X. Thu chaps 2

Recap on geo traces

let

$$D : \mathrm{Corr} = \mathrm{Corr}(C, V, H) \longrightarrow LC_\Lambda = \mathrm{linCat}_\Lambda$$

be a sheaf theory st $D(pt) = \mathrm{Mod}_\Lambda$

$pt \rightsquigarrow$ final obj of C

let
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X", "\wr"'] & Y \arrow[d, "\phi_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
be a comm diag in C, with $\phi_X \in \mathrm{Aut}\, X$

HYP 1) $\pi_X : X \to pt \in H$, $\Delta_X : X \to X \times X \in HR$

2) $f \in VR$ 3) $\Delta_{X/Y} : X \longrightarrow X \times_Y X \in V$

Set $X_1 = X \times_Y X$, with its natural (convolution) algebra structure in Corr. $\exists$ lax mon $\Rightarrow$

$$A := D(X_1) \in \mathrm{Alg}(LC_\Lambda)$$

$\phi_X, \phi_Y \rightsquigarrow \phi \circlearrowright X_1 \rightsquigarrow \phi := \phi_1^! : A \to A$

aut. of algebras in LC_Λ

New input: $Z \xrightarrow{g_1 \times g_2} Y \times Y$ map in $C \rightsquigarrow$

X_1-bimodule in Corr $\leftarrow$ $Q := X \times_Y Z \times_Y X \simeq (X \times X) \times_{Y \times Y} Z$

<u>Key ex</u> $Z = Y \xrightarrow{id \times \phi_Y} Y \times Y$, in which case $(x_1, z, x_2) \mapsto (\phi(x_1), x_2)$ gives an isom of X_1-bimodules $Q \simeq {}_\phi X_1$, so an isom of $A = D(X_1)$-bimodules in LC_Λ

$$D(Q) \simeq {}_\phi A$$

Back to general case, we want to describe

$$LC_\Lambda \ni Tr(A, D(Q)) = A \otimes_{A \otimes A^{rev}} D(Q)$$

(in special case above this is denoted $Tr(A, \phi)$)

<u>Key diagram</u>:

$$\begin{tikzcd}
 & X \times_{Y \times Y}^{f \times f} Z \arrow[dl, "q = f \times id"'] \arrow[dr, "\delta = \Delta_X \times id"] & \\
Y \times_{Y \times Y}^{\Delta} Z & & (X \times X) \times_{Y \times Y} Z \simeq Q
\end{tikzcd}$$

$\rightsquigarrow$ "DL functor"

$\downarrow$ Quan's talk $\quad ch = q_! \delta^* : D(Q) \longrightarrow D(Y \times_{Y \times Y} Z)$

<u>Th</u> (Zhu) $\exists$ factorization

$$\begin{tikzcd} D(\Omega) \arrow[r, "\mathrm{can}"] \arrow[drr, "\mathrm{Ch}"'] & \mathrm{Tr}(A, D(\Omega)) \arrow[r, "\beta"] & \mathrm{Tr}_{geo}(A, D(\Omega)) \arrow[d, "\alpha"] \\ & & D(Y \times_{Y \times Y} Z) \end{tikzcd}$$

with can, β the natural maps, α fully faithful & $\operatorname{Im}\alpha = \langle \operatorname{Im} \mathrm{Ch} \rangle$ (A-linear present. stable cat generated by $\operatorname{Im}\mathrm{Ch}$)

If $A \otimes_A A \xrightarrow[\boxtimes_A]{\simeq} D(X_1 \times_X X_1)$ and $A \otimes_A D(\Omega) \xhookrightarrow[\boxtimes_A]{ff} D(X_1 \times_X \Omega)$ then β is an iso & so

$\exists \quad \mathrm{Tr}(A, D(\Omega)) \xhookrightarrow{ff} D(Y \times_{Y \times Y} Z)$ with image $\langle \operatorname{Im}\mathrm{Ch} \rangle$

<u>Rk</u> In the special case $Z = Y \xrightarrow{id \times \phi} Y \times Y$ we have $Y \times_{Y \times Y} Z \simeq \mathcal{L}_\phi(Y)$ ("ϕ-fixed pts of Y")

$$X \times_{Y \times Y} Z \simeq X \times_Y \mathcal{L}_\phi(Y)$$

the diagram becomes

$$\begin{tikzcd} X \times_Y \mathcal{L}_\phi(Y) \arrow[r, "\delta"] \arrow[d, "q"] & \Omega = {}^{\phi}X_1 \\ \mathcal{L}_\phi(Y) & \end{tikzcd}$$

with

$$\begin{tikzcd} X \times_Y \mathcal{L}_\phi Y \arrow[r, "\delta"] \arrow[d, "pr_1"] & X_1 = X \times_Y X \arrow[d, "pr_1"'] \arrow[dr, "pr_2"] & \\ X \arrow[r, "\phi"] \arrow[rr, bend right, "id"'] & X & X \end{tikzcd}$$

In this case $A \otimes_\Lambda D(\mathfrak{X}) \xrightarrow[\text{ff}]{\Delta} D(X_1 \times \mathfrak{X})$ is a conseq of $A \otimes_\Lambda A \simeq D(X_1 \times X_1)$

Rk This applies this with several sheaf theories but today we work only with iC^*. This works with $iC^!$ in 9.50, but does not check the assumptions needed for the machine to work.

For iC^* $C = \mathrm{indArSt}^{\mathrm{op}}$ $V = \mathrm{Ael}$, $H = \mathrm{Sfor} = HR$ & repres proper maps are VR, so all above hyp will be ok in practice (not so clear for $iC^!$)

Hard part in applying th: proving that $A \otimes_\Lambda A \simeq D(X_1 \times X_1)$ & controlling dual. Ch.

Recollections on the stack of (tame) Langlands parameters

$\Lambda = \bar{\mathbb{Z}}_\ell$ or $\bar{\mathbb{F}}_\ell$ $\ell \neq p$ F NA local nes field $\mathbb{F}_q$ of char p

G/F conn reductive, for simplicity unramified (q-split & split/$\breve{F}$) (everything below works for G tamely ram)

$\mathrm{Gal}(\bar{F}/F$

Recall $\Gamma_{\tilde F/F} \overset{\text{fin}}{=}$ quot of Γ_F'' acting on $\hat G$. As $G_m =: \Gamma_{\tilde F/F}$ is cyclic, generated by the im of σ, a fixed lift of Frob in W_F

Fix $\tau \in W_F/\underbrace{P_F}_{\text{wild inertia}}$ top. generator of $I_F/P_F \simeq \hat{\mathbb{Z}}^{p'}$

$\rightsquigarrow i : \Gamma_q := \langle \sigma, \tau \mid \sigma\tau\sigma^{-1} = \tau^q \rangle \subset W_F^t = W_F/P_F$

so we're in the setting of the previous talk & get stacks $L_i = L_{G,i} \underset{\text{clopen}}{\supset} L_i^t = L^t_{G,i}$ over $\mathbb{Z}[\frac{1}{p}]$ whose BC to $\mathbb{Z}_\ell$ is can indep of i and classifies $\hat G$-conj classes of strongly continuous c-parameters (same ones for L_i^t)

$$\begin{tikzcd} W_F \arrow[r, "\varphi"] \arrow[d, "\mathrm{cor}"'] & {}^cG(A) \arrow[d, "r"] \\ \mathbb{Z}_\ell^\times \times \Gamma_{\tilde F/F} \arrow[r] & A^\times \times \Gamma_{\tilde F/F} \end{tikzcd} \qquad ({}^cG = \hat G \rtimes (G_m \times \Gamma_{\tilde F/F}))$$

(A classical $\mathbb{Z}_\ell$-alg)

let $L, L^t :=$ BC of L_i, L_i^t to Λ (so indep of i) Recall that $\exists$ closed schemes $L^{t\square}, L^{\square}$ over Λ (with $L^{t\square}$ fin pres) st

$L^t = L^{t\square}/\underbrace{\hat G}_{\text{really } \hat G \otimes \Lambda}$ $\qquad L = L^{\square}/\hat G$

and $L^{t\square} = L_i^{t\square} \otimes_{\mathbb{Z}[\frac{1}{p}]} \Lambda$ $\qquad$ inv of σ in $(G_{ad} \times \Gamma_{\tilde F/F})(\mathbb{Z}_\ell)$ ${}^c\hat G(\mathbb{Z}_\ell)$

$L_i^{\square t} = \{ (g_\tau, g_\sigma) \in \hat G \times \hat G \sigma \subset {}^cG \times {}^cG \mid g_\sigma g_\tau g_\sigma^{-1} = g_\tau^q \}$

This gives a map $L_i^{\square t} \xrightarrow{pr_1} \hat G$ $\rightsquigarrow$

$$L_i^t \longrightarrow \underset{\text{(adj act)}}{\hat G/\hat G} \longrightarrow \hat G//\hat G$$

$\cup$ $\{1\}$ $\leftarrow$ formal cpl at $\{1\}$

$\cup$ $\{1\}$

$\rightsquigarrow$ can define stacks $/\mathbb{Z}[\frac{1}{p}]$

$L_i^u = L_i^t \times_{\hat G//\hat G} \{1\} \subset L_i^{\hat u} = L_i^t \times_{\hat G//\hat G} \widehat{\{1\}}$

$\uparrow$ has nontriv der str $\qquad$ $\uparrow$ classical ind-stack

As usual drop i to denote BC to Λ

To apply the machine of geo traces need to describe L^t & its friends as ϕ-fixed points.

<u>Key point</u> giving a c-param $\varphi: W_F \to {}^cG(\Lambda)$

$\Leftrightarrow$ give a c-param $\check{\varphi}$ of I_F (namely $\check{\varphi} = \varphi|_{I_F}$) and an elem $g \in \hat{G}$ st

$\underline{g\,\phi(\check{\varphi})\,g^{-1} = \check{\varphi}}$ where

$$\phi(\check{\varphi})(\gamma) = \bar{\sigma}\,\check{\varphi}(\sigma^{-1}\gamma\sigma)\,\bar{\sigma}^{-1}$$

" $\check{\varphi}(\gamma) = \varphi(\sigma)\,\check{\varphi}(\sigma^{-1}\gamma\sigma)\,\varphi^{-1}(\sigma)$ "

namely g is determined by $\varphi(\sigma) = g\bar{\sigma}$

This suggests defining the stacks $\check{L}/\Lambda$ & $\check{L}^{t}/\Lambda$ in the same way as we defined L, L^{t} by replacing W_F by I_F (and Γ_q by $\mathbb{Z}^{\mathbb{Z}[\frac{1}{p}]} \subset \Gamma_q$), for instance

$$\begin{tikzcd}
\check{L}^{t} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R_{I_F^t,\, {}^cG} \underset{\mathbb{Z}_\ell}{\otimes} \Lambda \arrow[d] \\
\operatorname{Spec}\Lambda \arrow[r, "\mathrm{can}"'] & R_{I_F^t,\, G_m \times \Gamma_{\tilde{F}/F}} \underset{\mathbb{Z}_\ell}{\otimes} \Lambda
\end{tikzcd}$$

More concretely, evaluation at $\sigma^{-i} z\, \sigma^{i}$ ($i \in \mathbb{Z}$) gives $\check{L}^{t} \xrightarrow{\sim} \varprojlim_{g \mapsto g^q} \hat{G}/\hat{G}$

Using the general formula $\quad \hookrightarrow$ really $\hat{G} \otimes \Lambda$

for $R_{I^t_{F},H} \rightsquigarrow$

$$\check{L}^t = \hat{G}/\hat{G} \times_{\hat{G}/\!/\hat{G}} (\hat{G}/\!/\hat{G})^{\wedge,p}$$

$\rightsquigarrow \exists\ \check{L}^t \hookrightarrow \hat{G}/\hat{G}$ realizing $\check{L}^t$ as

$\check{\varphi} \longrightarrow \check{\varphi}(2)$

formal cpl of $\hat{G}/\hat{G}$ along suitable closed substacks

Th $\exists$ natural isom $L \simeq \mathcal{L}_\phi(\check{L})$

$L^t \simeq \mathcal{L}_\phi(\check{L}^t)$

Steinberg stacks $\quad B \subset G$ Borel $\rightsquigarrow \hat{B} \subset \hat{G}$

$\rightsquigarrow {}^cB \hookrightarrow {}^cG \quad$ Similar constructions for B yield diagram of stacks$/\Lambda$

$$\begin{tikzcd} \check{L}^t_B \arrow[r, hook] \arrow[d] & \hat{B}/\hat{B} \arrow[d] \\ \check{L}^t \arrow[r, hook] & \check{G}/\hat{G} \end{tikzcd}$$

& isom $L^t_B \simeq \mathcal{L}_\phi(\check{L}^t_B)$

$\check{L}^t_B \simeq \hat{B}/\hat{B} \times_{\hat{T}} \hat{T}^{\wedge,p}$

Define $\check{L}^u_B = \check{L}^t_B \times_{\check{L}^t_T} \check{L}^u_T \hookrightarrow \check{L}^{\hat{u}}_B = \check{L}^t_B \times_{\check{L}^t_T} \check{L}^{\hat{u}}_T$

We get proper quasi-smooth maps

$\check{\pi}^t : \check{L}^t_B \longrightarrow \check{L}^t$ $\qquad$ $\check{\pi}^u : \check{L}^u_B \longrightarrow \check{L}^t$ really $\check{L}^t$ NOT $\check{L}^u$

& commu diag

$$\begin{tikzcd} \check{L}^t_B \arrow[r, "\check{\pi}^t"] \arrow[d] & \check{L}^t \arrow[d] \\ \hat{B}/\hat{B} \arrow[r] & \hat{G}/\hat{G} \end{tikzcd} \qquad \begin{tikzcd} \check{L}^u_B \arrow[r, "\check{\pi}^u"] \arrow[d] & \check{L}^t \arrow[d] \\ \hat{U}/\hat{B} \arrow[r] & \hat{G}/\hat{G} \end{tikzcd}$$

will apply the general discussion on geo traces to one of the horiz maps above & theory $D = iC^*$

<u>Def</u> (Steinberg stacks)

• on spectral side $\check{S}^t := \check{L}^t_B \times_{\check{L}^t} \check{L}^t_B$

$\check{S}^u := \check{L}^u_B \times_{\check{L}^t} \check{L}^u_B$

• on group side

$S_G = \hat{B}/\hat{B} \times_{\hat{G}/\hat{G}} \hat{B}/\hat{B}$ $\qquad$ $S^u_G = \hat{U}/\hat{B} \times_{\hat{G}/\hat{G}} \hat{U}/\hat{B}$

These play the role of $X_1 = X \times_Y X$ in our abstract setup

Rk $S_G = \hat{G} \times^{\hat{B}} \hat{B}/\hat{G} \underset{\hat{G}/\hat{G}}{\times} \hat{G} \times^{\hat{B}} \hat{B}/\hat{G}$

$= \left(\hat{G} \times^{\hat{B}} \hat{B} \underset{\hat{G}}{\times} \hat{G} \times^{\hat{B}} \hat{B} \right)/\hat{G} = S_G^{\square}/\hat{G}$

where $S_G^{\square} = \underbrace{\left(\hat{G} \times^{\hat{B}} \hat{B}\right)}_{\text{Groth-Springer resolution}} \underset{\hat{G}}{\times} \left(\hat{G} \times^{\hat{B}} \hat{B}\right)$

classifies triples $(g, \bar{g}_1, \bar{g}_2) \in \hat{G} \times \hat{Fl} \times \hat{Fl}$, $\hat{Fl} = \hat{G}/\hat{B}$

st $g \in g_1 \hat{B} g_1^{-1} \cap g_2 \hat{B} g_2^{-1}$

Th S_G is a classical stack, lci (so quasi-sm) & $\exists$ embedding $\check{S}^t \xrightarrow{\hat{c}} S_G$ realizing $\check{S}^t$ as formal cpl of S_G, thus $\check{S}^t$ is classical quasi-smooth ind-stack. Also $\check{S}^u \simeq S_G^u$ & $\check{\pi}^t, \check{\pi}^u$ are proper

Apply the general machine $\leadsto$ corresp

$\tilde{L}^t := \check{L}_B \times L^t \xrightarrow{\delta^t} \check{S}^t$ and

$X \times_Y \mathcal{L}_\phi Y = \tilde{L}^t \xrightarrow{\tilde{\pi}^t} L^t = \mathcal{L}_\phi Y$

$$
\begin{tikzcd}
\tilde{L}^u = \check{L}^u_B \times_{\check{L}^t} L^t \arrow[r, "\delta^u"] \arrow[d, "\tilde{\pi}^u"] & \check{S}^u \\
L^t &
\end{tikzcd}
$$

⇝ DL functors $\qquad ch^t = \tilde{\pi}^t_* \delta^{t*}$

$ch^t : ic(\check{S}^t) \to ic(L^t) \qquad ch^u : ic(\check{S}^u) \to ic(L^t)$

which preserve categories of coherent sh since $\tilde{\pi}^t, \tilde{\pi}^u$ are proper

<u>Key obs</u> δ^t (resp δ^u) is a BC of the diag of $\check{L}^t_B$ (resp $\check{L}^u_B$), which is a BC of the diag of $\hat{B}/\hat{B}$ (resp $\hat{U}/\hat{B}$) & one can check (!) that $\Delta^* = \Delta^!$ for diag of $\hat{B}/\hat{B}$ & $\Delta^*, \Delta^!$ differ by a shift by $\dim \hat{T}$ for diag of $\hat{U}/\hat{B}$. Thus ⇒ $\delta^{t!} = \delta^{t*}$ & if we use $ic^!$ or ic^* to def convol prod on $ic(\check{S}^t)$ ~~the~~ the 2 monoidal structures are the same (& differ by $[\pm \dim \hat{T}]$ for $\check{S}^u$)

$\underline{\text{Th}}$ (Thm) $\mathrm{IC}(\check{S}^u) \underset{\wedge}{\otimes} \mathrm{IC}(\check{S}^u) \simeq \mathrm{IC}(\check{S}^u \underset{\wedge}{\boxtimes} \check{S}^u)$

similarly for $\check{S}^t$

$\underline{\text{Pf}}$ (for $\check{S}^u$)

$(g, \bar{g}_1, \bar{g}_2) \longmapsto (\bar{g}_1, \bar{g}_2)$

$S^u \hookrightarrow \mathcal{S} = \mathcal{S}_G = S_G^\square / \hat{G} \longrightarrow (\hat{\mathrm{Fl}} \times \hat{\mathrm{Fl}}) / \hat{G}$

$\check{S}^{u\,2}$

inside $\hat{\mathrm{Fl}} \times \hat{\mathrm{Fl}}$ look at the $\hat{G}$-orbit $\mathcal{O}(w)$ of $(1, w)$. Then $\hat{B}_w := \mathrm{Stab}_{\hat{G}}(1, w) = \hat{B} \cap \mathrm{Ad}_w \hat{B}$

The inverse im of $\mathcal{O}(w)/\hat{G}$ in S^u is a locl substack with reduced substack

$Z_w = Z_w^\square / \hat{B}_w \qquad Z_w^\square = \hat{U}_w := R_u(\hat{B}_w)$

(on which $\hat{B}_w$ acts by conj)

So we get a finite stratif (X_i) of S^u where $(X_i \setminus X_{i+1})_{red}$ are the Z_w's, which are smooth.

Lem (Thm 9.34) $X, Y \in \mathrm{AlgSt}^{\mathrm{ofp}}_{\Lambda}$ having finite stratif $X = X_0 \supset X_1 \supset \dots$
$Y = Y_0 \supset Y_1 \supset \dots$

st $\boxtimes_\Lambda$ is an equiv (for ic^*) for $(X_i \setminus X_{i+1})_{\mathrm{red}}$ & $(Y_j \setminus Y_{j+1})_{\mathrm{red}}$ $\Rightarrow$ it's an equiv for (X, Y)
(ie $ic(X) \otimes_\Lambda ic(Y) \xrightarrow[\boxtimes_\Lambda]{\simeq} ic(X \times Y)$)

Lem (Thm ch 8) If $\boxtimes_\Lambda$ is always ff (OK for ic^*) & π_X, Δ_X, $\pi_Y, \Delta_Y \in V \cap H$ & if $\boxtimes_\Lambda$ is an isom for (X, X) then it's also $\simeq$ for (X, Y)

Combine these 2 lemmas with the above discussion $\Rightarrow$ enough to see that
$$ic(Z_w) \otimes_\Lambda ic(Z_w) \simeq ic(Z_w \times Z_w)$$
We will see that if $f : Z_w = Z_w^\square / \hat{B}_w \to \mathbb{B}$, $\hat{B}_w$ is the nat map then
(1) $\mathrm{Perf}\, Z_w = \langle \mathrm{Im}\, f^* \rangle$
(2) $\mathrm{Perf}\, Z_w \times Z_w = \langle \mathrm{Im}\, (f \times f)^* \rangle$

so (as $\mathcal{Z}_w$ is sm) the same for $iC = \mathrm{ind}\,\mathrm{Coh}$

So it suffices to see that $\hat{\boxtimes}$ is an isom for $(\mathbb{B}\hat{B}_w, \mathbb{B}\hat{B}_w)$ & for $(\mathbb{B}(\hat{B}_w \times \hat{B}_w), \mathbb{B}(\hat{B}_w \times \hat{B}_w))$ It's actually an isom for $(\mathbb{B}G, \mathbb{B}G)$ $\forall G$ lin alg gr $/k$ (key pt: $\mathcal{O}(G)$ has a filt whose graded are $V_1 \boxtimes V_2$

as $G \times G$-rep

$V_i \in \mathrm{Rep}(G)$) See Thm 9.35

Will only prove (1) (pf for (2) identical)

$X := \mathcal{Z}_w \xrightarrow{f} Y := \mathbb{B}\hat{B}_w$

$$\begin{tikzcd}
X \arrow[dr, "\Delta_{X/Y}"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$F \in iC(X)$

$F = p_{2*}\Delta_{X/Y*}F \underset{\text{proj form}}{=} p_{2*}(\Delta_{X/Y*}1 \otimes p_1^* F)$ Supp we could prove

(3) $\Delta_{X/Y*}1 \in \langle \mathrm{Im}\, \alpha^* \rangle$

for simplicity supp $\Delta_{X/Y*} 1 = \varinjlim_{i} \alpha^* K_i$

$\Rightarrow F = \varinjlim p_{2*}(\alpha^* K_i \otimes p_1^* F) = \varinjlim p_{2*} p_1^* (f^* K_i \otimes F) = \varinjlim f_* f^* (f^* K_i \otimes F)$

$\in \langle \operatorname{Im} f^* \rangle$ and we win

To prove (3) use

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta_{X/Y}"] & Y = pt/\hat{B}_w \arrow[d, hook, "i"] & \\
\frac{\hat{U}_w \times \hat{U}_w}{\hat{B}_w} = X \times_Y X \arrow[r, "m"] \arrow[rr, bend right, "\alpha"'] & X = \hat{U}_w/\hat{B}_w \arrow[r, "f"] & Y
\end{tikzcd}$$

where m is induced by $\hat{U}_w \times \hat{U}_w \to \hat{U}_w$
$(x,y) \mapsto xy^{-1}$

($\hat{B}_w$-equiv map)

If $i_* 1_Y \in \langle \operatorname{Im} f^* \rangle \Rightarrow \Delta_{X/Y*} 1_X =$

$\Delta_{X/Y*} f^* 1_Y = m^* i_* 1_Y \in \langle \operatorname{Im} m^* f^* \rangle$
$= \langle \operatorname{Im} \alpha^* \rangle$

To see $i_* 1_Y \in \langle \operatorname{Im} f^* \rangle$ one constructs a finite filt $\hat{U}_w = Z_0 \supset Z_1 \supset \dots$ by normal

subgroups stable by $\hat{B}_w$ s.t. $\exists f_i \in \mathcal{O}(Z_i)$
$Z_{i+1} = V(f_i)$ & $\hat{B}_w \circlearrowright f_i$ by a character
Then prove by induction that $i_{i*} 1_{Z_i} \in \langle \mathrm{Im}\, f^* \rangle$ $\forall i$

<u>Big image for DL</u> Recall the corresp

$$\begin{tikzcd} \tilde{L}^u \arrow[r, "\delta^u"] \arrow[d, "\tilde{\pi}^u"'] & \check{S}^u \\ L^t & \end{tikzcd} \qquad \begin{tikzcd} \tilde{L}^t \arrow[r, "\delta^t"] \arrow[d, "\tilde{\pi}^t"] & \check{S}^t \\ L^t & \end{tikzcd}$$

with associated functors $ch^u = \tilde{\pi}^u_* \delta^{u*}$
$ch^t = \tilde{\pi}^t_* \delta^{t*}$
$\tilde{\pi}^u$ factors through $L^{\hat{u}}$ $\Rightarrow$ $\mathrm{Im}\, ch^u \subset$
$ic(L^{\hat{u}}) \subset ic(L^t)$

<u>Recap on sing support</u> $\otimes \mathrm{Sm}_\Lambda := \{X \in \mathrm{AlgSt}_\Lambda^{op}$
$| X \to \mathrm{Spec}\,\Lambda$ quasi-smooth$\}$ $\forall X \in \mathrm{clSm}_\Lambda$
$\exists$ clas stack $S(X) = \mathrm{Sing}(X)/\Lambda$ + map
$S(X) \xrightarrow{\alpha} X_{cl}$ s.t. $\forall x \in X_{cl}(k)$ (k field)

$\alpha^{-1}(x) \cong H^{-1}(x^* \mathbb{L}_{X/\Lambda})$

$\forall f : X \to Y$ in $\mathrm{QSm}_\Lambda \rightsquigarrow$ corresp

$$\begin{tikzcd} & S(Y)_X := S(Y) \times_{Y_{cl}} X_{cl} \arrow[dl, "pr_2"'] \arrow[dr, "S(f) = \mathrm{Sing}(f)"] & \\ S(Y) & & S(X) \end{tikzcd}$$

If $N \subset S(Y)$ is a closed conic subset $\rightsquigarrow$

$f^* N := \underbrace{\mathrm{Sing}(f)(N)}_{\text{Zhu's notation}}$ = smallest closed

conic subset $\supset$ image of $N \times_{Y_{cl}} X_{cl}$ by $S(f)$

For $M \subset S(X)$ closed conic & f proper ($\Rightarrow pr_2$ proper, so closed) $f_* M := pr_2(S(f)^{-1}(M))$

$\subset S(Y)$
cl conic

Th (Arinkin–Gaitsgory) f q-sm $\Rightarrow$

$$\langle f^*(\mathrm{Coh}_N Y) \rangle \subseteq \mathrm{Coh}_{f^*N} X$$

f proper $\Rightarrow \langle f_*(\mathrm{Coh}_M Y) \rangle \subseteq \mathrm{Coh}_{f_* M}$

with = if f ci

If Λ is a field of char 0 & X, Y are stacks "of global complete intersection" the inclusion are =

Prop (Thm 2.82, 2.85) We have

$$\langle \dim \delta^{u*} \rangle = \mathrm{Coh}_{\delta^{u*}(S(\check{S}^u))}(\tilde{L}^u)$$

$$\langle \dim \delta^{t*} \rangle = \mathrm{Coh}_{\delta^{t*}(S(\check{S}^t))}(\tilde{L}^t)$$

Pf for $\check{S}^u$

$$\begin{tikzcd}
\tilde{L}^u = X \times_{X \times Y} Y \arrow[r, "\delta^u = \delta_X \times \mathrm{id}"] \arrow[dr, "i"'] & (X \times X) \times_{Y \times Y} Y \simeq X \times X \simeq \check{S}^u \\
& A := \underbrace{(X \times_{\hat{B}} X)}_{(\hat{U} \times \hat{U})/\hat{B}} \times_{Y \times Y} Y \arrow[u, "f"']
\end{tikzcd}$$

$X := \hat{U}/\hat{B}$

$Y := \hat{G}/\hat{G}$

i is a q-sm ci, f is a sm map

$\Downarrow$

$S(f)$ isom

ker $i_C(A) = \langle \dim f^* \rangle$

Pf pull back the stratif on $\check{S}^u$

by f ⇝ stratif (A_i) on A st $(A_i \setminus A_{i+1})$ red

are the $Z'_w = Z^{1\square}_w / \hat{B}_w$ $Z'^{1\square}_w = \hat{U}_w \times \hat{B}$

+ suitable (explicit) act of $\hat{B}_w$

Now argue as in the pf of $IC(Z_w) \hat{\otimes} IC(Z_w)$

$\simeq IC(Z_w \times Z_w)$

So $\langle \dim \delta^{u*} \rangle = \langle \dim i^* f^* \rangle$

$= \langle i^* IC(A) \rangle = \langle i^* IC_{S(A)}(A) \rangle$

$= \quad IC_{i^*S(A)}$

prev th

As $S(f)$ imm $S(A) = f^* S(\check{S}^u)$

$\Rightarrow i^* S(A) = i^* f^* S(\check{S}^u) = \delta^{u*} S(\check{S}^u)$ ☒

One can show that

$S(L^t) = \{(\varphi, \xi) \mid \varphi \in L^t,$

$\{ \xi \in (\hat{\mathfrak{g}}^*)^{\varphi(I_F)=1,\ \varphi(\sigma)=1/q} \} \supset$

$N^t = \{(\varphi, \xi) \in S(L^t) \mid \xi \in \underbrace{\hat{\mathcal{N}}^*}_{\text{nilp cone}} \subset \hat{\mathfrak{g}}^*(\ldots$

Th (Zhu) 1) If Λ char 0 $N^t = S(L^t)$

2) $\tilde{\pi}^t_* \delta^{t*} S(\check{S}^t) = N^t$

$\tilde{\pi}^u_* \delta^{u*} S(\check{S}^u) = i^* S(L^{\hat{u}})$

$i : L^{\hat{u}} \hookrightarrow L^t$

Th (Zhu) If $\Lambda = \overline{\mathbb{Q}}_\ell \Rightarrow$

$Tr(ic(\check{S}^t), \phi) \simeq ic(L^t)$

$Tr(ic(\check{S}^u), \phi) = ic(L^{\hat{u}})$

If $\Lambda = \overline{\mathbb{F}}_\ell$ $\exists$ $Tr(ic(\check{S}^u), \phi)$

$\hookrightarrow ic(L^{\hat{u}})$ with image stable

under the action of indRep$(L^{\hat{u}})$

If $\hat{G}_{der}$ is simply-connected

the image $\supset$ ind $\mathrm{Perf}(\hat{L}^u)$

Pf general machine for $\check{L}^t_B \to \check{L}^t$

& $\check{L}^u_B \to \check{L}^t \rightsquigarrow$ ff & image gener.

by dual ch^u dual char o using the
dual ch^t of these

thm of AG above

$$\langle \mathrm{dual}\, ch^t \rangle = \langle \mathrm{dual}\ \tilde{\pi}^t_* \delta^{t*} \rangle$$

$$= \langle \tilde{\pi}^t_* \, iC_{\delta^{t*} S(\check{S}^t)} \, \check{L}^t \rangle$$

$$= \langle iC_{\tilde{\pi}^t_* \delta^{t*} S(\check{S}^t)} \, \check{L}^t \rangle$$

$\tilde{\pi}^t$ proper & Adams

$$= iC_{N^t}(L^t($$

$$= iC_{S(L^t)} L^t = iC(L^t)$$

If $A = E\ell$ & $E \in \mathrm{ind}\,\mathrm{Rep}\, L^{\hat{u}}$

$F \in \mathrm{im}\, ch^{u}$ $\quad F = \tilde{\pi}^{u}_{*} \delta^{u*} G$

$$E \otimes F = \tilde{\pi}^{u}_{*}\big(\underbrace{\tilde{\pi}^{u*} E}_{\substack{\in\, \mathrm{ind}\,\mathrm{Rep}\, \tilde{L}^{\hat{u}} \\ \text{no sing supp in } 0\text{-section}}} \otimes \underbrace{\delta^{u*} G}_{\substack{\text{sing supp} \\ \text{in } \delta^{u*} S / \check{S}^{u}}}\big)$$

$$\underbrace{\qquad\qquad\qquad}_{\in\ \mathrm{IC}_{\delta^{u*} S / \check{S}^{u}}}$$

$$\mathrm{IC}_{\delta^{u*} S / \check{S}^{u}} = \langle \mathrm{dee}\, \delta^{u*} \rangle$$

$$\in \tilde{\pi}^{u}_{*} \langle \mathrm{dee}\, \delta^{u*} \rangle \subset \langle \mathrm{dee}\, ch^{u} \rangle$$

Proof for $\hat{G}_{der}$ is skipped, the key pt is playing with BC in

$$\begin{tikzcd}
\tilde{L}^{u} \arrow[r] \arrow[d, "\tilde{\pi}^{u}"'] \arrow[dr, phantom, "\square"] & \check{L}^{u}_{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{\delta}/\hat{B} \arrow[r] \arrow[d, "\delta"] & \delta/\hat{G} \arrow[dl] \\
L^{t} \arrow[r] & \check{L}^{t} \arrow[r] & \hat{G}/\hat{G} &
\end{tikzcd}$$

and
$$\begin{tikzcd}
\mathcal{C}^u \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{U}/\hat{G} \arrow[d] \\
\mathcal{L}^{\hat{u}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{U}^{\wedge}/\hat{G} \arrow[d] \\
\mathcal{L}^t \arrow[r] & \hat{G}/\hat{G}
\end{tikzcd}$$

$\mathcal{U} \subset \hat{G}$ variety of unipotents

Since $\hat{G}_{der}$ sc $\Rightarrow$ $\mathcal{U} = \hat{G} \times_{\hat{G}/\!/\hat{G}} \{1\}$

Let $X = I_w \backslash LG / I_w$

$X^u = I_w^u \backslash LG / I_w^u$

Th (Bezrukavnikov, Arkhipov-B, B-Riche) $\exists$ Φ-equiv eq of monoidal cat

$$\operatorname{ind} Shv_{fg}(X) \simeq IC(\check{S}^{u}_{G})$$

$$Shv_{mon}(X^u) \simeq IC(\check{S}^{t}_{G})$$

Th (Hu) If $\Lambda = \overline{\mathbb{Q}}_\ell$ $\exists$

$$\begin{tikzcd}
\mathrm{Sh}_{\mathrm{mon}}(X^u) \arrow[r, "\simeq"] \arrow[d, "\mathrm{ch}^{\mathrm{mon}}_{LG,\phi}"'] & \mathrm{IC}(S_G^{\vee t}) \arrow[d, "a^t"] \\
\mathrm{Sh}^t(\mathrm{Isoc}_G) \arrow[r, "\simeq"] & \mathrm{IC}(L_G^t)
\end{tikzcd}$$

inducing $\mathrm{Sh}^{\hat{u}}(\mathrm{Isoc}_G) \simeq \mathrm{IC}(L_G^{\hat{u}})$

If $\Lambda = \overline{\mathbb{F}}_\ell$ $\exists$

$$\mathrm{ind}\,\mathrm{Sh}^{\mathrm{unip}}_{\mathrm{fg}}(\mathrm{Isoc}_G) \xhookrightarrow{\mathrm{ff}} \mathrm{IC}(L_G^{\hat{u}})$$

with image stable by ind Perf $L_G^{\hat{u}}$

(& containing it if $\hat{G}$ der sc)