H : conn. reductive gp $/F$

Defn • $\mathrm{Rep}^{\text{tame}}(H(F)) \subset \mathrm{Rep}(H(F))$

presentable Λ-linear stable subcat. gen. by

$$\text{c-ind}_{P^u}^{H(F)} \Lambda$$

with P^u pro-p-radical of a parahoric P.

(objects in $\mathrm{Rep}^{\text{tame}}(H(F))$ called depth 0 representations)

• $\mathrm{Rep}^{\widehat{\text{unip}}}(H(F)) \subset \mathrm{Rep}^{\text{tame}}(H(F))$ gen. by

$$\text{c-ind}_P^{H(F)} \pi_P$$

with $\pi_P \in \mathrm{Rep}^{\widehat{\text{unip}}}(L_P)$, L_P levi quot. of P

presentable Λ-linear stable cat. gen. by $\{R^{\text{mon},*}_{w,\hat{\mathfrak{n}}}\}$

• $\mathrm{Rep}^{\text{unip}}_{fg}(H(F)) \subset \mathrm{Rep}_{fg}(H(F))$ gen. by

$$\text{c-ind}_P^{H(F)} \pi_P$$

with $\pi_P \in \mathrm{Rep}^{\text{unip}}_c(L_P)$.

full idempotent complete stable cat. gen. by $\{R^*_w\}$.

tame local Langlands cor.

Defn $\mathrm{Shv}^{\text{tame}}(\mathrm{Isoc}_G) \subset \mathrm{Shv}(\mathrm{Isoc}_G)$

full subcat. spanned by obj's $\mathcal{F}$ s.t.

$\forall b \in B(G)$, $i_b^! \mathcal{F} \in \mathrm{Rep}^{\text{tame}}(G_b(F))$.

similarly define

$$\mathrm{Shv}^{\text{tame}}(\mathrm{Isoc}_{G,?b}), \quad ? = <, \leq.$$

then

$$\mathrm{Shv}^{\text{tame}}(\mathrm{Isoc}_G) = \operatorname{colim}_b \mathrm{Shv}^{\text{tame}}(\mathrm{Isoc}_{G,\leq b}).$$

Prop [Zhu, Prop. 4.1.26] (1) $\mathrm{Shv}^{\text{tame}}(\mathrm{Isoc}_G)$ is gen. by (as presentable Λ-linear cat) by the essential image of

$$\mathrm{ch}^{\text{mon}}_{LG,\phi} : \mathrm{Shv}_{\text{mon}}(I^u_\infty \backslash LG / I^u_\infty) \to \mathrm{Shv}(\mathrm{Isoc}_G).$$

(2) similar statement holds for

• $\mathrm{Shv}^{\widehat{\text{unip}}}$ & $\mathrm{ch}^{\hat{\mathfrak{n}}\text{-mon}}_{LG,\phi}$

• $\mathrm{Ind}\mathrm{Shv}^{\text{unip}}_{fg}$ & $\mathrm{ch}^{\text{unip}}_{LG,\phi}$.

Lem [Zhu, Lem 4.1.23] (1) $\mathcal{C}$: presentable Λ-linear cat. gen. by the essential image of $\mathrm{Ch}^{\text{mon}}_{LG,\phi}$.

Then $\mathcal{C}$ is gen. as presentable Λ-linear cat. by

$\{\tilde{R}^?_x\}_{x \text{ min. length in } \sigma\text{-conj. class}}$, $? = *$ or $!$

(2) similar statement holds for

• $\mathrm{Ch}^{\hat{\mathfrak{n}}\text{-mon}}_{LG,\phi}$ & $\{R^?_{x,\hat{\mathfrak{n}}}\}$

• $\mathrm{Ch}^{\text{unip}}_{LG,\phi}$ & $\{R^?_x\}$.

(here we consider the generated idempotent complete Λ-linear cat.

Pf We only show (i) and ? = *.

Recall for $x = s x' \sigma(s)$

with $\ell(x) = \ell(x') + 2$ & s simple refl.,

we have fiber seq.

$$\begin{cases} \tilde{R}^{+}_{x'} \to \tilde{R}^{+}_{x} \to \text{cone} \xrightarrow{+1} \\ \qquad\qquad\qquad\qquad\quad \| \\ \tilde{R}^{+}_{x'\sigma(s)} \to \tilde{R}^{+}_{x'\sigma(s)} \to \text{cone} \xrightarrow{+1} \end{cases}.$$

Recall there always exist a sequence $\forall w$

$$x = v_0 \xrightarrow[\sigma]{s_1} v_1 \xrightarrow[\sigma]{s_2} \cdots \xrightarrow[\sigma]{s_n} v_n \text{ ↞ min. length in } \sigma\text{-conj. class}.$$

#

Before proving Prp, recall v_n above can be chosen as $v_n = v$ as below:

$v = uw$ min. length in σ-conj. class

and a facet $\breve{f} \subset \overline{\breve{a}}$ s.t.

- w : σ-straight
- w min. in $W_{\breve{f}}\, w$, $u \in W_{\breve{f}}$
- $w\,\sigma(W_{\breve{f}})\,w^{-1} = W_{\breve{f}}$.

Then

$$(*) \quad \tilde{R}^{+}_{v} \simeq \underset{\substack{\uparrow \\ b \in B(G) \\ \text{corresp. to} \\ w \in B(\tilde{W})\ \text{str.}}}{(i_b)_*}\ \underset{\substack{\uparrow \\ \text{parahoric} \\ \text{ass. to } \breve{f}}}{c\text{-}\mathrm{ind}_{P_b}^{G_b(F)}}\Big(\underset{\substack{\uparrow \\ \text{DL induction for} \\ L_b \text{ Levi quot. of } P_b}}{\tilde{R}^{b,+}_{u}}\Big)\,[-\langle 2\rho, \nu_b\rangle].$$

And if x is of min. length in σ-conj. class

then $\tilde{R}^{+}_{x} = \tilde{R}^{+}_{v}$.

More precisely:

To w we associate

- recall $W \to X_*(T)^{I_F}_{\mathbb{Q}}$, $w \mapsto \tilde{\nu}_w$
- $\breve{M}_w := Z_{G_{\breve{F}}}(\tilde{\nu}_w)$ Levi of $G_{\breve{F}}$
- $\breve{M}_w \circlearrowleft \mathrm{Ad}_{\dot{w}}\sigma$ and

$$G_b(F) \simeq \breve{M}_w(\breve{F})^{\mathrm{Ad}_{\dot{w}}\sigma}$$

$$\Big(G_b(F) = \{ g \in G(\breve{F}) \mid g^{-1}\dot{w}\,\sigma(g) = \dot{w} \} \\ \Updownarrow \\ = \{ g \in \breve{M}_w(\breve{F}) \mid \mathrm{Ad}_{\dot{w}}\sigma(g) = g \} \Big)$$

To $\underset{\text{corresp. to } \breve{f}\text{, explained below}}{\breve{f}_{\breve{M}_w}} \subset \overline{\breve{a}_{\breve{M}_w}}$ stable under $\mathrm{Ad}_{\dot{w}}\sigma$

we associate

- $\breve{P}_{\breve{f}_{\breve{M}_w}}$ parahoric (over $\breve{O}$)

  then $P_b \simeq \breve{P}_{\breve{f}_{\breve{M}_w}}(\breve{O})^{\mathrm{Ad}_{\dot{w}}\sigma}$.

Relation between $\breve{f}$ & $\breve{f}_{\breve{M}_w}$:

- let ν_w : dominant conjugation of $\tilde{\nu}_w$
- $M := Z_G(\nu_w)$ Levi of G $/F$
- $\exists!\ y \in W_0$ min. in $y W_M$ s.t. $y\nu_w = \tilde{\nu}_w$,

  then $\underset{\circlearrowleft\ \mathrm{Ad}_{\dot{w}}\sigma}{\breve{M}_w} = y\,\underset{\circlearrowleft\ \mathrm{Ad}_{\dot{w}^+}\sigma}{M_{\breve{F}}}\,y^{-1}$ where $w^+ := y^{-1} w\,\sigma(y)$
- $\breve{f}_{\breve{M}_w} \subset \overline{\breve{a}_{\breve{M}_w}}$ corresp. to $\breve{f}_M \subset \overline{\breve{a}_M} \cup \overline{\breve{a}}$
- finally, $\breve{f} = y\,\breve{f}_M \subset \overline{\breve{a}}$.

[Zhu, Rmk 3.5]

Rmk/Lem $v = uw$ as above.

(1) u is of min. length in its $\mathrm{Ad}_{\dot w}\sigma$-conj. class in W_F

(2) if u' is a min. length element in the $\mathrm{Ad}_{\dot w}\sigma$-conj. class of u, then $u'w$ is a min. length element in σ-conj. class of uw.

Pf (1) if not then $\exists\ t \in W_F$ s.t.

$$\ell(t u w \sigma(t)^{-1} w^{-1}) < \ell(u)$$

$$\Rightarrow \ell(t u w \sigma(t)^{-1}) < \ell(uw) \quad \lightning$$

(2) $\ell(u'w) = \underset{\ell(u)}{\underset{\shortparallel}{\ell(u')}} + \ell(w) = \ell(uw)$. $\square$

Pf of Prop we only show (1).

$\mathcal{C} \subseteq \mathrm{Shv}^{\text{-tame}}(\mathrm{Isoc}_G)$:

by Lem & (*), enough to notice that for $v = uw$ as above

$$\tilde R_v^{*} \simeq (i_b)_*\left(c\text{-ind}_{P_b}^{G_b(F)} \tilde R_u^{b,*}\right)[\text{shift}]$$
$$\in \mathrm{Shv}^{\text{-tame}}(\mathrm{Isoc}_G) .$$

$\mathrm{Shv}^{\text{-tame}}(\mathrm{Isoc}_G) \subseteq \mathcal{C}$:

for any $\mathcal{F} \in \mathrm{Shv}^{\text{-tame}}(\mathrm{Isoc}_G)$,

$$\mathcal{F} = \operatorname{colim}_b\, (i_{\le b})_* i_{\le b}^! \mathcal{F}$$

iterated extension of $(i_b)_*(i_b^! \mathcal{F})$

$\Rightarrow$ enough to show

$$(i_b)_*\, c\text{-ind}_{P_b}^{G_b(F)}(\tau_{L_b}) \in \mathcal{C} .$$

a parahoric repn of L_b

By (finite) Deligne-Lusztig theory and (finite analogue of) Lem, it's enough to consider

$$\tau_{L_b} = \tilde R_u^{b,+},$$

u min. length in $\mathrm{Ad}_{\dot w}\sigma$-conj. class in W_F

determined by P_b

Rmk/Lem $\Rightarrow$

$$(i_b)_*\, c\text{-ind}_{P_b}^{G_b(F)}(\tilde R_u^{b,+}) \simeq \tilde R_{uw}^{+} .$$

($w \in B(\tilde W)_{st}$ corresp. to b)

#

Recall a geometric origin of categorical trace :

in some $\mathrm{Corr}(C)_{V;H}$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[loop below, "\phi_X"] & Y \arrow[loop below, "\phi_Y"] \end{tikzcd} \quad \text{s.t.} \quad f \circ \phi_X = \phi_Y \circ f$$

equivalence

$$L_{\phi_Y}(Y) := Y \underset{Y\times Y}{\times} \underset{(\mathrm{Id}, \phi_Y)}{Y} \quad \text{“fixed pt scheme”}$$

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] & X \times X \\ X \times_Y L_\phi(Y) \arrow[u] \arrow[r, "\delta"] \arrow[d, "q"] & X \underset{f\circ\phi_X,\, Y,\, f}{\times} X \arrow[u] \\ L_\phi(Y) & \end{tikzcd} \simeq (X \underset{\phi_X \circ \phi_X,\, Y,\, f,\, f}{\times} X) \text{ as } X \times_Y X\text{-bimodule}$$

a sheaf theory

$$\rightsquigarrow q_* \delta^* : D(X \times_Y X) \to D(L_\phi Y)$$

induces $\mathrm{Tr}(D(X \times_Y X), \phi) \to D(L_\phi Y)$

$$\mathrm{Tr}_{\text{geo}}(D(X \times_Y X), \phi)$$

Now, take

$$S_k \curvearrowright X = \mathbb{B} I_w^u \xrightarrow[\circlearrowleft\, \sigma]{} Y = \mathbb{B} LG$$

and the sheaf theory Shv_{mon}, get

$$ch^{mon}_{LG,\phi} : Shv_{mon}(I_w^u \backslash LG / I_w^u) \to Shv(LG/_{Ad_\sigma} LG)$$

$$LG/_{Ad_\sigma} LG = Isoc_G$$

which induces

$$Tr(Shv_{mon}(I_w^u \backslash LG / I_w^u), \sigma) \to Shv(Isoc_G).$$

<u>Thm</u> [Zhu, Thm 4.128] (1) It induces an equivalence

$$Tr(Shv_{mon}(I_w^u \backslash LG / I_w^u), \sigma) \xrightarrow{\sim} Shv^{tame}(Isoc_G).$$

(2) similarly,

$$Tr(Shv((I_w, \hat{u}) \backslash LG / (I_w, \hat{u})), \sigma) \xrightarrow{\sim} Shv^{\widehat{unip}}(Isoc_G)$$

$$Tr(IndShv_{fg}(I_w \backslash LG / I_w), \sigma) \xrightarrow{\sim} IndShv^{unip}_{fg}(Isoc_G).$$

<u>Pf</u> we only show (1).

- $Tr \xrightarrow{\sim} Tr_{geo}$ by Prop 8.71 (Prop 2 in my previous note)

$\ulcorner$ recall [Zhu, Prop 4.44] $\forall\, X \in PreStk_k$ with $H \circlearrowright X$, H torus

$$Shv_{mon}(I_w^u \backslash LG / I_w^u) \otimes_\Lambda Shv_{mon}(X) \overset{\boxtimes_\Lambda}{\simeq} Shv_{mon}(I_w^u \backslash LG / I_w^u \times X) \lrcorner$$

- fully-faithfulness $Tr_{geo} \hookrightarrow Shv(Isoc_G)$ by Prop 8.57 (Prop 1 in my previous note)

- essentially-surjectivity by Prop above. ∎