6-functor formalisms

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HM stands for the paper of Heyer and Mann on 6-functor formalisms and smooth representations, LZ for the paper of Liu-Zheng, S for Scholze's lecture notes on 6-functor formalisms.

0.1 The category of correspondences

Let C be a category (i.e. ∞-category) having finite limits, with final object $*$, and endow C with the Cartesian monoidal structure, for which it becomes a symmetric monoidal category. We start by introducing a bunch of dry definitions, but it is convenient to think of weakly stable (see below) classes as classes of morphisms f for which f^* (resp. $f_!$) are defined. More precisely in the context of Zhu's work H (for horizontal) is the class of morphisms f for which f^* is defined, and V (for vertical) is the class of morphisms for which $f_!$ is defined. So for a geometric setup (C, E), E is the class of morphisms f for which $f_!$ is defined, also called !-able morphisms. In practically all situations of Zhu's paper we will deal with geometric setups, but in the IndCoh^* theory this is not the case. In the ℓ-adic setting we will however always deal with geometric setups.

Definition 0.1. a) A (homotopy) class of morphisms E in C is called *weakly stable* if E contains all isomorphisms and is stable under compositions and arbitrary base change by morphisms in C. If E is weakly stable we let C_E be the wide subcategory of C with morphisms E.

b) A weakly stable class E is called *strongly stable* if moreover whenever $g, g \circ f \in E$ we also have $f \in E$ (in other words if $X \to S, Y \to S$ are in E, then any map $X \to Y$ over S is in E as well).

c) A *weak setup* is a triple (C, V, H) where C is a category with finite limits and V, H weakly stable classes in C.

d) A *geometric setup* is a weak setup in which V is strongly stable and H consists of all morphisms in C. We simply denote it (C, V), and we often write E instead of V. A morphism of geometric setups $(C, E) \to (C', E')$ is a functor $F : C \to C'$ inducing a functor $C_E \to C'_{E'}$ and such that $F(X \times_S Y) \simeq F(X) \times_{F(S)} F(Y)$ whenever $X \to S \in E$.

Remark 0.2. 1. A weakly stable class E is strongly stable if and only if E is closed under relative diagonals (i.e. whenever $f : X \to Y \in E$ we also have $\Delta_f : X \to X \times_Y X \in E$), if and only if C_E has fibre products and the inclusion $C_E \subset C$ preserves fibre products. See lemma 2.1.5 in HM.

Fix now two weakly weakly stable classes of morphisms V, H in C. A classical construction going back to Barwick associates to this a new category

$$\mathrm{Corr} = \mathrm{Corr}_{V;H}$$

having the same objects as C and 1-morphisms from X to Y are given by correspondences $X \leftarrow Z \rightarrow Y$, with $f : Z \rightarrow X \in H$ and $g : Z \rightarrow Y \in V$. Think of this correspondence as encoding the functor $g_! f^*$. Composition is defined as one expects, namely

$$(X_1 \leftarrow Y \rightarrow X_2) \circ (X_2 \leftarrow Z \rightarrow X_3) = X_1 \leftarrow Y \times_{X_2} Z \rightarrow X_3,$$

where the maps from the fiber product to X_1, X_3 are the projections to Y, Z composed with $Y \rightarrow X_1$ and $Z \rightarrow X_3$ (they are still in H and V).

The category Corr inherits a symmetric monoidal structure from C, given on objects by $X \otimes Y = X \times Y$ (product computed in C) and has the key property that it comes with symmetric monoidal (non-full) embeddings

$$C_H^{\mathrm{op}} \rightarrow \mathrm{Corr} \leftarrow C_V.$$

On objects both functors send X to X. The functor $C_H^{\mathrm{op}} \rightarrow \mathrm{Corr}$ sends $f : X \rightarrow Y$ to $Y \leftarrow X \rightarrow X$ ($X \rightarrow X$ being the identity of X), and the functor $\mathrm{Corr} \rightarrow C_V$ sends $f : X \rightarrow Y$ in C_V to $X \leftarrow X \rightarrow Y$.

Here are some other key features of Corr. Call $X \in C$ H-nice if the canonical map $\pi_X : X \rightarrow *$ is in H, and so does the diagonal $\Delta_X : X \rightarrow X \times X$. Thus for a geometric setup all X's are H-nice. If X is H-nice then π_X and Δ_X turn X into a commutative algebra object in C_H^{op}. Moreover any map $f : X \rightarrow Y$ between H-nice objects is in H and is a map $Y \rightarrow X$ of commutative algebra objects in C_H^{op}, making X an Y-module. If moreover $f \in V$ then f becomes a morphism of Y-modules in Corr (via the embedding of C_H^{op} in Corr).

Remark 0.3.

1. (HM lemma 2.2.11) Mapping (C, E) to $\mathrm{Corr}(C, E) := \mathrm{Corr}(C)_{\mathrm{All},E}$ gives a functor from the category of geometric setups to symmetric monoidal categories with lax symmetric monoidal functors.
2. Suppose that E is strongly stable and that $X \rightarrow * \in E$, then also $\Delta : X \rightarrow X \times X \in E$ and one checks (HM 2.3.9) that the correspondences $X \times X \leftarrow X \rightarrow *$ and $* \leftarrow X \rightarrow X \times X$ are the evaluation and unit exhibiting a self-duality of X in $\mathrm{Corr}(C, E)$. In particular, if $E = \mathrm{All}$ then $\mathrm{Corr}(C)$ is closed symmetric monoidal, since every object is self-dual, moreover we have $\underline{\mathrm{Hom}}(X, Y) = Y \times X$ in $\mathrm{Corr}(C)$. There are versions for arbitrary weak setups (C, H, V): if $\pi_X, \Delta_X \in V \cap H$, then X is dualizable in $\mathrm{Corr}(C)_{V,H}$ (again, the unit and counit of the duality datum are given by the maps π_X and Δ_X), and more precisely it is self-dual.

0.2 Target categories

We will need to work with several categories as targets of our 3 (or 6)-functor formalisms (that Zhu calls sheaf theories).

• the category $\widehat{\mathrm{Cat}}$ of not necessarily small categories, endowed with the Cartesian symmetric monoidal structure.

• the subcategory Pr^L of presentable categories with continuous (i.e. colimit-preserving) functors, endowed with Lurie's symmetric monoidal structure.

• the full subcategory $\mathrm{LC} = \mathrm{LinCat}$ of Pr^L consisting of presentable stable (or linear) categories. It is stable under the Lurie tensor product so becomes a symmetric monoidal category, and the functor $\mathrm{Pr}^L \to \widehat{\mathrm{Cat}}$ is lax symmetric monoidal.

• the sub-category $\mathrm{LC}^{\mathrm{cg}}$ of LC consisting of compactly generated linear categories, with continuous functors that preserve compact objects. It is also stable under the Lurie tensor product, thus a symmetric monoidal category.

• the subcategory $\mathrm{LC}^{\mathrm{perf}}$ of $\widehat{\mathrm{Cat}}$ of small stable idempotent complete categories, with exact functors. The ind-completion functor

$$\mathrm{Ind} : \mathrm{LC}^{\mathrm{perf}} \to \mathrm{LC}^{\mathrm{cg}}$$

is an equivalence, with quasi-inverse given by passage to the full subcategory of compact objects. We turn $\mathrm{LC}^{\mathrm{perf}}$ into a symmetric monoidal category by declaring Ind symmetric monoidal.

Let Λ be a commutative ring. The derived category Mod_Λ of Λ-modules is a commutative algebra object in LC and we let LC_Λ be the category of Mod_Λ-modules in LC. It is a symmetric monoidal category itself, with unit Mod_Λ, and we denote by $\otimes_\Lambda$ its monoidal structure. The objects of LC_Λ are called Λ-*linear categories* and the morphisms in LC_Λ are called Λ-linear functors. We similarly obtain $\mathrm{LC}^{\mathrm{cg}}_\Lambda$, etc.

0.3 Unwinding the definition of a 3-functor formalism

Definition 0.4. Let (C, V, H) be a weakly stable setup (for instance a geometric setup), with associated category of correspondences Corr. A 3-functor formalism on (C, V, H) is a lax symmetric monoidal functor

$$D : \mathrm{Corr} \to \widehat{\mathrm{Cat}}.$$

Such a formalism is called presentable (resp. Λ-linear, resp. Λ-linear compactly generated, etc) if it factors through a lax symmetric monoidal functor (still denoted) $D : \mathrm{Corr} \to \mathrm{Pr}^L$ (resp. $D : \mathrm{Corr} \to \mathrm{LC}_\Lambda$, resp. $D : \mathrm{Corr} \to \mathrm{LC}^{\mathrm{cg}}_\Lambda$).

Let D be a 3-functor formalism on (C, V, H). Composing with the (non-full) lax symmetric monoidal embeddings $C_H^{\mathrm{op}} \to \mathrm{Corr}$ and $C_V \to \mathrm{Corr}$ yields lax symmetric monoidal functors

$$D^* : C_H^{\mathrm{op}} \to \widehat{\mathrm{Cat}}, \quad D_! : C_V \to \widehat{\mathrm{Cat}}.$$

For $f : X \to Y \in C_H$ let $f^* = D^*(f) : D(Y) \to D(X)$ and for $f : X \to Y \in C_V$ let $f_! = D_!(f) : D(X) \to D(Y)$ the resulting functors, so that

$$f^* = D(Y \leftarrow X \to X), \ f_! = D(X \leftarrow X \to Y).$$

By definition, for any $X \in C$ we are given a canonical functor

$$\boxtimes : D(X) \times D(X) \to D(X \times X),$$

called external tensor product. If D is Λ-linear, it factors through a functor

$$\boxtimes_\Lambda : D(X) \otimes_\Lambda D(X) \to D(X \times X).$$

Remark 0.5. D is presentable if and only if all $D(X)$ are presentable and all $f_!, g^*$ are continuous. D is Λ-linear if and only if all $D(X)$ are Λ-linear categories and all $f_!, g^*$ are Λ-linear functors.

Recall that $X \in C$ is H-nice if $\pi_X : X \to *, \Delta_X : X \to X \times X$ are in H, and then X is a commutative algebra object in C_H^{op}, thus also in Corr. Since D is lax symmetric monoidal, it follows that $D(X)$ is naturally a symmetric monoidal category, whose unit is denoted 1_X. Concretely, $1_X = \pi_X^* 1_*$ and

$$F \otimes G = \Delta_X^*(F \boxtimes G), \quad F, G \in D(X).$$

Now take $f : X \to Y$ a map between H-nice objects. As f is a map between commutative algebra objects in C_H^{op}, f^* is symmetric monoidal, and endows $D(X)$ with a structure of $D(Y)$-module. If moreover $f \in V$ then f is a map of Y-modules in Corr, thus $f_!$ is a $D(Y)$-linear functor, which concretely translates into canonical isomorphisms (called projection formula)

$$f_!(F \otimes f^* G) \simeq f_! F \otimes G.$$

Definition 0.6. A 3-functor formalism D is called a 6-functor formalism if all functors $f_!$ (with $f \in V$), f^* (with $f \in H$) admit right adjoints, and if all symmetric monoidal categories $D(X)$ (with X H-nice) are closed, i.e. have internal Hom's.

Recall that any continuous functor between presentable categories has a right adjoint. Using this, one gets (see HM lemma 3.2.5 and Zhu 8.22 (2)):

Proposition 0.7. *A* 3*-functor formalism D is presentable (resp. Λ-linear) if and only if all $D(X)$ are in Pr^L (resp. LC_Λ) and all functors $f^*, g_!$ (for $f \in H, g \in V$) are continuous (resp. are Λ-linear), in which case D is automatically a* 6*-functor formalism.*

In particular for any Λ-linear D we can talk about the right adjoint f_* of f^* for all $f \in H$ and about the right adjoint $f^!$ of $f_!$ for all $f \in V$. Note however that these are not necessarily continuous, but they do interact well with inner Hom's (when these exist):

Proposition 0.8. *If $f : X \to Y$ is a map between H-nice objects then there are canonical isomorphisms*

$$\underline{\mathrm{Hom}}(F, f_* G) \simeq f_* \underline{\mathrm{Hom}}(f^* F, G).$$

If moreover $f \in V$ there are also natural isomorphisms

$$f^! \underline{\mathrm{Hom}}(F, G) \simeq \underline{\mathrm{Hom}}(f^* F, f^! G), \quad f_* \underline{\mathrm{Hom}}(F, f^! G) \simeq \underline{\mathrm{Hom}}(f_! F, G).$$

Proof. Let's do the second one, the others are done in a similar way. For arbitrary $A \in D(X)$ we compute, using adjunctions and the projection formula

$$\mathrm{Hom}(A, f^!\underline{\mathrm{Hom}}(F,G)) \simeq \mathrm{Hom}(f_!A, \underline{\mathrm{Hom}}(F,G)) \simeq \mathrm{Hom}(f_!A \otimes F, G)$$

$$\simeq \mathrm{Hom}(f_!(A \otimes f^*F), G) \simeq \mathrm{Hom}(A \otimes f^*F, f^!G) \simeq \mathrm{Hom}(A, \underline{\mathrm{Hom}}(f^*F, f^!G))$$

and we are done using Yoneda. □

0.4 Liu-Zheng and Mann's construction of 3-functor formalisms "for free"

Essentially all 3-functor formalisms constructed in real-life come from the following data:

- we are given a geometric setup (C, E) and a functor

$$D_0 : C^{\mathrm{op}} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}),$$

encoding the functors $(-)^*$ and $\otimes$. We write $f^* = D(f)$ for a morphism f in C.

- we are also given two **strongly** stable classes[1] of morphisms I and P such that all $f \in I \cap P$ are n-truncated for some $n \geq -2$ (depending on f) and of the form $f = pj$ with $p \in P, j \in I$ (we could also ask that any $f \in E$ is of the form $f = jp$). The critical assumptions we make are:

a) if $j : X \to Y \in I$ then j^* has a left adjoint $j_!$ compatible with base change and satisfying the projection formula. [2]

b) if $p : X \to Y \in P$ then p^* has a right adjoint p_* compatible with base change and satisfying the projection formula.[3]

c) the functors $j_!$ and p_* are compatible in cartesian diagrams, i.e. if $p : X \to Y \in P$ and $j : Y' \to Y \in I$ with corresponding base-changes $p' : X' \to Y'$ and $j' : X' \to X$, the Beck-Chevalley map $j_!p'_* \to p_*j'_!$ is an isomorphism.

In practice a) and c) are easy (and very often once a) and b) are proved part c) is automatic, for instance this is the case if all $j \in I$ are monomorphisms), and the whole work is in part b), i.e. proving proper base change. See HM prop. 3.3.3 for the proof of the following deep result.

Theorem 0.9. *(Liu-Zheng, Mann) Under the above assumptions there is a unique* 3-*functor formalism D on (C, E) such that $p_! = p_*$ for $p \in P$, $j_! = j_!$ for $j \in J$ and the proper base change isomorphism encoded by D is given by the Beck-Chevalley map.*

[1] We don't need to ask that I and P contain all isomorphisms and are stable under equivalences, but this happens in practice.

[2] This means that if $f : Y' \to Y$ is any map in C, with base change $j' : X' = Y' \times_Y X \to Y'$ and projection $f' : X' \to X$, then the Beck-Chevalley map $j'_!(f')^* \to f^*j_!$ is an isomorphism of functors, and moreover that the natural map $j_!(j^*F \otimes G) \to F \otimes j_!G$ is an isomorphism.

[3] i.e. for any $f : Y' \to Y$ in C with base change $p' : X' = Y' \times_Y X \to Y'$ and projection $f' : X' \to X$ the Beck-Chevalley map $f^*p_* \to (p')_*(f')^*$ is an isomorphism and the natural map $F \otimes p_*G \to p_*(f^*F \otimes G)$ is an isomorphism.

0.5 An example: étale sheaves

0.5.1 Étale (complexes of) sheaves

Take C the category of qcqs schemes, and E the class of morphisms $f : X \to Y$ of finite expansion in C, which concretely means that f factors $X \to X' \to Y$ with $X \to X'$ integral and $X' \to Y$ separated finitely presented, in particular E contains all separated finitely generated maps, but also perfections of such maps, or pro-(finite étale) maps of qcqs schemes. We take for I the class of open immersions and for P the class of "proper" maps in E, i.e. those which are universally closed. For D_0 we fix a *torsion* ring Λ and we let $D_0(X) = D(X_{\mathrm{et}}, \Lambda)$ be the derived ∞-category of sheaves of Λ-modules on X_{et}, which identifies with the category of $\mathrm{Mod}_\Lambda = D(\Lambda)$-valued hypersheaves on X_{et}.

Theorem 0.10. *The assumptions in the LZM machinery (theorem 0.9) are satisfied, and so there is a* 3*-functor formalism*

$$D : \mathrm{Corr}(\mathrm{Sch}^{\mathrm{qcqs}}, E) \to \mathrm{LC}_\Lambda$$

such that $j_!$ is the left adjoint of j^* for $j \in I$ and $p_!$ is the right adjoint p_* of p^* for $p \in P$.

Proof. This is Proposition 7.16 in Scholze's appendix to lecture VII. After proving an analogue of Nagata compactification in this setting (due to Hamacher), the tricky part is showing that for any $p : X \to Y \in P$ the functor p_* is continuous, satisfies the projection formula and is compatible with arbitrary pullbacks by morphisms of schemes. It turns out that p is automatically the composition of an integral map and a finitely presented proper map. By limit arguments we are reduced to the case when p is proper finitely presented. In this case writing Y as cofiltered inverse limit with affine transitions of schemes of finite type over $\mathbf{Z}$ the map p is the base-change of a proper map between schemes of finite type over $\mathbf{Z}$. In this case p_* has finite cohomological dimension, so we can reduce to complexes in D^+, and then to sheaves of Λ-modules, i.e. to the classical setting in SGA. □

Remark 0.11. Instead of taking a torsion ring Λ one could work as in Scholze's lecture with $D_{\mathrm{pf}}(\mathbf{Z})$-valued sheaves on X_{et}, where $D_{\mathrm{pf}}(\mathbf{Z})$ is the full sub-category of $D(\mathbf{Z})$ consisting of profinitely complete complexes of abelian groups, i.e. for which $A \simeq \varprojlim_n A/^L n$. We could also work with $D_{\mathrm{pf}}(\mathbf{Z})$-valued hypersheaves on X_{et}, resulting in a different functor formalism. Yet another option is to work with the left-completion of the category of $D_{\mathrm{pf}}(\mathbf{Z})$-valued sheaves (which is the same as the left-completion of the category of $D_{\mathrm{pf}}(\mathbf{Z})$-valued hypersheaves), resulting in yet another formalism. There are natural maps between these formalisms, and for schemes of finite type over $\mathbf{Z}$ or over algebraically closed fields there is no difference between sheaves and hypersheaves, and the category is already left-complete for the canonical t-structure.

0.6 Extending 3-functor formalisms

Let D be a 3-functor formalism on a geometric setup (C, E).

Definition 0.12. Let $f : X \to Y$ be a morphism in C, with Cech nerve $X_\bullet$, and maps $f_n : X_n \to Y$.

• f is a D^*-cover if the functor $(f_n^*)_n : D(Y) \to \varprojlim_{n\in\Delta} D(X_n)$ is an equivalence.

• Suppose that D is a 6-functor formalism and $f \in E$. We say that f is a $D^!$-cover if the functor $(f_n^!)_n : D(Y) \to \varprojlim_{n\in\Delta} D(X_n)$ is an equivalence.

We say that f is a universal $D^?$-cover if any base change of f along a map $Y' \to Y$ in C is a $D^?$-cover.

Proposition 0.13. *(HM 3.4.8) Let (C, E) and (C, E') be geometric setups with $E \subset E'$ and let D be a 3-functor formalism on (C, E). Under any of the following two assumptions D extends uniquely to a 3-functor formalism D' on (C, E') (which is presentable in the second case):*

• *any $f \in E'$ has a pullback by a universal D^*-cover that belongs to E.*

• *D is presentable and for any $f \in E'$ there is a universal $D^!$-cover g in E such that $f \circ g \in E$.*

Caution: the first condition is actually subtle, the most naive candidate for E' would be that of maps which belong to E after pullback by a suitable universal D^*-cover, but this does not seem to form a strongly stable class (I don't see any reason for stability by composition).

How to define $f_!$ for $f : X \to Y \in E'$? In the first case take a universal D^*-cover $g : Y' \to Y$ such that the pullback $f' : X' \to Y'$ is in E. Let $g_\bullet : Y'_\bullet \to Y$ and $g'_\bullet : X'_\bullet \to X'$ be the Cech nerves of g and of its base-change g'. The functor $f_!$ is defined by

$$D(X) \simeq \varprojlim_{n\in\Delta} D(X'_n) \to \varprojlim_{n\in\Delta} D(Y'_n) \simeq D(Y),$$

where the first map is induced by the $(g'_n)^*$'s (note that g' is a D^*-cover), the second by !-pushforward along $f'_n : X'_n \to Y'_n$ (which is in E by assumption) and the last is the inverse of the isomorphism induced by the g_n^*'s. By proper base change the $(f'_n)_!$ behave well with respect to transition maps, so this makes sense (it is of course not obvious how to make all this homotopy coherent, compatible with compositions etc).

In the second case take $f : X \to Y \in E$ and let $g : X' \to X$ be a universal $D^!$-cover such that $f \circ g \in E$. Let $g_\bullet : X'_\bullet \to X$ be the Cech nerve of g. Using the equivalence $(g_n^!)_n : D(X) \simeq \varprojlim_{n\in\Delta} D(X'_n)$ and adjunctions one deduces that for any $F \in D(X)$ the natural map

$$F \to \varinjlim_{n\in\Delta^{\mathrm{op}}} g_{n,!} g_n^! F$$

is an equivalence (the colimit exists since $D(X)$ is presentable). One then sets

$$f_! F := \varinjlim_{n\in\Delta^{\mathrm{op}}} (f \circ g_n)_! g_n^! F.$$

For the next technical but crucial result the context is as follows. Let τ be a **subcanonical** Grothendieck topology on C and let D be a 3-functor formalism on (C, E)

which is sheafy, i.e. the functor $D^* : C^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ satisfies τ-descent. Let $\mathrm{Sh}(C)$ be the category of τ-sheaves of anima on C and let E_r be the class of maps $f : X \to Y$ in $\mathrm{Sh}(C)$ which are representable in E, i.e. for any map $Z \to Y$ in $\mathrm{Sh}(C)$ with $Z \in C$ we have $X \times_Y Z \to Z \in E$. See HM 3.4.2 for the proof.

Proposition 0.14. *The obvious map $(C, E) \to (\mathrm{Sh}(C), E_r)$ is a morphism of geometric setups and D extends uniquely to a sheafy 3-functor formalism D' on $(\mathrm{Sh}(C), E_r)$, via right Kan extension along $\mathrm{Corr}(C, E) \to \mathrm{Corr}(\mathrm{Sh}(C), E_r)$. Moreover, the functor $(D')^* : \mathrm{Sh(C)}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ is obtained by right Kan extension along $C^{\mathrm{op}} \to \mathrm{Sh}(C)^{\mathrm{op}}$, and if D is presentable (resp. Λ-linear) so is D'.*

Note that we can always choose the trivial topology on C, then $\mathrm{Sh}(C)$ is the category of prestaks (i.e. (accessible) Ani-valued presheaves) on C.

Again, intuitively this is clear and making this precise is a pain. Namely for any $X \in \mathrm{Sh}(C)$ one defines (and is forced to do so)

$$D'(X) = \varprojlim_{Z \in C^{\mathrm{op}}/X} D(Z),$$

transitions being induced by pullbacks and the $f_!$ (for $f \in E_r$) are defined "componentwise".

0.7 Left and right adjoints in 2-categories

See the appendix D to HM for proofs of the results stated here. A 2-category (i.e. $(\infty, 2)$-category) is by definition a Cat-enriched category (this makes sense since Cat is symmetric monoidal for its Cartesian monoidal structure). Fix a 2-category K, for any objects $X, Y \in K$ we thus have a category $\mathrm{Fun}_K(X, Y)$, whose objects are called morphisms from X to Y (these are 1-morphisms) and whose morphisms are called 2-morphisms. Say a morphism $f : X \to Y$ is a left adjoint if there is a morphism $g : Y \to X$ (called a right adjoint of f) and 2-morphisms $\eta : \mathrm{id}_X \to gf$, $\varepsilon : fg \to \mathrm{id}_Y$ such that the composite $f \to fgf \to f$ (first map induced by η, second one by ε) is id in $\mathrm{Fun}_K(X, Y)$ and similarly $g \to gfg \to g$ is id in $\mathrm{Fun}_K(Y, X)$. For instance, a functor $F : C \to D$ of categories is a left adjoint (in usual sense) if and only if it's a left adjoint seen as 1-morphism in the 2-category of categories.

Remark 0.15. • If f has a right adjoint g, then g is essentially unique: for any two right adjoints g_1, g_2 of f the map $g_1 \to g_2 f g_1 \to g_2$ is an isomorphism (HM D.2.6). So we will speak about *the* right adjoint of f.

• Any 2-functor between 2-categories sends left adjoint morphisms to left adjoint morphisms.

Let $f : X \to Y$ be a morphism in K. For any object Z of K we get an induced functor $f_{*,Z} : \mathrm{Fun}_K(Z, X) \to \mathrm{Fun}_K(Z, Y)$ and for any morphism $h : Z \to T$ we get functors $h^*_X : \mathrm{Fun}_K(T, X) \to \mathrm{Fun}_K(Z, X)$ and $h^*_Y : \mathrm{Fun}_K(T, Y) \to \mathrm{Fun}_K(Z, Y)$. One then has the very practical criterion:

Proposition 0.16. *(HM D.2.8) $f : X \to Y$ is a left adjoint if and only if for any object Z of K the functor $f_{*,Z}$ has a right adjoint G_Z and for any morphism $h : Z \to T$ the natural transformation $h_X^* G_T \to G_Z h_Y^*$ is an isomorphism, and then the right adjoint of f is $G_Y(\mathrm{id}_Y) : Y \to X$.*

Moreover, it suffices to check the existence of G_Z only for $Z = X, Y$, and the fact that the natural map $G_Y(\mathrm{id}_Y) \circ f \to G_X(f)$ is an isomorphism (in $\mathrm{Fun}_K(X, X)$) after applying $\mathrm{Hom}(\mathrm{id}_Y, -)$.

Let I be a category. We say that K is compatible with I-limits (resp. I-colimits) if for all objects X, Y the category $\mathrm{Fun}_K(X, Y)$ has all limits indexed by I (resp. all colimits indexed by I) and for any morphism $h : Z \to T$ and any object X the functor $h_X^* : \mathrm{Fun}_K(T, X) \to \mathrm{Fun}_K(Z, X)$ preserves such limits (resp. colimits).

Proposition 0.17. *(HM D.4.13, D.4.9) a) If K is compatible with I-colimits and I^{op}-limits, then the full subcategory $\mathrm{Fun}_L(X, Y) \subset \mathrm{Fun}(X, Y)$ spanned by morphisms that are left adjoints has all I-colimits and the inclusion preserves them.*

b) If K is compatible with I-limits and $Y = \varprojlim_{i \in I} Y_i$ is a limit in C, then a morphism $f : X \to Y$ is a left adjoint as soon as all $X \to Y_i$ are left adjoints.

0.8 Suave duality and Verdier duality

Let D be a 3-functor formalism on a geometric setup (C, E). It turns out that one can associate to each object S of C a 2-category K_S of kernels, with objects the maps $f : X \to S$ in E and such that

$$\mathrm{Fun}_{K_S}(X, Y) = D(Y \times_S X).$$

The composition of $A : X \to Y$ and $B : Y \to Z$ (thus $A \in D(Y \times_S X)$ and $B \in D(Z \times_S Y)$) is

$$B \circ A = p_{13,!}(p_{23}^* A \otimes p_{12}^* B) \in D(Z \times_S X),$$

the p_{ij} being the projections from $Z \times_S Y \times_S X$. In particular $\mathrm{Fun}_{K_S}(X, S)$ and $\mathrm{Fun}_{K_S}(S, X)$ both identify with $D(X)$, i.e. we can see any $A \in D(X)$ as a morphism $X \to S$ and as a morphism $S \to X$ in K_S.

Remark 0.18.

1. (HM 4.1.5) The 2-category K_S comes equipped with canonical 2-functors, whose composition is D

$$\mathrm{Corr}((C_E)/S, \mathrm{All}) \to K_S \to \mathrm{Cat}_2,$$

the first functor sending X/S to X/S (on objects) and a correspondence $X \leftarrow Z \to Y$ to $u_!(1) \in D(Y \times_S X)$, where $u : Z \to Y \times_S X$ is the induced map. The second functor sends X/S to $D(X)$ and sends $A \in \mathrm{Fun}_S(X, Y) = D(Y \times_S X)$ to $p_{1,!}(A \otimes p_2^*(-)) : D(X) \to D(Y)$, where p_i are the projections from $Y \times_S X$. This explains the name category of kernels: any morphism $A \in \mathrm{Fun}_S(X, Y)$ gives rise to a kernel functor.

2. (HM 4.2.4) Any morphism $f : T \to S$ induces a 2-functor $f^* : K_S \to K_T$ sending X/S to X_T/T, where $X_T = X \times_S T$, and which on functor categories $D(Y \times_S X) \to D(Y_T \times_T X_T)$ is the obvious $*$-pullback functor. Similarly, any $g : T \to S$ in E induces a 2-functor $g_! : K_T \to K_S$ which on objects is the forgetful functor and on functor categories is given by $u_! : D(Y \times_T X) \to D(Y \times_S X)$, where $u : Y \times_T X \to Y \times_S X$ is the obvious map.

3. (HM 4.3.1) If $X' \to X$ is a universal D^*-cover with Cech nerve $X'_\bullet$, the natural 2-functor $K_X \to \varprojlim_{n \in \Delta} K_{X'_n}$ is fully faithful.

Definition 0.19. Let $f : X \to S$ be a map in E. We say that $A \in D(X)$ is f-suave if seen as a morphism $X \to S$ in K_S, A is a left adjoint. We call $D_f(A) \in D(X) = \mathrm{Fun}_{K_S}(S, X)$ the corresponding right adjoint. We say that f is suave (or D-suave) if the monoidal unit $1_X \in D(X)$ is f-suave, and we say that f is smooth (or D-smooth) if f is suave and $\omega_f := D_f(1_X)$ is invertible in $D(X)$.

The next result resumes the key stability properties of this notion:

Theorem 0.20. *Let D be a 3-functor formalism on (C, E), $f : X \to S \in E$ and $A \in D(X)$.*

1. *(Verdier duality) If A is f-suave then so is $D_f(A)$ and the natural map $A \to D_f(D_f(A))$ is an isomorphism.*

2. *($*$-locality on the target) Let $g : S' \to S$ be a map in C with base changes $g' : X' = X \times_S S' \to X$ and $f' : X' \to S'$. If A is f-suave then $(g')^*A$ is f'-suave. The converse holds if g is a universal D^*-cover.*

3. *(limits/colimits stability) Let I be a category such that all $D(X)$ have colimits indexed by I and limits indexed by I^{op}, and these are preserved by $\otimes$ separately in each variable, and by the f^* and $f^!$ for $f \in E$. Then f-suave objects are stable under I-colimits and I^{op}-limits. In particular, for a stable formalism f-suave objects are stable under taking fibers, cofibers and retracts.*

Proof. 1) is formal, see HM 4.4.4. The key point is that $D(X \times_S Y) \simeq D(Y \times_S X)$ induces an identification $K_S \simeq K_S^{\mathrm{op}}$.

For 2), stability by pullback follows since the 2-functor $g^* : K_S \to K_{S'}$ sends A to $(g')^*A$ and preserves adjunctions, as does any 2-functor. For the second part, note that $K_S \to \varprojlim_{n \in \Delta} K_{X'_n}$ is a fully faithful 2-functor (where $X'_\bullet$ is the Cech nerve of g), since g is of universal D^*-descent (remark 0.18). By assumption the image of $A : X \to S$ in any $K_{X'_n}$ is a left adjoint, and then it follows formally that A itself is a left adjoint.

Part 3) is a simple consequence of propositions 0.17 and 0.16. □

The next result is incredibly powerful and underlies most non formal properties of suave maps:

Theorem 0.21. *Consider a 3-functor formalism D on a geometric setup (C, E) and let $h : X \to S, g : Y \to S$ be in E and let $f : X \to Y$ be a map over S. If $A \in D(Y)$ is g-suave and $B \in D(X)$ is f-suave then $f^*A \otimes B$ is h-suave and*

$$D_h(f^*A \otimes B) \simeq f^*D_g(A) \otimes D_f(B).$$

Proof. There is a 2-functor $K_Y \to K_S$ sending $B \in D(X) = \mathrm{Fun}_{K_Y}(X, Y)$ to $u_!B \in D(Y \times_S X) = \mathrm{Fun}_{K_S}(X, Y)$, where $u : X \to Y \times_S X$ is (f, id). Thus $u_!B$ is a left adjoint, with right adjoint $u_!D_f(B)$, and so $A \circ u_!B$ is also a right adjoint, with the obvious right adjoint. Next one computes in K_S

$$A \circ u_!B = f^*A \otimes B.$$

□

Corollary 0.22. *Let D be a 3-functor formalism on (C, E).*

1. *Suave maps are stable under composition and base change. If $f : X \to Y$ is a morphism over S (with $X \to S, Y \to S \in E$) and $X \to S$ is suave and $Y \to S$ has suave diagonal, then f is suave.*
2. *Let $f : X \to Y$ be a morphism over S such that $g : Y \to S \in E$ and $h : X \to S \in E$.*
   - *If g is suave then all f-suave objects A are h-suave and there are natural isomorphisms $D_h(A) \simeq D_f(A) \otimes f^*\omega_g$.*
   - *if f is suave then any dualizable object $A \in D(X)$ is f-suave with $D_f(A) \simeq A^\vee \otimes \omega_f$.*
   - *if Δ_f is suave, then all h-suave objects A are f-suave and there are natural isomorphisms $D_f(A) \simeq D_h(A) \otimes \omega_{(f,\mathrm{id}_Y)}$, where*
   - *if Δ_f is suave then any f-suave object A is dualizable in $D(X)$ and $A^\vee \simeq D_f(A) \otimes \omega_{\Delta_f}$, and if h is suave then so is g.*
3. *($*$-locality for suave pullback) Suppose that f is suave and let $A \in D(Y)$. If A is g-suave then f^*A is h-suave and $D_h(f^*A) \simeq f^*D_g(A) \otimes \omega_f$. Conversely, if f^*A is h-suave, f is a universal D^*-cover and D is compatible with colimits then A is g-suave and $D_g(A) \simeq f_!D_h(f^*A)$.*

Proof. These are all immediate consequences of the above two theorems. □

Let $f : X \to S \in E$ and $A \in D(X)$, seen as a morphism $A : X \to S$ in K_S. Then the corresponding functor $A_* : \mathrm{Fun}_{K_S}(Z, X) = D(X \times_S Z) \to \mathrm{Fun}_{K_S}(Z, S) = D(Z)$ is simply $p_{2,!}(p_1^*A \otimes (-))$, where p_i are the projections from $X \times_S Z$. If D is a 6-functor formalism, a simple computation shows that this has a right adjoint $G_Z = \underline{\mathrm{Hom}}(p_1^*A, p_2^!(-))$ and we have $G_S(\mathrm{id}_S) = \underline{\mathrm{Hom}}(A, f^!1)$. Since $\mathrm{Hom}(\mathrm{id}_X, -) = \mathrm{Hom}(1, \Delta^!(-))$ (where $\Delta : X \to X \times_S X$ is the diagonal of f), proposition 0.16 yields:

Theorem 0.23. *Let D be a 6-functor formalism and let $f : X \to S \in E$ and $A \in D(X)$.*

- *if A is f-suave there is a canonical isomorphism*

$$D_f(A) \simeq \underline{\mathrm{Hom}}(A, f^! 1),$$

inducing isomorphisms of functors

$$D_f(A) \otimes f^*(-) \simeq \underline{\mathrm{Hom}}(A, f^!(-)), \quad D_f(A) \boxtimes (-) \simeq \underline{\mathrm{Hom}}(p_1^* A, p_2^!(-))$$

for any $Z \to S \in E$, with projections $p_1 : X \times_S Z \to X, p_2 : X \times_S Z \to Z$.

- *Conversely, if for $Z = X$ the natural map*

$$\underline{\mathrm{Hom}}(A, f^! 1) \boxtimes A \to \underline{\mathrm{Hom}}(p_1^* A, p_2^! A)$$

is an isomorphism (even after applying $\mathrm{Hom}(1, \Delta^!(-))$), then A is f-suave.

Corollary 0.24. *Let D be a 6-functor formalism on (C, E) and let $f : X \to S \in E$.*

1. *If f is suave then there is a natural isomorphism $\omega_f \simeq f^! 1$ inducing natural isomorphisms*

   $$\omega_f \otimes f^* \simeq f^!, \quad f^* \simeq \underline{\mathrm{Hom}}(\omega_f, f^!).$$

   Moreover the formation of $f^!$ commutes with arbitrary $*$-pullback and for any map $g : S' \to S$ in C with base changes $g' : X' = X \times_S S' \to X$ and $f' : X' \to S'$ there are natural Beck-Chevalley isomorphisms

   $$f^* g_* \simeq g'_*(f')^*, \quad (g')^* f^! \simeq (f')^! g^*,$$

   If moreover $g \in E$ then there are natural Beck-Chevalley isomorphisms

   $$f^! g_! \simeq f'_!(g')^!, \quad (f')^* g^! \simeq (g')^! f^*.$$

2. *Conversely, if the formation of $f^! 1$ commutes with base change along f, i.e. the natural map $p_1^* f^! 1 \to p_2^! 1$ is an isomorphism (where p_1, p_2 are the projections from $X \times_S X$), then f is suave.*

Proof. Most of these results follow by taking $A = 1$ in the above theorem. To see that $f^* g_* \simeq g'_*(f')^*$, use that $f^* \simeq \underline{\mathrm{Hom}}(\omega_f, f^!)$ and similarly with f', combined with prop. 0.8.

□

Theorem 0.25. *Let $f : X \to S \in E$ be suave, where D is a 6-functor formalism. If $D(X)$ has all countable limits and colimits and if f^* is conservative, then f is a universal D^* and $D^!$-cover.*

Proof. The key is Lurie's Beck-Chevalley conditions for descent, combined with suave base-change (part 2 of the above corollary). This reduces us to checking only the following things: $f^!$ is conservative (OK since f^* is so and $f^* = \underline{\mathrm{Hom}}(\omega_f, f^!)$) and $f^!$ preserves colimits over $f^!$-split simplicial objects in $D(X)$, which is the case simply because $f^! \simeq \omega_f \otimes f^*$ preserves all small colimits. This proves $D^!$-descent, hence also universal !-descent as the hypotheses are preserved under base-change. For D^*-descent one needs to check that f^* preserves f^*-split totalizations of cosimplicial objects in $D(X)$, which holds since $f^* \simeq \underline{\mathrm{Hom}}(\omega_f, f^!)$.

□