\,X$ is coherent if:
- it is t-bounded
- each $H^{i}(\mathcal{F}) \in (\mathrm{QCoh}\,X_{cl})^{\heartsuit}$ are coherent.

$$\boxed{\begin{cases} \mathrm{IndCoh}\,X & := \text{ind completion of } \mathrm{Coh}\,X \\ X \in \mathrm{AlgStk}^{\mathrm{afp}}_{\Lambda} & \end{cases}}$$

Remark 9.20: 1) $\mathrm{Perf}(X) \subseteq \mathrm{Coh}\,X \iff \mathcal{O}_X \in \mathrm{Coh}\,X$ i.e. X eventually coconnective

- when X scheme $\mathrm{Coh}\,X = \mathrm{Perf}\,X \iff X$ classical & regular.

2) One can define $(\mathrm{QC}^{!}(X))_{X \in \mathrm{PStk}_{\Lambda}}$ as the right Kan ext. of IndCoh on affine spaces

$$\mathrm{IndCoh}\,X \longrightarrow \mathrm{QC}^{!}(X)$$

is an equivalence when $X \in \mathrm{AlgSp}^{\mathrm{afp}}_{\Lambda}$

or Λ classical $\oplus$ $X \in \mathrm{AlgStk}^{\mathrm{afp}}_{\Lambda}$ w/ affine diagonal

3) $(\mathrm{IndCoh}\,X)^{\omega} = \mathrm{Coh}X$.

II) t-structure on IndCoh X

$\mathrm{IndCoh}X^{\leq 0} := \mathrm{Ind}(\mathrm{Coh}X^{\leq 0})$.

· let $\Psi_X : \mathrm{IndCoh}X \longrightarrow \mathrm{QCoh}X$ be the ind completion of the inclusion $\mathrm{Coh}X \subseteq \mathrm{QCoh}X$

· Ψ_X is t-exact.

Lemma 9.21 The functor Ψ restricts to an equivalence

$$\Psi_X^{\geq m} : \mathrm{IndCoh}(X)^{\geq m} \cong \mathrm{QCoh}(X)^{\geq m}, \forall m$$

$$\Psi_X^{+} : \mathrm{IndCoh}(X)^{+} \cong \mathrm{QCoh}(X)^{+}$$

Sketch proof: · One can reduce to the case $m = 0$

· $\mathrm{Coh}(X)^{\geq 0}$ is compact in $\mathrm{QCoh}(X)^{\geq 0}$ and $\mathrm{QCoh}(X)^{\geq 0}$ is generated by $\mathrm{Coh}(X)^{\geq 0}$ w/ filtered colim

III) Internal Hom

There is a monoidal action

$$\mathrm{Perf}(X) \otimes \mathrm{Coh}(X) \to \mathrm{Coh}(X)$$
$$(\mathcal{E}, \mathcal{F}) \longmapsto \mathcal{E} \otimes \mathcal{F}$$

taking the ind completion one has:

$$- \otimes \mathcal{F} : \mathrm{IndPerf}(X) \longrightarrow \mathrm{IndCoh}(X)$$

admits a right adjoint.

$$\underline{\mathrm{Hom}}(\mathcal{F}, -) : \mathrm{IndCoh}(X) \longrightarrow \mathrm{IndPerf}(X)$$

when X is eventually coconnective ($\mathcal{O}_X \in \mathrm{Coh}X$)

$$\mathrm{Perf}X \subseteq \mathrm{Coh}X$$

$$\Xi_X : \mathrm{IndPerf}X \subseteq \mathrm{IndCoh}X$$

fully faithful embedding

when $X \in \mathrm{AlgSp}^{afp}_{\Lambda}$, $\mathrm{IndPerf}(X) = \mathrm{QCoh}(X)$

$$\Xi_X \quad \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X) \quad \Psi_X$$

$$\Psi_X = \underline{\mathrm{Hom}}(\mathcal{O}_X, -)$$

III) Functoriality of IndCoh

let $X, Y \in \mathrm{AlgStk}^{afp}_{\Lambda}$. $X \xrightarrow{f} Y$

lemma 9.26 (1) There is a unique functor $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}X \longrightarrow \mathrm{IndCoh}Y$

whose restriction to

$\mathrm{IndCoh}\,X^{+} \simeq \mathrm{QCoh}\,X^{+}$ coincide

with the functor $f_*|_{\mathrm{QCoh}X^{+}} : \mathrm{QCoh}X^{+} \to \mathrm{QCoh}Y^{+}$

<u>construction</u> f_* is right t-exact

$$\mathrm{Coh}\,X \xrightarrow{f_*} \mathrm{QCoh}\,Y^{+} \underset{\Psi_Y}{\simeq} \mathrm{IndCoh}\,Y^{+} \cap \mathrm{IndCoh}\,Y$$

ind complete to obtain

$$\mathrm{IndCoh}\,X \longrightarrow \mathrm{IndCoh}\,Y$$

<u>lemma 9.26 (2)</u> $X \xrightarrow{g} Y$ w/ g of finite tor amplitude (g^* is right t-exact up to a shift)

There exists a unique functor $g^{\mathrm{IndCoh}\,*} : \mathrm{IndCoh}\,Y \to \mathrm{IndCoh}\,X$ such that its restriction to $\mathrm{QCoh}\,Y^{+}$ coincide w/

$$g^*|_{\mathrm{QCoh}Y^{+}} : \mathrm{QCoh}Y^{+} \to \mathrm{QCoh}X^{+}.$$

<u>construction</u> same idea

$$\mathrm{Coh}\,Y \xrightarrow{g^*} \mathrm{QCoh}\,X^{+} \simeq \mathrm{IndCoh}\,X^{+} \cap \mathrm{IndCoh}\,X.$$

Lemma 9.26(3)

$$\begin{tikzcd} X' \arrow[r, hook, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

The Beck Chevalley map

$$g^{\mathrm{IndCoh},*} \circ f_*^{\mathrm{IndCoh}} \longrightarrow (f'_*)^{\mathrm{IndCoh}} \circ (g')^{\mathrm{IndCoh},*}$$

is an isomorphism.

sketch proof: it is an isomorphism on $\mathrm{Coh}(X)$, all computation remains in $\mathrm{QCoh}(Y')^+$.

Remark: → f_*^{IndCoh} is defined for all morphism in $\mathrm{AlgStk}_\Lambda^{\mathrm{aft}}$ (unlike f_*^{QCoh})

→ Denote by $R\Gamma^{\mathrm{IndCoh}}(X, \_)$: $(\pi_X)_*^{\mathrm{IndCoh}}: \mathrm{IndCoh}(X) \longrightarrow \mathrm{Mod}_\Lambda$

→ when f_*^{QCoh} is defined we have

$$\Psi_Y \circ f_*^{\mathrm{IndCoh}} \cong f_* \circ \Psi_X.$$

IV) The !-pullback

If $f : X \to Y$ is a proper morphism between algebraic stacks afp/Λ (resp. in $\mathrm{AlgSp}_\Lambda^{afp}$) then f_*^{IndCoh} sends $\mathrm{Coh}X$ to $\mathrm{Coh}Y$ so it is continuous.

$\boxed{\text{We denote by } f^{\mathrm{IndCoh},!} \text{ the right adjoint of } f_*^{\mathrm{IndCoh}} \text{ when } f \text{ is proper}}$

<u>Lemma</u> 9.27 Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cart. diagram in $\mathrm{AlgStk}_\Lambda^{afp}$ w/ f resp. and proper.

Then f' is proper and the Beck-Chevalley map is an isomorphism

$$(g')_*^{\mathrm{IndCoh}} \circ (f')^{\mathrm{IndCoh},!} \cong f^{\mathrm{IndCoh},!} \circ g_*^{\mathrm{IndCoh}}.$$

if g is of finite amplitude

$$(g')^{\mathrm{IndCoh},*} \circ f^{\mathrm{IndCoh},!} \cong (f')^{\mathrm{IndCoh}!} g^{\mathrm{IndCoh}*}$$

Proposition 9.28 Let $f: X \to Y$ be a morphism in $\mathrm{AlgStk}_\Lambda^{afp}$

1) if $X \in \mathrm{AlgSp}_\Lambda^{afp}$ and if f is proper and surjective (on the geometric points)

↳ then the essential image of f_*^{IndCoh} generates $\mathrm{IndCoh}(X)$

↳ then $f^{\mathrm{IndCoh},!}$ is conservative.

(φ iso $\iff$ $f^{\mathrm{IndCoh}!}\varphi$ iso)

2) $X \in \mathrm{AlgSp}_\Lambda^{afp}$ f smooth covering.

$$\mathrm{IndCoh}(Y) = \lim_{\text{*-pullback}} \mathrm{IndCoh}(X_\bullet)$$

3) f smooth covering.

$$\mathrm{IndCoh}(Y)^+ = \lim \mathrm{IndCoh}(X_\bullet)^+$$

Lemma 9.29 Let $f: X \to Y$ be a morphism in $\mathrm{AlgSp}_\Lambda^{afp}$ of finite tor amplitude

$$f_* \mathcal{E} \otimes \mathcal{F} \cong f_*^{\mathrm{IndCoh}} (\mathcal{E} \otimes f^{\mathrm{IndCoh},*} \mathcal{F})$$

we don't know if that holds for $\mathrm{AlgStk}_\Lambda^{afp}$

fails for $pt \to BH$, $\mathrm{Coh}(BH) = \mathrm{Perf}(BH)$
$= \mathrm{Rep}_c(H, \Lambda)$

but $\mathrm{IndCoh}(pt) \to \mathrm{IndCoh}(BH)$ is not essentially surjective.

H = finite group such that $|H| = 0$ in Λ.

V) The exterior product

Proposition 9.3.1 (1)

The exterior product

$$\boxtimes : QCoh(X) \otimes QCoh(Y) \longrightarrow QCoh(X \times Y)$$

sends coherent to coherent and induces a fully faithful embedding.

$$\boxtimes : IndCoh(X) \otimes IndCoh(Y) \longrightarrow IndCoh(X \times Y)$$

which admits a right adjoint $\boxtimes^R$.

$\boxtimes$ sends Coh to Coh

since $(Coh X)^\heartsuit = (Coh X_{cl})^\heartsuit$ we are reduced to the case X & Y classical.

then $\mathcal{F} \boxtimes \mathcal{G} = p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}$

& $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are of finite tor dimension so $\mathcal{F} \boxtimes \mathcal{G}$ is coherent if $\mathcal{F}$ and $\mathcal{G}$ are.

Proposition 9.3.1 (2)

let $f : X \to Z$ be a morphism in $AlgStk^{afp}$. Then the following diagram is commutative

$$\begin{tikzcd}
\mathrm{IndCoh}(X)\hat{\otimes}\mathrm{IndCoh}(Y) \arrow[r, "\boxtimes"] \arrow[d, "f_*^{\mathrm{IndCoh}}\otimes id"'] & \mathrm{IndCoh}(X\hat{\times}Y) \arrow[d, "(f\times id)_*^{\mathrm{IndCoh}}"] \\
\mathrm{IndCoh}(Z)\hat{\otimes}\mathrm{IndCoh}(Y) \arrow[r, "\boxtimes"] & \mathrm{IndCoh}(Z\hat{\times}Y)
\end{tikzcd}$$

if f is of finite amplitude then the following diagram is commutative

$$\begin{tikzcd}
\mathrm{IndCoh}(X)\hat{\otimes}\mathrm{IndCoh}(Y) \arrow[r, "\boxtimes"] & \mathrm{IndCoh}(X\hat{\times}Y) \\
\mathrm{IndCoh}(Z)\hat{\otimes}\mathrm{IndCoh}(Y) \arrow[r, "\boxtimes"] \arrow[u, "f^{\mathrm{IndCoh},*}\otimes id"] & \mathrm{IndCoh}(Z\hat{\times}Y) \arrow[u, "(f\times id)^{\mathrm{IndCoh},*}"']
\end{tikzcd}$$

proof: it is enough to show it on Coh

$$\begin{cases} f_*^{\mathrm{IndCoh}}\mathcal{F}\boxtimes\mathcal{G} \cong (f\times id)_*^{\mathrm{IndCoh}}(\mathcal{F}\boxtimes\mathcal{G}) \\ \mathcal{F}\in \mathrm{Coh}\,X \quad , \quad \mathcal{G}\in\mathrm{Coh}\,Y. \end{cases}$$

all sheaves in the formula are bounded below so this formula is deduced from the formula on QCoh^+.

VI) The Sheaf theory IndCoh*

Reminder $\mathrm{IndArtStk}_\Lambda^{afp}$ = ind Artin stack almost of finite presentation.

ftor := morphisms representable in AlgStk that are of finite tor dimension

Theorem 9.32 There is a sheaf theory

$$\mathrm{IndCoh}^* : \mathrm{Corr}(\mathrm{IndArtStk}_\Lambda^{afp})_{All, ftor} \longrightarrow \mathrm{Lincat}_\Lambda$$

sending X to $\mathrm{IndCoh}^*(X) = \mathrm{IndCoh}(X)$ and $Y \xleftarrow{g} Z \xrightarrow{f} X$ to $f_*^{\mathrm{IndCoh}} \circ g^{\mathrm{IndCoh},*}$

sketch proof : First we construct

$$\mathrm{Corr}(\mathrm{AlgStk}_\Lambda^{afp})_{All, iso} \longrightarrow \mathrm{Lincat}_\Lambda$$

step 1 : from the theory of QCoh

$$\mathrm{QCoh}_*: \mathrm{Corr}(\mathrm{AlgStk}_\Lambda^{afp})_{all, iso} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

cat of all categories.

we deduce a sheaf theory QCoh_*^+

$$\mathrm{QCoh}_*^+: \mathrm{Corr}(\mathrm{AlgStk}_\Lambda^{afp})_{all, iso} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

$$X \longmapsto (\mathrm{QCoh}\, X)^+$$

step 2: lemma 9.26 and proposition 9.31 gives a sheaf theory on the homotopy category.

$$h\,\mathrm{Corr}(\mathrm{AlgStk}_\Lambda^{qfp})_{All,iso} \longrightarrow h\underline{\mathrm{Lincat}}^{t,+}$$

subcategory of Lincat consisting of category equipped w/ a t-structure, accessible, compatible w/ filtered colim and compactly gen.

step 3 use the two following results.

(1) the functor $\mathrm{Lincat}^{t,+} \longrightarrow \widehat{\mathrm{Cat}}_\infty$

$(\mathcal{C}, \mathcal{C}^{\leq 0}) \longmapsto \mathcal{C}^+$

is lax symmetric monoidal.

(2) let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ a map of operad and $h\mathcal{D}^\otimes \to h\mathcal{C}^\otimes$ a map of operad then

$$\mathcal{D}^\otimes := h\mathcal{D}^\otimes \times_{h\mathcal{C}^\otimes} \mathcal{C}^\otimes \to \mathcal{C}^\otimes$$ is a map of operad.

Applying (2) to $\mathcal{C} = \mathrm{Corr}(\mathrm{AlgStk}_\Lambda^{qfp})_{all,iso}$

$\mathcal{D} = \mathrm{Lincat}^{t,+}$

$\mathcal{O} = \widehat{\mathrm{Cat}}_\infty$

we have.

$$\mathrm{Corr}(\mathrm{AlgStk}_\Lambda^{qfp})_{All,iso} \xrightarrow{\mathrm{Ind}\,\mathrm{Coh}_*} \mathrm{Lincat}.$$

step 4 we apply a left Kan extension along $\mathrm{AlgStk}_\Lambda^{afp} \longrightarrow \mathrm{Ind\,ArStk}_\Lambda^{afp}$ and we obtain

$$\mathrm{Corr}(\mathrm{Ind\,ArtStk}_\Lambda^{afp})_{All, iso} \longrightarrow \mathrm{Lincat}.$$

for horizontal arrows we can do the same and deduce

$$\mathrm{Corr}(\mathrm{Ind\,ArtStk}_\Lambda^{afp})_{All, fbr} \longrightarrow \mathrm{Lincat}$$

$\square$

VII) Formal completion

let $X \in \mathrm{AlgStk}_\Lambda^{afp}$ and $\hat{\imath} : \hat{Z} \to X$ be the formal completion of X along a closed subset Z.

then $\hat{Z} = \mathrm{colim}_a Z_a$ for $\iota_a : Z_a \to X$ closed embedd.

by definition (left Kan extension)

$$\mathrm{IndCoh}(\hat{Z}) = \mathrm{colim}_a \mathrm{IndCoh}(Z_a)$$

$*$-pushforward.

the functor $(\hat{\imath})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\hat{Z}) \longrightarrow \mathrm{IndCoh}(X)$

preserves compact objects.

so it is continuous and admits a right adjoint $(\hat{i})^{\mathrm{IndCoh},!}$

proposition 9.33 Let $U \subset X$ be the open complement. Then

$$\mathrm{IndCoh}(\hat{Z}) \xrightarrow{F} \mathrm{IndCoh}(X) \xrightarrow{G} \mathrm{IndCoh}(U)$$

is a localization sequence

(arrows admit right adjoint and $id \to F^R \circ F$ and $G \circ G^R \to id$ are iso.

moreover $F(F^R(c)) \to c \to G^R(G(c))$ is a fiber sequence)

VIII) When is $\boxtimes$ an equivalence?

$$\mathrm{IndCoh}\,X \underset{\Lambda}{\otimes} \mathrm{IndCoh}\,Y \xrightarrow{\boxtimes} \mathrm{IndCoh}(X \underset{\Lambda}{\times} Y)$$

corollary: $\boxtimes$ is an equivalence if X admits finite filtration

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots$$

$$Y = Y_0 \supset Y_1 \supset Y_2 \supset \cdots$$

by closed substacks such that

$$\mathrm{IndCoh}((X_i \setminus X_{i+1})_{red}) \underset{\wedge}{\otimes} \mathrm{IndCoh}((Y_i \setminus Y_{i+1})_{red})$$

$$\downarrow$$

$$\mathrm{IndCoh}((X_i \setminus X_{i+1})_{red} \underset{\wedge}{\times} (Y_j \setminus Y_{i+1})_{red})$$

is essentially surjective.

We can deduce other case using the following lemma:

<u>lemma 8.20</u>: We suppose $\boxtimes$ fully feithfull. Let $X \in \mathcal{C}$ such that

$$\pi_X : X \longrightarrow pt \in \text{Vertical}$$

$$\Delta_X : X \longrightarrow X \times X \in \text{Horizontal}.$$

then the following are equivalent

(1) $D(X) \underset{D(pt)}{\otimes} D(Y) \longrightarrow D(X \times Y)$ is an equiv for Y such that

$\pi_Y \in \text{Ver}$

$\Delta_Y \in \text{Horiz}.$

(2) $D(X) \underset{D(pt)}{\otimes} D(X) \longrightarrow D(X \times X)$ is an equivalenc.

is fully faithfull and $(\Delta_X)_* \Lambda_X$ belongs to the essential image of $\boxtimes_{D(pt)}$

<u>proposition</u> 9.35

(1) $X, Y \in \mathrm{AlgSp}_{\Lambda}^{op}$, X is smooth over Λ and Y is regular

(2) $X = Y = BG$ where G is a smooth affine algebraic group over Λ a field.

<u>sketch proof</u> : case (1): since $X \times_{\Lambda} Y$ is regular, the statement follows from the QCoh case.

case (2), we use (3). it is enough to show that. $\mathcal{O}_G$ belongs to

$$\mathrm{IndCoh}(BG) \otimes \mathrm{IndCoh}(BG).$$

regarded as a $G \times G$ rep.

but $\mathcal{O}_G$ admits a filtration with associated graded being $V_1 \boxtimes$ where V_1, V_2 are representation of G.