DAG

- Λ : ordinary comm. ring

CAlg_Λ : cat. of animated Λ-rings
(i.e. simplicial comm. Λ-rings)

$\overset{\text{Dold-Kan}}{\underset{\text{over } \mathbb{Q}}{\Longleftrightarrow}}$ connective comm. dg-Λ-algs

$\mathrm{CAlg}_\Lambda^\heartsuit$: cat. of ordinary comm. Λ-rings

$A \in \mathrm{CAlg}_\Lambda \rightsquigarrow \pi_0(A) \in \mathrm{CAlg}_\Lambda^\heartsuit$.

- prestack : (accessible) functor $\mathrm{CAlg}_\Lambda \to \underset{\infty\text{-Gpd}}{\overset{}{\mathrm{Ani}}}$ ($\mathrm{Ani} = \infty$-Gpd)

$\rightsquigarrow$ PreStk_Λ

stack : a prestack which is a sheaf w.r.t. étale topology on CAlg_Λ .

via left Kan extension

$$\mathrm{PreStk}_\Lambda \to \mathrm{PreStk}_\Lambda^{cl} \subset \mathrm{Fun}(\mathrm{CAlg}_\Lambda^\heartsuit, \mathrm{Ani})$$
$$X \mapsto X_{cl}$$

e.g. $(\mathrm{Spec}\, A)_{cl} = \mathrm{Spec}\, \pi_0(A)$.

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$$\begin{tikzcd} \mathrm{CAlg}_\Lambda^{op} \arrow[r, hook] \arrow[dr, "|\cdot|"'] & \mathrm{PreStk}_\Lambda \arrow[d, "\text{left Kan extension}"] \\ & \mathrm{Top} \end{tikzcd}$$
$A \mapsto |\mathrm{Spec}\, \pi_0(A)|$

i.e. $|X| = \varinjlim_{\mathrm{CAlg}_\Lambda^{op}/X} |\mathrm{Spec}\, \pi_0(R)|$

$= |X_{cl}|$.

$Y \to X$ is an open embedding of prestacks

if $\exists$ open $U \subset |X|$ s.t. $\forall A \in \mathrm{CAlg}_\Lambda$,

$$Y(A) \simeq X(A) \times_{\mathrm{Map}(|\mathrm{Spec} A|, |X|)} \mathrm{Map}(|\mathrm{Spec} A|, U) .$$

scheme : a stack admitting an open covering by affine schemes

X : scheme $\rightsquigarrow X_{cl}$: ordinary scheme

algebraic space : a stack admitting an étale covering by schemes

- $A \to B$ flat if $\pi_0(A) \xrightarrow{\text{flat}} \pi_0(B)$ & $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \simeq \pi_i(B)$
- $A \to B$ Zariski open, étale, smooth, faithfully flat if flat & $\pi_0(A) \to \pi_0(B)$ is so
- could define Zariski open, étale ... morphisms for schemes, as they are local properties
- extends to representable morphisms between stacks

algebraic stack : a stack X s.t.

$X \to X \times_\Lambda X$ is representable by algebraic space

and $\exists$ smooth surjection $U \to X$ with
i.e. surjection after $|\cdot|$
U alg. space

$$\mathrm{Sch}_\Lambda \subset \mathrm{AlgSp}_\Lambda \subset \mathrm{AlgStk}_\Lambda$$

- $A \in \mathrm{CAlg}_\Lambda$ is called

<u>almost of finite presentation</u> (afp) if

$$\tau_{\le n} A \in \tau_{\le n} \mathrm{CAlg}_\Lambda = \{ R \mid \pi_m(R) = 0 \}$$

is compact.

e.g. if Λ noetherian, A is afp iff

$\pi_0(A)$ is f.g. and $\pi_i(A)$ is f.g.

module of $\pi_0(A)$.

$$\mathrm{PreStk}_\Lambda^{\mathrm{lafp}} := \mathrm{Fun}(\mathrm{CAlg}_\Lambda^{\mathrm{afp}}, \mathrm{Ani})$$

$$\underset{\substack{\text{via} \\ \text{left Kan extension}}}{\subset} \mathrm{PreStk}_\Lambda .$$

$$\mathrm{Sch}_\Lambda^{\mathrm{afp}} := \mathrm{Sch}_\Lambda^{\mathrm{qcqs}} \cap \mathrm{PreStk}_\Lambda^{\mathrm{lafp}}$$

$$\mathrm{AlgSp}_\Lambda^{\mathrm{afp}} := \mathrm{AlgSp}_\Lambda^{\mathrm{qcqs}} \cap \mathrm{PreStk}_\Lambda^{\mathrm{lafp}}$$

$$\mathrm{AlgStk}_\Lambda^{\mathrm{afp}} := \mathrm{AlgStk}_\Lambda^{\mathrm{qcqs}} \cap \mathrm{PreStk}_\Lambda^{\mathrm{lafp}} .$$

- ind-scheme

(resp. ind-algebraic space, ind-alg. stk) :

$X \in \mathrm{PreStk}_\Lambda^{\mathrm{lafp}}$ s.t.

- nilcomplete, i.e. $X(A) = \varprojlim_n X(\tau_{\le n} A)$
- $X = \varinjlim_i X_i$ (filtered colimit) with $X_i \in \mathrm{Sch}_\Lambda^{\mathrm{afp}}$

(resp. $\mathrm{AlgSp}_\Lambda^{\mathrm{afp}}$, $\mathrm{AlgStk}^{\mathrm{afp}}$), and

closed transition maps.

$$\mathrm{IndSch}_\Lambda^{\mathrm{afp}} \subset \mathrm{IndAlgSp}_\Lambda^{\mathrm{afp}} \subset \mathrm{IndArStk}_\Lambda^{\mathrm{afp}}$$

<u>Example</u> (1) $X \in \mathrm{AlgStk}_\Lambda^{\mathrm{afp}}$, $Z \overset{\text{closed}}{\subset} X$.

$$X^\wedge_Z(A) := X(A) \times_{\mathrm{Map}(|\mathrm{Spec} A|, |X|)} \mathrm{Map}(|\mathrm{Spec} A|, |Z|) .$$

Then $X^\wedge_Z = \varinjlim_i Z_i$.

Z_i: afp closed substack of X s.t. $|Z_i| = |Z|$.

(2) H : group stack $\circlearrowright$ X : stack

$$X/H := (X/H)_{\mathrm{ét}} \in \mathrm{Stk}_\Lambda$$

$$\mathbb{B}H := \mathrm{pt}/H .$$

<u>QCoh</u>

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$$\begin{tikzcd} \mathrm{CAlg}_\Lambda \arrow[r, hook] \arrow[dr] & \mathrm{PreStk}_\Lambda^{\mathrm{op}} \arrow[d, "\mathrm{QCoh}"] \\ & \mathrm{Lincat}_\Lambda \end{tikzcd}$$
right Kan extension, $A \mapsto \mathrm{Mod}_A$

i.e. $\mathrm{QCoh}(X) = \varprojlim_{A \in \mathrm{CAlg}_{\Lambda/X}^{\mathrm{op}}} \mathrm{Mod}_A$.

also

$$\begin{tikzcd} \mathrm{CAlg}_\Lambda \arrow[r, hook] \arrow[dr] & \mathrm{PreStk}_\Lambda^{\mathrm{op}} \arrow[d, "\mathrm{Perf}"] \\ & \mathrm{Lincat}_\Lambda \end{tikzcd}$$
$A \mapsto \mathrm{Perf}_A = \{\text{perfect complexes in } \mathrm{Mod}_A\}$.

$f : X \to Y \rightsquigarrow$

$$\begin{array}{cccc} f^* : & QCoh(Y) & \to & QCoh(X) \\ & \cup & & \cup \\ & Perf(Y) & \to & Perf(X) \end{array} .$$

- $\exists$ t-str. s.t.

$$QCoh(X)^{\leq 0} = \varprojlim_{A \in CAlg^{op}_{\Lambda/X}} Mod_A^{\leq 0}$$

and f^* is right t-exact.

$\Rightarrow$ f_* (right adjoint to f^*) is left t-exact.

<u>Rmk</u> It gives a sheaf theory

$$Corr(PreStk_\Lambda)_{HR; All} \to LinCat_\Lambda .$$

if $X \in AlgStk_\Lambda$,

$$QCoh(X)^{\heartsuit} = QCoh(X_{cl})^{\heartsuit} .$$