Sheaves on classifying stacks and the local Langlands category

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1 Sheaves on classifying stacks

1.1 Classifying stacks of locally profinite groups

We fix a locally profinite group H and an algebra of coefficients Λ (with the usual properties required to define the theory $\mathrm{Sh}^! = \mathrm{Sh}^!(-, \Lambda)$). The Λ-module

$$\mathscr{H} = C_c^\infty(H, \Lambda)$$

will be seen as a representation of H for the left regular action. It also has an action of H by right translation, commuting with the left regular action.

Hypothesis: H has a Λ-valued left Haar measure, i.e. a map $\int_H : \mathscr{H} \to \Lambda$ such that $\int_H f(hx)dx = \int_H f(x)dx$ for all $h \in H$ and for which there is a compact open subgroup K of H such that $\mathrm{vol}(K) := \int_K dk \in \Lambda^\times$. Call such K a good subgroup of H. Let $\mathrm{Good}(H)$ be the set of good subgroups of H. Note that if K is good, then any open subgroup of K is good, so there is a basis of neighborhoods of 1 in H consisting of good subgroups.

1.1.1 The category $\mathrm{Rep}(H)$

Let $\mathscr{A}_H$ be the Grothendieck abelian category of smooth representations of H on Λ-modules and let

$$\mathrm{Rep}(H) = D(\mathscr{A}_H)$$

be its derived ∞-category.

Proposition 1.1. *The representations $c - \mathrm{ind}_K^H \Lambda$ for $K \in \mathrm{Good}(H)$ form a set of compact projective generators of $\mathscr{A}_H$, therefore $\mathrm{Rep}(H)$ is compactly generated and left complete for its canonical t-structure.*

Proof. Let $V \in \mathscr{A}_H$. The inclusion $V^K \subset V$ has a splitting given by[1]

$$p : V \to V^K, \quad p(v) = \mathrm{vol}(K)^{-1} \int_K k.vdk,$$

[1] To give a meaning to the integral, pick K' open in K which fixes v and express the integral as a finite sum.

showing that the functor $(-)^K$ is exact on $\mathscr{A}^H$ and that we have a natural isomorphism of Λ-modules

$$V_K \simeq V^K$$

between K-invariants and K-coinvariants. Moreover by Frobenius reciprocity

$$\mathrm{Hom}_{\mathscr{A}_H}(c - \mathrm{ind}_K^H \Lambda, -) = (-)^K.$$

Since this is exact and commutes with colimits (by the above discussion), $c - \mathrm{ind}_K^H \Lambda$ is compact projective. Since for any $V \in \mathscr{A}_H$ we have $V = \varinjlim_{K \in \mathrm{Good}(H)} V^K$, it is clear that these representations generate $\mathscr{A}_H$. This proves the first part. The second part is a standard consequence of the first.

□

1.1.2 Interpretation in terms of QCoh when H is profinite

Suppose now that H is profinite and let X_H be the corresponding affine group scheme over Λ, thus $\underline{H}_\Lambda = \mathrm{Spec}(C^\infty(H, \Lambda))$, where we see now $C^\infty(H, \Lambda)$ as a commutative Λ-algebra via pointwise multiplication. Let $B\underline{H}_\Lambda$ be the classifying stack in the fpqc topology.

Proposition 1.2. *There is a symmetric monoidal equivalence*

$$\mathrm{QCoh}(B\underline{H}_\Lambda) \simeq \mathrm{Rep}(H)$$

such that the pullback functor $\mathrm{QCoh}(B\underline{H}_\Lambda) \to \mathrm{Mod}_\Lambda$ *corresponds to the forgetful functor* $\mathrm{Rep}(H) \to \mathrm{Mod}_\Lambda$.

Proof. By the discussion in example 9.13 in Zhu's paper (more precisely by lemma 9.8) we know that $\mathrm{QCoh}(B\underline{H}_\Lambda)$ is the left completion of the derived category of $\mathrm{QCoh}(B\underline{H}_\Lambda)^\heartsuit$, the latter being identified with the category of algebraic representations of $\underline{H}_\Lambda$ on Λ-modules, i.e. the category of comodules over $\Lambda[\underline{H}_\Lambda] = C^\infty(H, \Lambda)$. As is well-known, this is the same as $\mathscr{A}_H$. Thus $\mathrm{QCoh}(B\underline{H}_\Lambda)$ is the left-completion of $D(\mathscr{A}_H) = \mathrm{Rep}(H)$, and we conclude by proposition 1.1. □

Remark 1.3. The description in terms of comodules can also be deduced from Barr-Beck-Lurie.

Corollary 1.4. *There is a symmetric monoidal equivalence*

$$\varprojlim_{n \in \Delta} \mathrm{Mod}_{C^\infty(H^n, \Lambda)} \simeq \mathrm{Rep}(H).$$

Proof. Since the theory QCoh satisfies fpqc descent (this follows from Lurie's version of Grothendieck's classical result on fpqc descent for modules over rings), there is a natural symmetric monoidal equivalence

$$\mathrm{QCoh}(B\underline{H}_\Lambda) \simeq \varprojlim_{n \in \dagger\Delta} \mathrm{QCoh}(\underline{H^n}_\Lambda).$$

It remains to observe that

$$\mathrm{QCoh}(\underline{H^n}_\Lambda) = \mathrm{Mod}_{\Lambda[\underline{H^n}_\Lambda]} = \mathrm{Mod}_{C^\infty(H^n,\Lambda)}.$$

□

1.1.3 The étale case

Here we allow H locally profinite, but with a Λ-valued left Haar measure. Any profinite set S gives a perfect k-scheme $\underline{S}$ given by the spectrum of the perfection of $C^\infty(S,k)$. If S is a locally profinite set we can write S as a filtered colimit of profinite sets S_i with transitions maps being pullbacks of inclusions of finite sets, thus we can associate to S an ind-scheme

$$\underline{S} = \varinjlim_i \mathrm{Spec} C^\infty(S_i,k)^{\mathrm{perf}}$$

over k, such that for any perfect k-algebra R we have

$$\underline{S}(R) = C^{\mathrm{cont}}(|\mathrm{Spec}(R)|, S).$$

Theorem 1.5. *(Zhu 3.51, 3.52)*
a) There is a symmetric monoidal equivalence

$$\mathrm{can}_H : \mathrm{Sh}^!(B\underline{H}) \simeq \mathrm{Rep}(H)$$

such that for any $K \in \mathrm{Good}(H)$ *there are natural identifications*

$$\mathrm{For} \circ \mathrm{can}_H \simeq \mathrm{can}_K \circ \iota_K^!, \quad \mathrm{can}_H \circ \iota_{K,*} \simeq c - \mathrm{ind}_K^H \circ \mathrm{can}_K,$$

where $\iota_K : B\underline{K} \to B\underline{H}$ *is the obvious map,* $\mathrm{For} : \mathrm{Rep}(H) \to \mathrm{Rep}(K)$ *and* $c - \mathrm{ind}_K^H : \mathrm{Rep}(K) \to \mathrm{Rep}(H)$ *are the forgetful and compact induction functor.*

b) The category $\mathrm{Sh}^!(B\underline{H})$ *is compactly generated and for any perfect prestack* X *over* k *with* $\mathrm{Sh}^!(X)$ *dualizable the external tensor product*

$$\mathrm{Sh}^!(B\underline{H}) \otimes_\Lambda \mathrm{Sh}^!(X) \to \mathrm{Sh}^!(B\underline{H} \times X)$$

is an equivalence.

Proof. a) I will just give a sketch of the main ideas, since the details are painful (and not really given in the paper...). The proof is done in several stages. One first proves that when H is profinite and $\mathrm{vol}(H) \in \Lambda^\times$ then the obvious map $f : \mathrm{pt} = \mathrm{Spec}(k) \to B\underline{H}$ satisfies descent (universally). Note that f is ps (pro-smooth), even pro-étale. Indeed, whenever $S \to B\underline{H}$ is a map from $S \in \mathrm{Sp}$ (i.e. a perfect qcqs algebraic space over k) the pullback $\pi : E \to S$ is a $\underline{H}$-torsor and so if we write $H = \varprojlim_i H_i$ with H_i finite groups then we can write π as inverse limit of $\underline{H}_i$-torsors $\pi_i : E_i \to S$. Now ps maps are HR, so f is HR. By proposition 8.30 (a simple application of comonadic Barr-Beck-Lurie) it suffices to prove that $\omega_S \to \pi_\flat \omega_E$ has a section (recall that $\pi_\flat$ is the right adjoint of $\pi^!$). Now since π is pro-étale,

$$\pi_\flat \omega_E \simeq \pi_* \omega_E \simeq \varinjlim_i \pi_{i,*}\omega_{E_i}$$

and it suffices to construct compatible splittings $\pi_{i,*}\omega_{E_i} \to \omega_S$. But since π_i is pfp finite, in particular pfp proper, we have $\pi_{i,*} \simeq \pi_{i,!}$ and the co-unit yields a map $s_i : \pi_{i,*}\omega_{E_i} \to \omega_S$. One checks that if one multiplies s_i by $\mathrm{vol}(\ker(H \to H_i))$ one gets a compatible system of splittings.

Keep H profinite with $\mathrm{vol}(H)$ invertible. By the previous paragraph we obtain a natural isomorphism

$$\mathrm{Sh}^!(B\underline{H}) \simeq \varprojlim_{n\in\Delta} \mathrm{Sh}^!(\underline{H^n}).$$

Now if $S = \varprojlim_i S_i$ is a profinite set written as cofiltered inverse limit of finite sets we have

$$\mathrm{Sh}^!(\underline{S}) = \mathrm{Sh}^!(\varprojlim_i \underline{S}_i) \simeq \varinjlim_i \mathrm{Sh}^!(\underline{S}_i).$$

Since each $\underline{S}_i$ is a finite disjoint union of pt, we obtain

$$\mathrm{Sh}^!(\underline{S}) \simeq \varinjlim_i \mathrm{Mod}_{\mathrm{Fct}(S_i,\Lambda)} \simeq \mathrm{Mod}_{\varinjlim_i \mathrm{Fct}(S_i,\Lambda)} = \mathrm{Mod}_{C^\infty(S,\Lambda)}.$$

It follows that there is a natural isomorphism

$$\mathrm{Sh}^!(B\underline{H}) \simeq \varprojlim_{n\in\Delta} \mathrm{Mod}_{C^\infty(H^n,\Lambda)}$$

and we conclude that this is isomorphic to $\mathrm{Rep}(H)$ by corollary 1.4.

Now we deal with the general case. Fix $K \in \mathrm{Good}(H)$ and the obvious map

$$\iota_K : B\underline{K} \to B\underline{H}.$$

One checks (Zhu, 3.50) using fpqc descent and some extra arguments that ι_K is ind-pfp finite, thus also ind-pfp proper and so it satisfies descent, giving

$$\mathrm{Sh}^!(B\underline{H}) \simeq \varprojlim_{n\in\Delta} \mathrm{Sh}^!(\mathrm{Hk}_n) \simeq \varinjlim_{n\in\Delta^{\mathrm{op}}} \mathrm{Sh}^!(\mathrm{Hk}_n),$$

where $\mathrm{Hk}_\bullet$ is the Cech nerve of ι_K and transitions are $()^!$ in the limit and $()_*$ in the colimit. Explicitly,

$$\mathrm{Hk}_n \simeq \underline{K}\backslash(\underline{H}/\underline{K})^n,$$

where $\underline{K}$ acts diagonally on $(\underline{H}/\underline{K})^n$. By Barr-Beck-Lurie $\varinjlim_{n\in\Delta^{\mathrm{op}}} \mathrm{Sh}^!(\mathrm{Hk}_n)$ is the category of modules over the monad $T = p_{2,*}p_1^!$ acting on $\mathrm{Sh}^!(B\underline{K})$, where $p_1, p_2 : \mathrm{Hk}_1 \to B\underline{K}$ are the two projections. One checks that under the equivalence $\mathrm{Sh}^!(B\underline{K}) \simeq \mathrm{Rep}(K)$ proved above this monad gets identified with the monad $T' = \mathrm{res} \circ c - \mathrm{ind}_K^H$ acting on $\mathrm{Rep}(K)$, thus $\mathrm{Sh}^!(B\underline{H})$ gets identified with the category of modules over T' acting on $\mathrm{Rep}(K)$. Using again Barr-Beck-Lurie this is nothing but $\mathrm{Rep}(H)$.

b) Since $\mathrm{Sh}^!(X)$ is dualizable, tensoring with it commutes with limits, and by the presentation $\mathrm{Sh}^!(B\underline{H}) \simeq \varprojlim_{n\in\Delta} \mathrm{Sh}^!(\mathrm{Hk}_n)$ it suffices to check that the external tensor product induces an equivalence

$$\mathrm{Sh}^!(\mathrm{Hk}_n) \otimes_\Lambda \mathrm{Sh}^!(X) \simeq \mathrm{Sh}^!(\mathrm{Hk}_n \times X).$$

Now as we observed in a), each Hk_n is a disjoint union of classifying stacks of good subgroups of H. Thus it suffices to prove that for any good subgroup K of H the external tensor product induces an equivalence

$$\mathrm{Sh}^!(B\underline{K}) \otimes_\Lambda \mathrm{Sh}^!(X) \simeq \mathrm{Sh}^!(B\underline{K} \times X).$$

But we saw in a) that $f : \mathrm{pt} \to B\underline{K}$ is of universal descent, and since tensoring with $\mathrm{Sh}^!(X)$ commutes with limits it suffices to prove the equivalence

$$\mathrm{Sh}^!(\underline{K^n}) \otimes_\Lambda \mathrm{Sh}^!(X) \simeq \mathrm{Sh}^!(\underline{K^n} \times X).$$

Writing $K^n = \varprojlim_i S_i$ with S_i finite sets, and using again that everything commutes with filtered colimits we reduce to proving that

$$\mathrm{Sh}^!(\underline{S_i}) \otimes_\Lambda \mathrm{Sh}^!(X) \simeq \mathrm{Sh}^!(\underline{S_i} \times X),$$

which is clear since $\underline{S_i}$ is simply a finite disjoint union of pt. $\square$

1.2 Bernstein-Zelevinsky duality

Let $C = \mathrm{Rep}(H)$, then we have just seen that

$$C \otimes_\Lambda C \simeq \mathrm{Sh}^!(B\underline{H}) \otimes_\Lambda \mathrm{Sh}^!(B\underline{H}) \simeq \mathrm{Sh}^!(B\underline{H} \times B\underline{H}) \simeq \mathrm{Sh}^!(B\underline{H \times H}) \simeq \mathrm{Rep}(H \times H),$$

noting that $\mathrm{Sh}^!(B\underline{H})$ is dualizable since it is compactly generated. Consider now

$$\mathscr{H} = C_c^\infty(H, \Lambda)$$

seen as $H\times H$-representation via the left and right regular action of H. This is an object of C, giving rise to a functor

$$\mathscr{H} : \mathrm{Mod}_\Lambda \to C \otimes_\Lambda C.$$

Let

$$\Delta_H : H \to \Lambda^\times$$

be the modulus character, described by

$$\int_H f(xh)dx = \Delta_H(h) \int_H f(x)dx, \quad f \in \mathscr{H}.$$

This character is trivial on any compact open subgroup of H.

Proposition 1.6. *$\mathscr{H}$ is the unit in a duality datum between C and itself, with evaluation map*

$$C \otimes_\Lambda C \to C \to \mathrm{Mod}_\Lambda, \quad A \boxtimes B \mapsto (A \otimes B \otimes \Delta_H^{-1})_H,$$

where $()_H$ means derived H-coinvariants. Thus

$$\lambda : C \to \mathrm{Mod}_\Lambda, \quad V \mapsto (V \otimes \Delta_H^{-1})_H$$

defines a Frobenius structure on C. The induced self-duality

$$D^{\mathrm{BZ}} : C^\vee \simeq C$$

is described on compact objects $A \in C^\omega$ by

$$D^{\mathrm{BZ}}(A) = \mathrm{Hom}_C(A, \mathscr{H}),$$

where we use the right regular action of H to see $\mathscr{H}$ as object of C and the left regular action to endow $\mathrm{Hom}_C(A, \mathscr{H})$ with a structure of object of C. Moreover all $c - \mathrm{ind}_K^H \Lambda$ are self-dual.

Proof. Checking that this is a duality datum comes down, after unwinding definitions, to the following

Lemma 1.7. *For any $V \in \mathscr{A}_H$ there is a natural isomorphism*

$$(\mathscr{H} \otimes V \otimes \Delta_H^{-1})_H \simeq V$$

induced by sending $f \otimes v$ to $\int_H f(h)hvdh$. Here we endow $\mathscr{H}$ with the right regular action.

Proof. One immediately checks that the map $u : \mathscr{H} \otimes V \otimes \Delta_H^{-1} \to V$ sending $f \otimes v$ to $\int_H f(x)xvdx$ satisfies $u(h.(f \otimes v)) = \Delta_H^(h)u(f \otimes v)$ and thus descends to a map $(\mathscr{H} \otimes V \otimes \Delta_H^{-1})_H \simeq V$, which is H-equivariant if we endow the left-hand side with the action of H induced from the left regular action of H on $\mathscr{H}$. If we endow $\mathscr{H}$ with the right regular representation then $\mathscr{H} \simeq \varinjlim_K c - \mathrm{ind}_K^H$, and we claim that

$$(c - \mathrm{ind}_K^H \Lambda \otimes V \otimes \Delta_H^{-1})_H \simeq V^K$$

via the above map. Once we know this we can take the colimit over all K and conclude. Now the map $c - \mathrm{ind}_K^H \Lambda \otimes V \to c - \mathrm{ind}_K^H (V|_K)$ sending $f \otimes v$ to $h \mapsto f(h)h^{-1}v$ is an isomorphism, and we have functorial isomorphisms (where $V_K = c - \mathrm{ind}_K^H \Lambda$)

$$\mathrm{Hom}_\Lambda((V_K \otimes_A)_H, M) = \mathrm{Hom}_{\Lambda[H]}(V_K \otimes A, M_{\mathrm{triv}}) = \mathrm{Hom}_{\Lambda[H]}(c - \mathrm{ind}_K^H (A|_K), M_{\mathrm{triv}})$$
$$= \mathrm{Hom}_{\Lambda[K]}(A|_K, M_{\mathrm{triv}}) = \mathrm{Hom}_\Lambda((A|_K)_K, M),$$

thus we obtain a canonical identification

$$(V_K \otimes A)_H \simeq (A|_K)_K$$

Using that Δ_H is trivial on K and the isomorphism $V_K \simeq V^K$ we conclude. □

At this point we know that λ is a Frobenius structure. By formal nonsense (see Zhu 7.39), since C is symmetric monoidal we know that the induced duality D^{DZ} on C^ω is an involution and we also know that for all $A \in C^\omega$ we have an isomorphism of functors

$$\mathrm{Hom}_C(A, -) \simeq (D^{\mathrm{BZ}}(A) \otimes (-) \otimes \Delta_H^{-1})_H,$$

in particular using the lemma

$$\mathrm{Hom}_C(A, \mathscr{H}) \simeq (D^{\mathrm{BZ}}(A) \otimes \mathscr{H} \otimes \Delta_H^{-1})_H \simeq D^{\mathrm{BZ}}(A),$$

yielding the formula for $D^{\mathrm{BZ}}(A)$. Also

$$D^{\mathrm{BZ}}(c - \mathrm{ind}_K^H \Lambda) = \mathrm{Hom}_H(c - \mathrm{ind}_K^H \Lambda, \mathscr{H}) \simeq \mathscr{H}^K \simeq c - \mathrm{ind}_K^H \Lambda.$$

□

Zhu gives a geometric construction of the above Frobenius structure and duality. Let $K \in \mathrm{Good}(H)$ and consider the map

$$\pi_K : Y := B\underline{K} \to X := B\underline{H},$$

with Cech nerve $\mathrm{Hk}_\bullet$. As we saw in the previous sections, the obvious map $f : \mathrm{pt} \to Y$ is a placid atlas for Y. Since $f^!(\omega_Y) = \omega_{\mathrm{pt}}$ is the (generalized) constant (co)sheaf Λ, thus $\Lambda_0 := \omega_Y$ is a generalized constant sheaf on Y. The map $d_0 : \mathrm{Hk}_n \to \mathrm{Hk}_0 = Y$ is ind-pfp, thus we get a generalized constant sheaf $\Lambda_n = d_0^* \Lambda_0$ on Hk_n and a corresponding Frobenius structure

$$\mathrm{R\Gamma}(\mathrm{Hk}_n, -) = \mathrm{Hom}(\Lambda_n, -) : \mathrm{Sh}^!(\mathrm{Hk}_n) \to \mathrm{Mod}_\Lambda.$$

One checks that these Frobenius structures are compatible with face maps in the simplicial stack $\mathrm{Hk}_\bullet$, yielding a Frobenius structure

$$\mathrm{R\Gamma}^{\mathrm{BZ}}(X, -) : \mathrm{Sh}^!(X) = \varinjlim_{n \in \Delta^{\mathrm{op}}} \mathrm{Sh}^!(\mathrm{Hk}_n) \to \mathrm{Mod}_\Lambda.$$

Theorem 1.8. *(Zhu 3.53, 3.56) Via the identification* $\mathrm{Sh}^!(X) \simeq \mathrm{Rep}(H)$*, the Frobenius structure* $\mathrm{R\Gamma}^{\mathrm{BZ}}(X, -)$ *gets identified with the Frobenius structure*

$$\mathrm{BZ} : \mathrm{Rep}(H) \to \mathrm{Mod}_\Lambda, \quad \mathrm{BZ}(V) = (V \otimes \Delta_H^{-1})_H,$$

where $()_H$ *means derived* H*-coinvariants. The induced self-duality*

$$D^{\mathrm{BZ}} : \mathrm{Sh}^!(X)^\vee \simeq \mathrm{Sh}^!(X)$$

corresponds on compact objects to the usual Bernstein-Zelevinsky duality

$$D^{\mathrm{BZ}} : \mathrm{Rep}(H)^{\omega,\mathrm{op}} \simeq \mathrm{Rep}(H)^\omega, \quad D^{\mathrm{BZ}}(V) = \mathrm{Hom}_{\mathrm{Rep}(H)}(V, \mathscr{H}),$$

where we see $\mathscr{H}$ *as object of* $\mathrm{Rep}(H)$ *via the left regular action and we use the right translation action of* H *on* $\mathscr{H}$ *to turn* $D^{\mathrm{BZ}}(V)$ *into a* H*-representation.*

2 The local Langlands category

2.1 Recollections on the geometry of Isoc_G and $\mathrm{Sht}^{\mathrm{loc}}$

Let F be a non-archimedean local field with residue field k_F of characteristic $p \neq \ell$. Let $\breve{F}$ be the completion of the maximal unramified extension of F and let k be its residue field, an algebraic closure of $\mathbf{F}_p$. All geometric objects will leave over k (so we work in the usual setting with perfect schemes, algebraic spaces, prestaks etc over k).

Fix a quasi-split reductive group G over F and a pinning (B, T, e). Associated to this data we have the extended Weyl group $\tilde{W}$, a quotient of $N_G(T)(\breve{F})$ endowed with an action of Frobenius σ, as well as an Iwahori group scheme $\mathscr{I}$ in G.

Let LG be the loop group of G and let

$$\mathrm{Iw} = L^+\mathscr{I}$$

be the positive loop group of $\mathscr{I}$. As functors on perfect k-algebras R we have

$$\mathrm{LG}(R) = G(W_{\mathscr{O}_F}(R)[1/\varpi]),\ \mathrm{Iw}(R) = \mathscr{I}(W_{\mathscr{O}_F}(R)),$$

where $W_{\mathscr{O}_F}(R) = W(R) \otimes_{W(k_F)} \mathscr{O}_F$ and ϖ is a uniformizer of F.

The quotient

$$\mathrm{Fl} = \mathrm{LG}/\mathrm{Iw}$$

turns out to be a an ind-pfp scheme, more precisely filtered colimit

$$\mathrm{Fl} = \varinjlim_{w\in\tilde{W}} \mathrm{Fl}_{\leq w}$$

with transitions pfp closed embeddings. This follows from work of Zhu and Bhatt-Scholze. Explicitly, the Iw-action on Fl has orbits indexed by $\tilde{W}$ and if Fl_w is the corresponding orbit then $\mathrm{Fl}_{\leq w} = \overline{\mathrm{Fl}_w}$ is its closure. Define

$$\mathrm{LG}_w = \pi^{-1}(\mathrm{Fl}_w), \quad \mathrm{LG}_{\leq w} = \pi^{-1}(\mathrm{Fl}_{\leq w}),$$

where $\pi : \mathrm{LG} \to \mathrm{Fl}$ is the obvious projection. Thus

$$\mathrm{LG} = \varinjlim_{w\in\tilde{W}} \mathrm{LG}_{\leq w}.$$

Then LG_w and $\mathrm{LG}_{\leq w}$ are perfect affine schemes (Zhu 3.8).

Remark 2.1. Iw is an essentially pu perfect affine scheme, since

$$\mathrm{Iw} = \varprojlim_n \mathrm{Iw}^n, \quad \mathrm{Iw}^n(R) = \mathscr{I}(W_{\mathscr{O}_F}(R)/\varpi^n),$$

Iw^n being pfp schemes over k and the transitions being unipotent. Similarly, $\mathrm{LG}_{\leq w}$ and LG_w are placid, by writing them

$$\mathrm{LG}_? = \varprojlim_n \mathrm{LG}_?/H_n$$

with unipotent transitions and $H_n = \ker(\mathrm{Iw} \to \mathrm{Iw}^n)$. It follows that LG is an ind-placid scheme.

The Frobenius automorphism of $W_{\mathscr{O}_F}(-)$ induces a Frobenius σ on LG, hence an action

$$\mathrm{Ad}_\sigma : \mathrm{LG} \times \mathrm{LG} \to \mathrm{LG}, \mathrm{Ad}_\sigma(h, g) = hg\sigma(h)^{-1}.$$

Write

$$X = \mathrm{Isoc}_G = \mathrm{LG}/\mathrm{Ad}_\sigma(\mathrm{LG}), X^{\mathrm{loc}} = \mathrm{Sht}^{\mathrm{loc}} = \mathrm{LG}/\mathrm{Ad}_\sigma(\mathrm{Iw})$$

for the (étale) quotient stacks of G-isocrystals and of local shtukas (with Iwahori level). The (obvious) Newton map

$$\mathrm{Nt} : X^{\mathrm{loc}} \to X$$

is ind pfp-proper (see Benjamin's talk or Zhu's 3.28). We have

$$X^{\mathrm{loc}} = \varinjlim_{w \in \tilde{W}} X^{\mathrm{loc}}_{\leq w},$$

each

$$X^{\mathrm{loc}}_{\leq w} = \mathrm{LG}_{\leq w}/\mathrm{Ad}_\sigma \mathrm{Iw}$$

being very placid since $\mathrm{LG}_{\leq w}$ is a placid perfect scheme. Define similarly

$$X^{\mathrm{loc}}_{w} = \mathrm{LG}_{w}/\mathrm{Ad}_\sigma \mathrm{Iw}.$$

It follows from the previous discussion that X^{loc} is ind-very placid and X is sifted very placid, with atlas given by Nt. Let $\mathrm{Hk}_\bullet$ be the simplicial stack given by the Cech nerve of Nt. As we have already seen in Benjamin's talk

$$\mathrm{Hk}_n = \varinjlim_{w=(w_0,\dots,w_n)\in \tilde{W}^{n+1}} \mathrm{Hk}_{n,\leq w},$$

with

$$\mathrm{Hk}_{n,\leq w} = (\mathrm{LG}_{\leq w_0} \times^{\mathrm{Iw}} \mathrm{LG}_{\leq w_1} \times^{\mathrm{Iw}} \dots \times^{\mathrm{Iw}} \mathrm{LG}_{\leq w_n})/\mathrm{Ad}_\sigma(\mathrm{Iw}),$$

each $\mathrm{Hk}_{n,\leq w}$ being very placid. Thus each Hk_n is ind-very placid.

2.2 The local Langlands category: compact generation and self-duality

The local Langlands category is

$$\mathrm{LL} := \mathrm{Sh}^!(X).$$

We will also need

$$\mathrm{LL}^{\mathrm{loc}} := \mathrm{Sh}^!(X^{\mathrm{loc}}).$$

Theorem 2.2. *(Zhu 3.65, 3.82,*

a) LL *and* $\mathrm{LL}^{\mathrm{loc}}$ *are compactly generated and* $(\mathrm{Nt}_*, \mathrm{Nt}^!)$ *induce an adjunction between* $\mathrm{LL}^{\mathrm{loc}}$ *and* LL.

b) For $? \in \{\emptyset, \mathrm{loc}\}$ *there is a canonical Frobenius structure*

$$\mathrm{R\Gamma}^{\mathrm{can}}(X^?, -) : \mathrm{LL}^? \to \mathrm{Mod}_\Lambda$$

such that
$$R\Gamma^{\mathrm{can}}(X^{\mathrm{loc}}, -) = R\Gamma^{\mathrm{can}}(X, \mathrm{Nt}_*(-)).$$
The resulting self-dualities
$$D^? : (\mathrm{LL}^?)^\vee \simeq \mathrm{LL}^?$$
have the property that there is a canonical isomorphism on subcategories of compact objects
$$\mathrm{Nt}_* \circ D^{\mathrm{loc}} \simeq D \circ \mathrm{Nt}_*.$$

Proof. (sketch)

a) Note that each $\mathrm{Sh}^!(\mathrm{Hk}_n) = \varinjlim_w \mathrm{Sh}^!(\mathrm{Hk}_{n,\leq w})$ is compactly generated since each $\mathrm{Sh}^!(\mathrm{Hk}_{n,\leq w})$ is so (because $\mathrm{Hk}_{n,\leq w}$ is very placid). In particular $\mathrm{LL}^{\mathrm{loc}} = \mathrm{Sh}^!(\mathrm{Hk}_0)$ is compactly generated.

On the other hand by ind-pfp proper descent we can write
$$\mathrm{LL} \simeq \varprojlim_{n\in\Delta} \mathrm{Sh}^!(\mathrm{Hk}_n) \simeq \varinjlim_{n\in\Delta^{\mathrm{op}}} \mathrm{Sh}^!(\mathrm{Hk}_n),$$
the transitions being given by $()^!$ along face maps in the limit description and by $()_*$ along face maps in the colimit description. Now the face and degeneracy maps in $\mathrm{Hk}_\bullet$ are ind-pfp proper, and for an ind-pfp proper map d the functor d_* is left adjoint to $d^!$. It follows from this that Nt_* and $\mathrm{Nt}^!$ are left adjoint. One also deduces that LL is compactly generated from the colimit description.

b) This is quite a lot more painful. The idea is to construct a compatible system $(\Lambda_n^{\mathrm{can}})_{n\in\Delta})$ of generalized constant sheaves on Hk_n, giving rise to Frobenius structures $R\Gamma^{\mathrm{can}}(\mathrm{Hk}_n, -) : \mathrm{Sh}^!(\mathrm{Hk}_n) \to \mathrm{Mod}_\Lambda$ such that for all maps $\alpha : [m] \to [n]$ with corresponding map $d_\alpha : \mathrm{Hk}_n \to \mathrm{Hk}_m$ we have canonical isomorphisms
$$R\Gamma^{\mathrm{can}}(\mathrm{Hk}_n, -) \simeq R\Gamma^{\mathrm{can}}(\mathrm{Hk}_m, (d_\alpha)_*(-)).$$
For $n = 0$ we get a Frobenius structure $R\Gamma^{\mathrm{can}}(X^{\mathrm{loc}}, -)$ via $X^{\mathrm{loc}} = \mathrm{Hk}_0$. We obtain a functor $R\Gamma^{\mathrm{can}}(X, -)$ by patching the various $R\Gamma^{\mathrm{can}}(\mathrm{Hk}_n, -)$ via the presentation
$$\mathrm{LL} \simeq \varinjlim_{n\in\Delta^{\mathrm{op}}} \mathrm{Sh}^!(\mathrm{Hk}_n).$$
It is compatible via Nt_* with $R\Gamma^{\mathrm{can}}(X, -)$ by construction, and we have to check that it induces a Frobenius structure. First, observe that each $R\Gamma^{\mathrm{can}}(\mathrm{Hk}_n, -)$ induces a self-duality D_n on $\mathrm{Sh}^!(\mathrm{Hk}_n)$, and these are compatible with the boundary maps (because the Frobenius structures on the Hk_n are compatible), yielding a self-duality D on LL such that $D \circ \mathrm{Nt}_* \simeq \mathrm{Nt}_* \circ D^{\mathrm{loc}}$ on compact objects, where D^{loc} is the self-duality on $\mathrm{LL}^{\mathrm{loc}}$ induced by $R\Gamma^{\mathrm{can}}(X^{\mathrm{loc}}, -)$. This formally implies that
$$\mathrm{Hom}_{\mathrm{LL}}(F, G) \simeq R\Gamma^{\mathrm{can}}(X, D(F) \otimes^! G)$$
for $F \in \mathrm{LL}^\omega$, thus $R\Gamma^{\mathrm{can}}(X, -)$ is indeed a Frobenius structure with associated self-duality D. Indeed, to check the above isomorphism we may assume that $F = \mathrm{Nt}_*(F_{\mathrm{loc}})$

with $F_{\mathrm{loc}} \in \mathrm{LL}^{\mathrm{loc}}$ compact, since (co)sheaves of this form generate LL^{ω}. Then we compute

$$\mathrm{Hom}_{\mathrm{LL}}(F, G) = \mathrm{Hom}_{\mathrm{LL}^{\mathrm{loc}}}(F_{\mathrm{loc}}, \mathrm{Nt}^!(G)) \simeq \mathrm{R}\Gamma^{\mathrm{can}}(X^{\mathrm{loc}}, D^{\mathrm{loc}}(F^{\mathrm{loc}}) \otimes^! \mathrm{Nt}^!(G))$$

$$\simeq \mathrm{R}\Gamma^{\mathrm{can}}(X, \mathrm{Nt}_*(D^{\mathrm{loc}}(F^{\mathrm{loc}}) \otimes^! \mathrm{Nt}^!(G))) \simeq \mathrm{R}\Gamma^{\mathrm{can}}(X, \mathrm{Nt}_*(D^{\mathrm{loc}}(F^{\mathrm{loc}})) \otimes^! G) \simeq \mathrm{R}\Gamma^{\mathrm{can}}(X, D(\mathrm{Nt}_*(F)) \otimes^! G),$$

as desired.

This being said, let us explain how to construct the various Λ_n^{can} on

$$\mathrm{Hk}_n = \varinjlim_{w=(w_0,\ldots,w_n)\in\tilde{W}^{n+1}} \mathrm{Hk}_{n,\leq w},$$

or equivalently we need to construct generalized constant sheaves $\Lambda_{n,\leq w}^{\mathrm{can}}$ on $\mathrm{Hk}_{n,\leq w}$ that are $()^*$-compatible with respect to transition maps. There is a natural map

$$\delta : X^{\mathrm{loc}} \to Y := \mathrm{Iw}\backslash \mathrm{LG}/\mathrm{Iw}$$

restricting to maps

$$\delta_w : X_w^{\mathrm{loc}} \to Y_{w^{-1}} := \mathrm{Iw}\backslash \mathrm{LG}_{w^{-1}}/\mathrm{Iw}, \quad \delta_{\leq w} : X_{\leq w}^{\mathrm{loc}} \to Y_{\leq w^{-1}} := \mathrm{Iw}\backslash \mathrm{LG}_{\leq w^{-1}}/\mathrm{Iw}.$$

Here w^{-1} is simply related to a choice of normalization of δ. This induces a map

$$\delta_{n,\leq w} : \mathrm{Hk}_{n,\leq w} \simeq (\mathrm{LG}_{\leq w_0} \times^{\mathrm{Iw}} \mathrm{LG}_{\leq w_1} \times^{\mathrm{Iw}} \ldots \times^{\mathrm{Iw}} \mathrm{LG}_{\leq w_n})/\mathrm{Ad}_\sigma(\mathrm{Iw}) \to Y_{\leq w_n^{-1}} \times \ldots \times Y_{w_0^{-1}}$$

and we define

$$\Lambda_n^{\mathrm{can}} = \delta_{n,\leq w}^!(\Lambda_{\leq w_n^{-1}}^{\mathrm{can}} \boxtimes_\Lambda \boxtimes \ldots \boxtimes \Lambda_{\leq w_0^{-1}}^{\mathrm{can}}),$$

where we still need to define generalized constant sheaves $\Lambda_{\leq w}^{\mathrm{can}}$ on $Y_{\leq w} := \mathrm{Iw}\backslash \mathrm{LG}_{\leq w}/\mathrm{Iw} = \mathrm{Iw}\backslash \mathrm{Fl}_{\leq w}$ for $w \in \tilde{W}$. Now $\mathrm{Fl}_{\leq w}$ is a pfp scheme over k, so has a natural generalized constant sheaf $\Lambda_{\mathrm{Fl}_{\leq w}}^{\mathrm{can}}$ obtained as $*$-pullback of ω_{pt} along $\mathrm{Fl}_\lambda \to \mathrm{pt}$. This sheaf is Iw-equivariant and we want to descend it to $Y_{\leq w}$ and get $\Lambda_{\leq w}^{\mathrm{can}}$. This is somewhat tricky since we are taking quotients by big group schemes. But since $\mathrm{Fl}_{\leq w}$ is pfp, for large enough n the action of Iw on $\mathrm{Fl}_{\leq w}$ factors through $\mathrm{Iw}^n = \mathrm{Iw}/\mathrm{Iw}^{(n)}$ The group $\mathrm{Iw}^{(n)}$ is pu, and one can then show that !-pullback induces an equivalence

$$\mathrm{Sh}^!(\mathrm{Iw}^n\backslash \mathrm{Fl}_{\leq w}) \simeq \mathrm{Sh}^!(\mathrm{Iw}\backslash \mathrm{Fl}_{\leq w}).$$

Now we can descend $\Lambda_{\mathrm{Fl}_{\leq w}}^{\mathrm{can}}$ to $\mathrm{Sh}^!(\mathrm{Iw}^n\backslash \mathrm{Fl}_{\leq w})$ and define $\Lambda_{\leq w}^{\mathrm{can}}$ as the corresponding object in $\mathrm{Sh}^!(\mathrm{Iw}\backslash \mathrm{Fl}_{\leq w})$. One has to check that these generalized constant sheaves are compatible with the simplicial structure, which comes down more or less to corresponding statements for affine flag varieties, see Zhu 3.80 for the (not so many) details.

□

Remark 2.3. It is expected that $\mathrm{R}\Gamma^{\mathrm{can}}(X, -)$ is simply the left adjoint to the functor $()^! : \mathrm{Mod}_\Lambda \to \mathrm{LL}$ induced by the projection $X \to \mathrm{pt}$.

2.3 Compact objects, Newton stratification

Fix $b \in B(G)$ and recall that He's theorem allows one to identify $B(G)$ with the set $B(\tilde{W})_{\rm str}$ of σ-straight σ-conjugacy classes in $\tilde{W}$. Let w_b be the associated σ-conjugacy class and *choose* once and for all a representative $\tilde{w}_b$ in $N_G(T)(\breve{F}) \subset G_b(\breve{F})$ for a σ-straight element in w_b. This allows to describe the F-points of the usual group G_b attached to F (which is actually well-defined up to conjugacy...) by

$$G_b(F) = \{g \in G(\breve{F}) |\, g^{-1}\tilde{w}_b\sigma(g) = \tilde{w}_b\}$$

and yields an Iwahori subgroup

$$I_b = \mathscr{I}(\mathscr{O}_{\breve{F}}) \cap G_b(F).$$

Note that the notation is again abusive since this really depends on the choice of the representative $\tilde{w}_b$. We will abuse again notation and write

$$X_b^{\rm loc} := X_{\tilde{w}_b}^{\rm loc}$$

and define similarly $X_{\leq b}^{\rm loc}$. Finally, recall that there are inclusions

$$j_b : X_b \to X_{\leq b}, \quad i_{\leq b} : X_{\leq b} \to X, i_{<b} : X_{<b} \to X, i_b : X_b \to X.$$

The next deep theorem appeared in Benjamin's talk:

Theorem 2.4. *(Zhu) Keep the above notations.*

a) j_b is an affine qc open immersion and $i_{\leq b}$ is a pfp closed embedding, $X_{\leq b}$ being the closure of X_b.

b) The Newton map $\mathrm{Nt} : X^{\rm loc} \to X$ restricts to a map $\mathrm{Nt}_b : X_b^{\rm loc} \to X_b$. The choice of $\tilde{w}_b$ induces morphisms $\mathrm{pt} \to X_b^{\rm loc}, \mathrm{pt} \to X_b$ which in turn induce isomorphisms

$$B_{\rm profet}(I_b) \simeq X_b^{\rm loc}, \quad B_{\rm proet}(G_b(F)) \simeq X_b$$

such that the corresponding map Nt_b is induced by the inclusion $I_b \subset G_b(F)$.

c) The map Nt_b is the base change of $X_{\leq b}^{\rm loc} \to X$ along $\iota_b : X_b \to X$.

It follows from this theorem and the results on classifying stacks that we have canonical identifications

$$\mathrm{LL}_b := \mathrm{Sh}^!(X_b) \simeq \mathrm{Rep}(G_b(F)), \quad \mathrm{LL}_b^{\rm loc} := \mathrm{Sh}^!(X_b^{\rm loc}) \simeq \mathrm{Rep}(I_b)$$

such that $\mathrm{Nt}_{b,*}$ corresponds to $c - \mathrm{Ind}_{I_b}^{G_b(F)}(-)$ and $\mathrm{Nt}_b^!$ corresponds to the forgetful functor.

Proposition 2.5. *Let $? \in \{b, \leq b, < b\}$, then we have adjoint pairs $(i_{?,!}, i_?^!)$ and $(i_?^*, i_{?,*})$ between $\mathrm{LL}_?$ and LL, with $i_{?,*}$ and $i_{?,!}$ fully faithful. Also there are natural isomorphisms of functors*

$$i_{\leq b,*} \simeq i_{\leq b,!}, \quad i_{<b,*} \simeq i_{<b,!},$$

as well as canonical fiber sequences (on LL*)*

$$i_{<b,*}i^!_{<b} \to i_{\leq b,*}i^!_{\leq b} \to i_{b,*}i^!_b$$

$$i_{b,!}i^*_b \to i_{\leq b,*}i^*_{\leq b} \to i_{<b,*}i^*_{<b}.$$

Also one has canonical fiber sequences on $\mathrm{LL}_{\leq b}$

$$i_{<b,*}i^!_{<b} \to \mathrm{id} \to j_{b,*}j^!_b.$$

Proof. By the above discussion we have a qc open immersion $j_b : X_b \to X_{\leq b}$ with closed complement $\alpha : X_{<b} \to X_{\leq b}$, as well as

$$i_{<b} = i_{\leq b}\alpha, \quad i_b = i_{\leq b}j_b.$$

Now use the excision sequences (Zhu 10.177, using that we are working with sifted very placid stacks)

$$\alpha_*\alpha^! \to \mathrm{id} \to j_{b,*}j^!_b, \quad j_{b,!}j^!_b \to \mathrm{id} \to \alpha_*\alpha^*$$

and the isomorphisms $\alpha_* \simeq \alpha_!$ and $j^!_b \simeq j^*_b$. Also note that $\alpha_*, j_{b,*}, j_{b,!}$ are fully faithful, with essential image given by $\ker j^!_b$, $\ker \alpha^!$, respectively $\ker \alpha^* F$. $\square$

As a first application we prove[2]

Proposition 2.6. *For any perfect prestack Y over k with $\mathrm{Sh}^!(Y)$ dualizable the external tensor product induces an equivalence*

$$\mathrm{LL} \otimes_\Lambda \mathrm{Sh}^!(Y) \simeq \mathrm{Sh}^!(X \times Y).$$

Proof. Since $\mathrm{LL} = \varinjlim \mathrm{LL}_{\leq b}$ and $\mathrm{Sh}^!(X \times Y) \simeq \varinjlim_b \mathrm{Sh}^!(X_{\leq b} \times Y)$, it suffices to prove the statement for $X_{\leq b}$ instead of X. Since $\mathrm{Sh}^!(X_{\leq b})$ is compactly generated, thus dualizable, the functor is fully faithful (something we have already seen), so it suffices to see that its essential image generates $\mathrm{Sh}^!(X_{\leq b} \times Y)$. This is proved by induction, using the excision sequence

$$i_{<b,*}i^!_{<b} \to \mathrm{id} \to j_{b,*}j^!_b$$

and using that $(-)_*$ and $()^!$ commute with external tensor product, as well as the fact that we know already the result for X_b instead of X, as $X_b \simeq B_{\mathrm{proet}}(G_b(F))$. A crucial point here is the property of the partial order on $B(G)$ that for any b there are only finitely many b' with $b' \leq b$, which allows one to perform the induction. $\square$

The following result is much deeper. Recall that a sequence of functors $F : A \to B, G : B \to C$ in LC_Λ is a semi-orthogonal decomposition of B if F, G have Λ-linear right adjoints F^R, G^R, the adjunctions induce isomorphisms $\mathrm{id} \simeq F^R \circ F, G \circ G^R \simeq \mathrm{id}$, $G \circ F \simeq 0$, G^R has a Λ-linear right adjoint and finally there is a fiber sequence

$$FF^R \to \mathrm{id} \to G^R G.$$

[2]The hypothesis that $\mathrm{Sh}^!(Y)$ is dualizable does not appear in Zhu's paper, but a closer look to its proof shows that it uses a result for classifying stacks, which requires this hypothesis.

Theorem 2.7. *(Zhu) a) The functors $i_{b,*}, i_{b,!}, i_b^!, i_b^*, j_{b,*}, i_{<b,*}$ preserve compactness.*

b) $F \in \mathrm{LL}$ is compact if and only if $i_b^! F$ is compact for all b and vanishes for almost all b, or equivalently $i_b^* F$ is compact for all b and vanishes for almost all b.

c) The functors $i_{<b,*} : \mathrm{LL}_{<b} \to \mathrm{LL}_{\leq b}$ and $j_b^! : \mathrm{LL}_{\leq b} \to \mathrm{LL}_b$ induce a semi-orthogonal decomposition of $\mathrm{LL}_{\leq b}$ and the functors $j_{b,!} : \mathrm{LL}_b \to \mathrm{LL}_{\leq b}$, $i_{<b}^* : \mathrm{LL}_{\leq b} \to \mathrm{LL}_{<b}$ induce a second semi-orthogonal decomposition.

Proof. a) For $i_{b,!}, i_b^*$ this is trivial since their right adjoints are continuous. Also $i_{<b,*} \simeq i_{<b,!}$ has a continuous right adjoint, so it preserves compactness. Next, since $i_{\leq b,*}$ is fully faithful and $i_b = i_{\leq b} \circ j_b$, we have $j_{b,*} = i_{\leq b}^* i_{b,*}$ and so it suffices to see that $i_{b,*}$ and $i_{b,!}$ preserve compactness. Let

$$i_b^{\mathrm{loc}} : X_b^{\mathrm{loc}} \to X^{\mathrm{loc}}, \quad i_{\leq b}^{\mathrm{loc}} : X_{\leq b}^{\mathrm{loc}} \to X^{\mathrm{loc}}$$

and let $\alpha = \mathrm{Nt} \circ i_{\leq b}^{\mathrm{loc}} : X_{\leq b}^{\mathrm{loc}} \to X$ be the restriction of Nt and $\beta : X_b^{\mathrm{loc}} \to X_{\leq b}^{\mathrm{loc}}$ be the inclusion, so that $i_{\leq b}^{\mathrm{loc}} \circ \beta = i_b^{\mathrm{loc}}$. As we have already seen, Nt_b is the base-change of α along i_b.

Let us prove that $i_{b,*}$ preserves compactness. As LL_b^ω is generated by $\mathrm{Nt}_{b,*}(\mathrm{LL}_b^{\mathrm{loc},\omega})$ with $\mathrm{LL}_b^{\mathrm{loc}} = \mathrm{Sh}^!(X_b^{\mathrm{loc}})$, it suffices to see that

$$i_{b,*}\mathrm{Nt}_{b,*} \simeq \mathrm{Nt}_* i_{b,*}^{\mathrm{loc}}$$

preserves compactness. Since Nt_* has a continuous right adjoint $\mathrm{Nt}^!$, it preserves compactness, thus it suffices to see that $i_{b,*}^{\mathrm{loc}}$ preserves compactness. This is hard and will be done below (see the theorem below). Similarly, to see that $i_b^!$ preserves compactness we use that LL^ω is generated by $\mathrm{Nt}_*(\mathrm{LL}^{\mathrm{loc},\omega})$ to reduce to checking that $i_b^! \mathrm{Nt}_*$ preserves compactness. Now by base-change and using that $i_{\leq b,*}$ is fully faithful with right adjoint $i_{\leq b}^!$ we have

$$i_b^! \mathrm{Nt}_* \simeq i_b^! \mathrm{Nt}_* i_{\leq b,*}^{\mathrm{loc}} i_{\leq b}^{\mathrm{loc},!} \simeq i_b^! \alpha_* i_{\leq b}^{\mathrm{loc},!} = \mathrm{Nt}_{b,*} \beta^! i_{\leq b}^{\mathrm{loc},!} = \mathrm{Nt}_{b,*} i_b^{\mathrm{loc},!}.$$

Now Nt_b is ind-pfp proper, thus $\mathrm{Nt}_{b,*}$ preserves compactness and we are reduced to proving that $i_b^{\mathrm{loc},!}$ preserves compactness, which is again hard and done below (see the theorem below).

b) If F is compact then we know that $i_b^* F$ and $i_b^! F$ are compact for all b. We need to see that they are zero for almost all b. We will give the proof for i_b^* and defer the proof for $i_b^!$ to the next section. Since $\mathrm{LL} = \varinjlim_b \mathrm{LL}_{\leq b}$, there are finitely many $b_1, \ldots, b_n$ with $F \in \cup_{i=1}^n \mathrm{LL}_{\leq b_i}$ (the point is that $B(G)$ is not a filtered poset, so we can't say that $F \in \mathrm{LL}_{\leq b}$ for some b). If $i_b^* F \neq 0$ for some b, then necessarily $b \leq b_i$ for some i and for each i there are only finitely many such b's.

Conversely, assume that $i_b^* F$ is compact for all b, and vanishes for almost all b. Then the support of F is concentrated on finitely many strata and we may assume that $F = i_{\leq b,*} G$ for some $G \in \mathrm{LL}_{\leq b}$. We then have a fiber sequence

$$i_{b,!} i_b^* F \to F = i_{\leq b,!} i_{\leq b}^* F \to i_{<b,!} i_{<b}^* F$$

and it suffices to see that the extremal terms are compact. But i_b^*F is compact and $i_{b,!}$ preserves compactness, and by induction $i_{<b}^*F$ is compact and $i_{<b,!}$ preserves compactness, so we are done.

For $i_b^!$ the argument involves duality, which will be seen in the next talk.

c) This follows formally from a) and the previous discussion concerning excision. $\square$

Remark 2.8. As Simon remarked, part b) is more subtle than expected and the argument given is not really complete.

Theorem 2.9. *(Zhu 10.148, 10.152, 10.173) Let H be a perfect affine group scheme over k sitting in an exact sequence*

$$1 \to H_0 \to H \to H' \to 1$$

of perfect affine group schemes over k with H_0 pro-unipotent and H' pfp over k. Let H act on an ind-placid space U and let $f : U \to X = U/H$ be the natural map.

a) The right adjoint $f_\flat$ to $f^!$ preserves compactness and $\mathrm{Sh}^!(X)^\omega$ is generated by the essential image of $\mathrm{Sh}^!_c(U)$ under $f_\flat$.

b) For any representable pfp morphism $\varphi : Y \to X$ the functors φ_* and $\varphi^!$ preserve compactness.