Gpe de travail Zhu

G. DOSPINESCU - 05/06/26 - 10h

<u>Recap:</u> k, Λ as usual

Sp = perfect qcqs alg. spaces / k

PSt = perfect prestacks over k

<u>placid space:</u> $X \in Sp$ s.th. $X \to *$ is essentially ps (= pro-smooth)

(i.e. $X \xrightarrow[ps]{} X'$, $X' \to$ pfp over $*$)

Have nice theory of constructible sheaves:

for f pfp, $f^!, f_*$ have LA $f_!, f^*$ that preserve constructibles

Also $\underline{\mathrm{Hom}}(F, G) \in S_c^!(X) \subset S^!(X)$ $\forall F, G \in S_c^!(X)$

[$S^!: \mathrm{Corr}(Sp)_{pfp; all}$ extends to $\mathrm{Corr}(Sp)_{ess\,proét; all}$]

For f proét., $f_* = RA(f^!)$

$f_!$ also exists for f ess.-pu

- X qc stack is called <u>quasi-placid</u> if it has a q-placid atlas $S \xrightarrow{f} X$ (i.e. repr. surj. ps map)

f is <u>placid</u> if f is also of universal homological descent ($S^!(X) = \varprojlim_{n \in \Delta} S^!(S_n)$ after any base change)

f is <u>very placid</u> if f is also essentially pu

X is (very) placid if it has a (very) placid atlas.

- X q-placid $\to$ $S^!(X)^\omega \subset S_c^!(X)$

If X very placid with very placid atlas $f: S \to X$,

then $S^!(X)$ is compactly generated

$S^!(X)^\omega$ = smallest stable idemp.-complete subcat. containing $f_!F$ for $F \in S^!_c(S)$.

If f is pu, $S^!(X)^\omega = S^!_c(X)$.

<u>Interlude on ind-stuff:</u>

P property of maps in Sp.

$f: X \to Y$ in PSt is ind-P if $\forall S \to Y$, $S \in Sp$, $X \times_Y S \to S$ can be written as filt. colim. $X \times_Y S = \varinjlim X_i$ where $X_i \to S$ are maps in Sp with property P, and $X_i \to X_j$ are pfp closed immersions.

formal procedure extends $S^!$ to Corr(PSt, Ind-Essproet)

For $f: X = \varinjlim X_i \longrightarrow Y$, ($Y$ a space)

$$f_*: \underset{\varinjlim_{(-)_*} S^!(X_i)}{\overset{\shortparallel}{S^!(X)}} \longrightarrow S^!(Y)$$

induced by functors $S^!(X_i) \xrightarrow[(f_i)_*]{} S^!(Y)$

with $f_i: X_i \to X \xrightarrow{f} Y$

<u>Easy to see:</u>
For f ind-pfp proper in PSt, $f_* = LA(f^!)$

Thm: (10.106) If f surjective and ind-pfp proper, f is of universal homol. descent.

<u>Rmk</u> (10.108): If f rep. ess.-pu, f is of universal homol. descent.

(seems dubious --)

Def: $\mathcal{P} \in \{$q-placid, placid, very-placid$\}$

A stack X is called ind-$\mathcal{P}$ if as a prestack X is a filt. colim. of (qc) stacks X_i which are $\mathcal{P}$ and s.t. $X_i \to X_j$ are pfp closed immersion.

Thm (10.155)

Y placid (very placid) stack and $f: X \to Y$ ind-pfp/ ind-pfp proper then ~~f~~ X is ind-placid (ind-very placid). Actually f is a filt. colimit of repr. pfp/pfp proper maps $X_i \to Y$ with $X_i \to X_j$ pfp closed immersion

Def: X is a sifted placid/ sifted very placid stack if $\exists$ atlas $V \xrightarrow{f} X$ with f surj. ind-pfp proper

V: ind-placid/ ind-very placid

Rmk: By Thm 10.106, such f is of univ. hom. descent. By Thm. 10.155, all V_n in the Cech nerve of f are ind-(very-) placid, boundary maps are ind-pfp proper

$\left("X = \varinjlim_{n \in \Delta^{op}} V_n" \right)$

Also $\mathcal{S}^!(X) = \varprojlim_{\substack{n \in \Delta \\ (-)^!}} \mathcal{S}^!(V_n) = \varinjlim_{\substack{n \in \Delta^{op} \\ (-)_*}} \mathcal{S}^!(V_n)$.

2 key examples:

- H locally-profinite gp with a Λ-valued Haar measure
  $\rightsquigarrow \underline{H}$ gp scheme over k representing $S \to C^0(|S|, H)$.

  $B\underline{H} := B_{fpqc}\underline{H}$: classifying stack for fpqc $\underline{H}$-torsors

  This is sifted placid

  Idea: $K \leq H$ suff. nice compact open subgp

  $$\underbrace{B\underline{K}}_{\text{ind-placid}} \xrightarrow{f} B\underline{H} \quad \text{is ind-pfp finite and surjective}$$

- G over E non arch. local field of res. char. $p \neq \ell$
  ϖ unif. of E

  LG : $\left\{ \begin{array}{l} \text{perfect } k\text{-alg.} \longrightarrow \mathrm{Gr} \\ \qquad R \longmapsto G(W_{\mathcal{O}_E}(R)[\tfrac{1}{\varpi}]) \end{array} \right.$

  is an ind-scheme.

  Frob on $R \rightsquigarrow$ Frob φ on $W_{\mathcal{O}_E}(R)[\frac{1}{\varpi}]$

  and so $\varphi \circlearrowright LG$.

  $\rightsquigarrow$ φ-twisted conjugation action of LG on LG

  $\mathrm{Isoc}_G = LG/\mathrm{Ad}_\varphi LG$ is a sifted very placid stack.

  Atlas is provided by Newton map $\mathrm{Nt}: \mathrm{Sht}_J \to \mathrm{Isoc}_G$

  $$\mathrm{Sht}_J = LG/\mathrm{Ad}_\varphi \underbrace{L^+J}_{\substack{\text{positive loop gp} \\ \text{of an Iwahori } J \text{ of } G}}$$

Local Langlands category: $\mathcal{S}^!(\mathrm{Isoc}_G)$.

<u>Rmk:</u> f rep. ind-pfp or ess. pu map between sifted very placid stacks, then we shall have $f_! = LA(f^!)$ compatible with !-pullback by ps maps.

If f is rep. pfp we have $f^* = LA(f_*)$ with similar base change properties.

<u>Thm:</u> X sifted very placid.

$j : U \hookrightarrow X$ qc open embedding with closed complement $i : Z \to X$. Then Z, U are sifted very placid, i is pfp and $i_* = i_!$, $j^*, j^!$ are fully faithful with essential image those $F \in S'(X)$ s.th. $j^! F = 0$, $i^! F = 0$, $i^* F = 0$.

We have fiber sequence

$$i_* i^! F \to F \to j_* j^! F$$

$$j_! j^* F \to F \to i_* i^* F.$$

<u>Verdier duality</u>

X pfp space $\rightsquigarrow$ $\omega_X = \pi_X^! \Lambda$ for $\pi_X : X \to \operatorname{Spec} k$

and canonical Verdier duality $\mathbb{D}_X : \begin{cases} Sh_c^*(X)^{op} \xrightarrow{\sim} Sh_c^*(X) \\ F \mapsto \underline{\mathrm{Hom}}(F, \omega_X) \end{cases}$

$\mathbb{D}_X(\Lambda) = \omega_X$, $\mathbb{D}_X(\omega_X) = \Lambda$.

for $f : X \to Y$ between pfp spaces, $f^!$ and f^* are intertwined by $\mathbb{D}$

$f_!$ and f_* ________

<u>Rmk:</u> $S^*_c(X)^{op} = S^!_c(X)$

Hence can interpret $\mathbb{D}_X$ as a duality $S^!_c(X)^{op} \xrightarrow{\sim} S^!_c(X)$

s.th. $\mathrm{Hom}_{S^!_c(X)}(F, G) = R\Gamma(X, \mathbb{D}(F) \otimes^! G)$

$$:= \mathrm{Hom}_{S^!_c(X)}(\Lambda, \mathbb{D}(F) \otimes^! G).$$

$\rightsquigarrow$ Get Frobenius structure on $S^!_c(X)$:

$$\begin{cases} S^!_c(X) \longrightarrow \mathrm{Mod}_\Lambda \\ F \longmapsto \mathrm{Hom}(\Lambda, F). \end{cases}$$

X placid space.

$$r : X \xrightarrow[ps]{} X' \quad \begin{matrix} \uparrow \\ \text{pfp} \end{matrix}$$

$\eta := r^* \omega_{X'} \in S^*_c(X)$

(⚠ this is not canonical!)

<u>Hard thm:</u> If η' is obtained from another $r' : X \xrightarrow[ps]{} X''$ pfp

then $\eta' = \eta \otimes \mathcal{L}$ with $\mathcal{L} \in Sh^*_c(X)$ invertible.

Key geom. input: $\forall$

$$\begin{tikzcd} & X \arrow[dl, "r"', "ps"] \arrow[dr, "r'", "ps"'] & \\ X' \ \text{pfp} & & X'' \ \text{pfp} \end{tikzcd}, \quad \exists \quad \begin{tikzcd} & X \arrow[d, "ps"] \arrow[ddl, bend right, "r"'] \arrow[ddr, bend left, "r'"] & \\ & X''' \arrow[dl, "sm"'] \arrow[dr, "sm"] & \\ X' & & X'' \end{tikzcd}$$

A generalized dualizing sheaf on X: anything of the form $\eta \otimes \mathcal{L}$ with $\mathcal{L} \in S^*_c(X)$ invertible

For such η we get $\quad S^*_c(X)^{op} \xrightarrow{D^\eta} S^*_c(X)$

$$F \longmapsto \underline{\mathrm{Hom}}(F, \eta)$$

constr. because X placid

r^* commutes with r^*

$$\Rightarrow \quad \mathbb{D}^\eta(r^*A) \simeq r^*(\mathbb{D}_{X'}A)$$

$\Rightarrow$ $\mathbb{D}^\eta$ is an involution (as it is on X' and on all X^i with $X = \varprojlim X_i$, $X_i \xrightarrow{sm} X'$)

Base-change properties $\rightarrow$ this is <u>weakly functorial</u>:

$f : X \to Y$ map between placid spaces. η gen. dualizing sheaf on Y.

- f pfp $\Rightarrow$ $\eta' = f^!\eta$ is a gen. dual. sh. on X and $\begin{pmatrix} f^! \text{ and } f^* \\ f_! \text{ and } f_* \end{pmatrix}$ are intertwined by $\mathbb{D}^\eta, \mathbb{D}^{\eta'}$

- f (weakly) pr $\Rightarrow$ $\eta' = f^*\eta$ is gen. dualiz. sheaf and $f^* \longleftrightarrow f^*$ via $\mathbb{D}^\eta$ and $\mathbb{D}^{\eta'}$.

As before $\mathbb{D}^\eta$ can be seen as $\mathbb{D}^\eta : S_c^!(X)^{op} \xrightarrow{\sim} S_c^!(X)$

We still have $\mathrm{Hom}_{S_c^!(X)}(F, G) = \mathrm{Hom}_{S_c^!(X)}(\Lambda^\eta, \mathbb{D}^\eta(F) \overset{!}{\otimes} G)$

where Λ^η is η seen in $S_c^!(X) = S_c^*(X)^{op}$.

Call Λ^η is a generalized constant sheaf on X.

Still have weak functoriality as before.

<u>Thm:</u> $U \xrightarrow{f} X$ quasi-placid atlas, U placid space $\rightarrow$ $\Lambda^\eta \in S_c^!(X)$ s.th. $f^!\Lambda^\eta$ is a generalized constant sheaf on U.

<u>Rmk:</u> In practice this can be constructed explicitly.

$\exists$ self-duality involution $\mathbb{D}^\eta : \mathcal{S}_c^!(X)^{op} \xrightarrow{\sim} \mathcal{S}_c^!(X)$
s.th. the same formula for $\mathrm{Hom}(F, G)$ as above holds.

Upshot: For X quasi-placid stack, $\mathcal{S}^!(X)$ is self-dual with Frobenius structure $\begin{cases} \mathcal{S}^!(X) \longrightarrow \mathrm{Mod}_\Lambda \\ F \longmapsto \mathrm{Hom}(\Lambda^\eta, F) \end{cases}$

for a gener. constant sheaf Λ^η.
This is weakly functorial

What about ind-qplacid stacks?

$X = \varinjlim X_i$ ind-qplacid
$\uparrow$ qpl., transition maps pfp closed im.

$\mathcal{S}^!(X) = \varinjlim \mathcal{S}^!(X_i)$ but $\mathcal{S}_c^!(X) \neq \varinjlim \mathcal{S}_c^!(X_i)$

$X_i \xrightarrow[\alpha_{ij}]{} X_j$ is pfp c.i.
$\Rightarrow$ $\mathcal{S}_c^!(X_i) \xrightarrow[(\alpha_{ij})_*]{} \mathcal{S}_c^!(X_j)$ is ff

$\Rightarrow$ get $\underbrace{\varinjlim \mathcal{S}_c^!(X_i)}$ $\xhookrightarrow[ff]{}$ $\mathcal{S}^!(X)$
$\uparrow$ independent of presentation
is called $\mathcal{S}_{fg}^!(X)$

$\leadsto$ get $\boxed{\mathrm{Ind}\,\mathcal{S}_{fg}^!(X)}$

A generalized cst sh. on X is a collection $\Lambda^\eta := (\Lambda_i^\eta)$

of gen. cst. sh. on X and isom. $\alpha_{ij}^* \Lambda_j^\eta \simeq \Lambda_i^\eta$

satisfying "usual conditions".

$\mathrm{Hom}(\Lambda_i^\eta, -) : S_c^!(X_i) \to \mathrm{Mod}_\Lambda$ are compatible

$\rightsquigarrow$ get Frobenius structure $S_{fg}^!(X) \xrightarrow[R\Gamma^\eta(X,-)]{} \mathrm{Mod}_\Lambda$

and $\mathbb{D}^\eta : S_{fg}^!(X)^{op} \xrightarrow{\sim} S_{fg}^!(X)$ s.th.

$$\mathrm{Hom}(F, G) = R\Gamma^\eta(X, \mathbb{D}^\eta F \otimes^! G)$$

So for ind-placid X, Ind $S_{fg}^!(X)$ is <u>self-dual</u>.

(when we can find a generalized cst sheaf)

<u>Smooth rep^{ns}</u>

H loc. profin. gp

$\mathcal{H} := C_c^\infty(H, \Lambda)$: cp supported smooth (i.e. loc. cst) $f : H \to \Lambda$

$H \times H \curvearrowright \mathcal{H}$ via $(h_1, h_2) \cdot f(x) = f(h_1^{-1} x h_2)$

See it for now as H-rep. via $(h_1 \cdot f)(x) = f(h_1^{-1} x)$

<u>Key hyp</u>: $\exists$ Λ-valued left Haar measure $\int_H : \mathcal{H} \to \Lambda$

i.e. H-equiv. s.th. $\exists$ $K \leq H$ compact open s.th.

$\mathrm{vol}\, K := \int_H 1_K(x)\, dx \in \Lambda^\times$. Call such K a <u>good subgp</u>

(If $K' \leq K$ open with K good, then K' is good)

A_H = sm rep^{ns} of H on Λ-modules

This is a Groth. abelian cat. with cp proj. generators

$V_K = \text{c-Ind}_K^H \Lambda \simeq \mathcal{H}^K$ (use right act. of H on $\mathcal{H}$ to make it an H-rep.)
for K good

$\text{Hom}_{A_H}(V_K, -) \underset{\substack{\text{Frob.}\\ \text{rec.}}}{=} \text{Hom}_{A_K}(\Lambda, -) = (-)^K$ left exact.

Actually this is exact and colimit-preserving thanks to

<u>Lemma:</u> $\forall v \in A_H$, $\quad V_K \xrightarrow{\sim} V^k$

$$\begin{array}{c} \| \quad [v] \longmapsto \int_K kv\,dk \\ V/\underset{k\in K, v \in V}{\text{span}(kv - v)} \end{array}$$

(i.e. use idempotent $\frac{1}{\text{vol}K} 1_K \in \mathcal{H}$)

$\text{Rep}(H) := \mathcal{D}(A_H)$ derived ∞-cat. of A_H is left comp. & compactly generated
(V_K's are compact generators)

<u>Q6h</u>:

Suppose H is profinite.
$H \rightsquigarrow \underline{H}_\Lambda$ affine flat gp. scheme over Λ

If $H = \varprojlim H_i$, $\underline{H}_\Lambda = \varprojlim \underline{H_i}_\Lambda$
fin. gp

$$\mathcal{O}(\underline{H}_\Lambda) = \varinjlim \mathcal{O}(\underline{H_i}_\Lambda)$$
$$= \varinjlim \text{Fct}(H_i, \Lambda)$$
$$= \mathcal{H} = C^\infty(H, \Lambda)$$

$\text{Alg}(\underline{H}_\Lambda) = \mathcal{O}(\underline{H}_\Lambda)$-comodules
alg. reps

$= \text{smooth } \Lambda\text{-reps of } H = A_H$

for V sm. rep. of H we have

$V \longrightarrow LC(H, V) \simeq LC(H, \Lambda) \otimes_\Lambda V = \mathcal{H} \otimes V$
$v \longmapsto (h \mapsto h\cdot v)$

$\rightsquigarrow$ makes V into a comodule over $\mathcal{O}(\underline{H}_\wedge) = \mathcal{H}$

$D(\mathrm{Alg}(\underline{H}_\wedge)) = \mathrm{Rep}(H)$

<u>Rmk 9.13</u> $\Rightarrow$ $\mathrm{QCoh}(\mathcal{B}_{fpqc}\underline{H}_\wedge) \simeq$ left completion of $D(\mathrm{Alg}(\underline{H}_\wedge))$

$= \mathrm{Rep}(H)$

and is comp. gen.

<u>For $S^!, S^*$</u>

Suppose H profinite $\rightsquigarrow$ $\underline{H}$ sch. over k

$\rightsquigarrow B\underline{H} = B_{fpqc}\underline{H}$ classifying stack

<u>Thm:</u> $pt = \mathrm{Spec}\,k \xrightarrow{f} B\underline{H}$ is of universal descent for both $S^*, S^!$

Čech nerve of this map: $(\underline{H}^n)_{n\in\Delta}$

$$\Rightarrow \quad S^*(B\underline{H}) = \varprojlim_{\substack{n\in\Delta \\ (-)^*}} S^*(\underline{H}^n)$$

$$S^!(B\underline{H}) = \varprojlim_{\substack{n\in\Delta \\ (-)^!}} S^!(\underline{H}^n).$$

Now we need to understand

$S^*(\underline{K}),\ S^!(\underline{K})$ for K profinite gp.

$K = \varprojlim K_i$ (finite gp) $\Rightarrow$ $\underline{K} = \varprojlim \underline{K_i}$

$S^*(\underline{K}) = \varinjlim S^*(\underline{K_i})$ and similarly for $S^!$

$S^*(\underline{K_i}) = \mathrm{Mod}_{\mathrm{Fct}(K_i,\Lambda)} = \mathrm{IndPerf}_{\mathrm{Fct}(K_i,\Lambda)}$

$\Rightarrow S^*(K_i) = \mathrm{ind}\big(\underbrace{\varinjlim \mathrm{Perf}_{\mathrm{Fct}(K_i,\Lambda)}}_{= \mathrm{Perf}_{\varinjlim \mathrm{Fct}(K_i,\Lambda)} = \mathrm{Perf}_{\mathcal{H}}}\big)$

$= \mathrm{IndPerf}_{\mathcal{H}}$

$= \mathrm{Mod}_{\mathcal{H}}$

pb somewhere here?

In the end we should get $S^*(\underline{K}) = \mathcal{D}(\text{non degenerate } \mathcal{H}\text{-modules})$

i.e. s.th. $\mathcal{H}M = M$

$\xrightarrow{\sim} \mathrm{Rep}(K)$

Upshot: $\forall K$ profinite, $S^*(\underline{K}) \xrightarrow{\sim} \mathrm{Rep}(K) \xleftarrow{\sim} S^!(\underline{K})$

So $S^!(B\underline{H}) = \varprojlim_{n\in\Delta} S^!(\underline{H}^n)$

$= \varprojlim_{n\in\Delta} \mathcal{D}(\text{smooth } H^n\text{-rep}^{ns})$

$\underset{\text{(Heyer-Mann)}}{=} \mathcal{D}(\text{comodule over } \mathcal{H})$

$= \mathrm{Rep}(H)$

Upshot:

$$\begin{tikzcd} S^*(B\underline{K}) \arrow[r, "\simeq"] \arrow[dr, "(-)^*"'] & \mathrm{Rep}(K) & S^!(B\underline{K}) \arrow[l, "\simeq"'] \arrow[dl, "(-)^!"] \\ & \mathrm{Mod}_\Lambda & \end{tikzcd}$$

$\mathrm{Mod}_\Lambda = S^*(pt) = S^!(pt)$

Locally profinite case:

Take $\underline{K} \subseteq H$ good, and consider $B\underline{K} \xrightarrow{f} B\underline{H}$

this is ind-pfp finite surj. ($\Rightarrow$ of univ. descent)

Cech nerve: $(HK_\bullet)$:

$$HK_n = \underbrace{B\underline{K} \times_{B\underline{H}} \cdots \times_{B\underline{H}} B\underline{K}}_{(n+1)\text{-times}}$$

$$= \underline{K} \backslash (\underline{H}/\underline{K})^{n+1}$$

diag.

$$= \bigsqcup_{x \in K\backslash (H/K)^{n+1}} B\underline{H_x}, \quad H_x \text{ stab. of } x$$

good subgp of H

($\Rightarrow$ BH_x placid, with placid atlas $* \to B\underline{H_x}$)

$$= \varinjlim_{\substack{I \subset H\backslash (H/K)^{n+1} \\ \text{finite}}} \bigsqcup_{x \in I} B\underline{H}_x$$

ind-placid stack.

$$S^!(B\underline{H}) = \lim_{n \in \Delta} \underbrace{S^!(HK_n)}_{\text{understood} \to}$$

$$\varinjlim_{I} \bigoplus_{x \in I} \underbrace{S^!(B\underline{H_x})}_{= \text{sm rep}^s \text{ of } \underline{H}_x}$$

Barr-Beck-Lurie applied to both $S^!(B\underline{H})$ and

$$\mathrm{Rep}(H) \underset{\mathrm{Res}}{\overset{\mathrm{c\text{-}Ind}_K^H}{\rightleftarrows}} \mathrm{Rep}(K) : \quad \mathrm{Rep}H = \mathrm{LMod}_{\mathrm{Res\,c\text{-}Ind}} \mathrm{Rep}(K)$$

$$S^!(B\underline{H}) = \mathrm{LMod}_T(S^!(B\underline{K}))$$

and $T \leftrightarrow \mathrm{Res\,c\text{-}Ind}$ under $S^!(B\underline{K}) \simeq \mathrm{Rep}(K)$.