$$\boxed{\widetilde{R}^{!}_{\dot w} = Ch^{mon}_{LG,\phi}\left(\widetilde{\Delta}^{mon}_{\dot w^{-1}}\right)} \qquad \boxed{\widetilde{R}^{*}_{\dot w} = Ch^{mon}_{LG,\phi}\left(\widetilde{\nabla}^{mon}_{\dot w^{-1}}\right)}$$

Here we have

(cf. Lang's thm)

$$pr^{\sigma}_{\dot w} : LG_w/Ad_\sigma I_w \longrightarrow \mathcal{J}_k \cdot \dot w / Ad_\sigma \mathcal{J}_k \simeq B\underbrace{\mathcal{J}_k^{\bar w \sigma}}$$

(finite étale gp scheme over k)

$$:= \ker\left(\begin{matrix} \mathcal{J}_k \longrightarrow \mathcal{J}_k \\ s \longmapsto s^{-1}\,\bar w\sigma(s)\end{matrix}\right)$$

($\bar w$ = image of w in W_0)

Lemma: (4.63) $\boxed{\widetilde{R}^{?}_{\dot w} = Nt_*(\tilde i_w)_?\,(pr^{\sigma}_{\dot w})^{!}\,\Lambda[S_k^{\bar w\sigma}]}$

Pf: In fact we have

$$\begin{tikzcd}
 & LG_w/Ad_\sigma I_w^u \arrow[r] \arrow[dl] \arrow[dr] \arrow[ddl, bend right] \arrow[ddr, phantom, "\square"] & LG/Ad_\sigma I_w^u \arrow[rr, "\delta^u"] \arrow[dr, phantom, "\square"] & & I_w^u\backslash LG/I_w^u \\
 & & & I_w^u\backslash LG_w/I_w^u \arrow[ur] \arrow[dl] \arrow[dr, "pr_{\dot w}"] & \\
LG_w/Ad_\sigma I_w \arrow[d, "\tilde i_w"'] \arrow[rr] \arrow[drr, bend right, "pr^{\sigma}_{\dot w}"'] & & \left(I_w^u\backslash LG_w/I_w^u\right)/Ad_\sigma \mathcal{J}_k \arrow[d] \arrow[rr, phantom, "\square"] & & \mathcal{J}_k \arrow[dll] \\
\underbrace{LG/Ad_\sigma I_w}_{=Sht^{loc}} \arrow[d] & & B(\mathcal{J}_k^{\bar w\sigma}) & & \\
Isoc_G & & & &
\end{tikzcd}$$

includes $(-)^{-1}$ to account for inverse in the def. of δ^u

and $\left(\mathcal{J}_k \longrightarrow B(\mathcal{J}_k^{\bar w\sigma})\right)_* \widetilde{Ch} = \Lambda[\mathcal{J}_k^{\bar w\sigma}]$

[cf. Lemma 4.39]

This gives the case $? = *$.
For the case $?=!$, one uses that $(-)_* = (-)_![\dim \mathcal{J}_k]$ on mon. sh. for $LG_{(w)}/Ad_\sigma I_w^u \to LG_{(w)}/Ad_\sigma I_w$. (cf Lemma 4.27) $\square$
(+ same argument as for $R^{!}_{\dot w}$)

Prop. (4.59)

The objects $\tilde{R}^*_{\dot w}$ and $\tilde{R}^!_{\dot w}$ are compact

Pf: We consider the diagram

$$\begin{tikzcd}
Iw^u \dot w Iw^u / \mathrm{Ad}_\sigma Iw^u \arrow[r, "\pi_h"] \arrow[d, "\pi_d"'] \arrow[dr, phantom, "\lrcorner"] & \dot w \arrow[d] \\
LG_w / \mathrm{Ad}_\sigma Iw \arrow[r, "pr^\sigma_{\dot w}"'] & B\mathcal{G}_k^{\dot w\sigma}
\end{tikzcd}$$

By base change, $(pr^\sigma_{\dot w})^! \Lambda[\mathcal{G}_k^{\dot w \sigma}] \simeq (\pi_d)_* \underbrace{(\pi_h)^! \Lambda}$

compact because

$Shv\left(Iw^u \dot w Iw^u / \mathrm{Ad}_\sigma Iw^u\right)^\omega$

$= Shv_c\left(Iw^u \dot w Iw^u / \mathrm{Ad}_\sigma Iw^u\right)$

(very placid stack with pu atlas)

(cf Prop. 10.144)

preserve compacts because π_d is proper

Hence this object is compact.

Pushforward along the ind-pfp proper map $Sht^{loc}_{\leq w} \hookrightarrow Sht^{loc} \to Isoc_G$ preserves compact objects. So it suffices to check that $*$- and $!$-pushforward along $Sht^{loc}_w \to Sht^{loc}_{\leq w}$ preserves compact objects. This is clear for $!$. The case of $*$ follows, using Verdier duality. [there is some work needed here, which we won't explain.] □

Rmk: R^*_w and $R^!_w$ are NOT compact (they belong to Shv_{fg}.)

In the next statement we will also consider for s simple reflection $w \in \tilde{W}$

$$R^{mon,!}_{\dot{w},s} := Ch^{mon}_{LG,\phi}\left(\Delta^{mon}_{\dot{w}^{-1}}(\tilde{Ch}_s)\right),$$

~~(for s simple reflection)~~

$$R^{mon,*}_{\dot{w},s} := Ch^{mon}_{LG,\phi}\left(\nabla^{mon}_{\dot{w}^{-1}}(\tilde{Ch}_s)\right).$$

Lemma (4.66)

Let $w, w' \in \tilde{W}$, s simple reflection s.th. $w = s w' \sigma(s)$, ~~We have fiber sequences in Shv~~ $l(w) = l(w') + 2$.

(1) We have fiber sequences in $Shv_{f.g.}(Isoc_G)$

$$R^*_{w'} \to R^*_w \to R^*_{w'\sigma(s)} \oplus R^*_{w'\sigma(s)}[1]$$

$$R^!_{w'\sigma(s)} \oplus R^!_{w'\sigma(s)}[-1] \to R^!_w \to R^!_{w'}$$

(2) We have fiber sequences in $Shv(Isoc_G)^\omega$

$$\tilde{R}^*_{w'} \to \tilde{R}^*_w \to R^{mon,*}_{w'\sigma(s),\sigma(s)}[1]$$

$$R^{mon,!}_{w'\sigma(s),\sigma(s)} \to \tilde{R}^!_w \to \tilde{R}^!_{w'}$$

$$\tilde{R}^*_{w\sigma(s)} \to \tilde{R}^*_{w'\sigma(s)} \to R^{mon,*}_{w'\sigma(s),\sigma(s)}[1]$$

$$R^{mon,!}_{w'\sigma(s),\sigma(s)} \to \tilde{R}^!_{w'\sigma(s)} \to \tilde{R}^!_{w'\sigma(s)}$$

Pf: We have $\tilde{\nabla}^{mon}_{\dot{w}^{-1}} \simeq \tilde{\nabla}^{mon}_{\sigma(s)} *^u \tilde{\nabla}^{mon}_{\dot{w}'^{-1}} *^u \tilde{\nabla}^{mon}_{s}$

by Prop. 4.47(1). Next we consider the diagram:

$$
\begin{tikzcd}
 & \dfrac{LG_{s} \times^{Iw^u} LG_{w'} \times^{Iw^u} LG_{\sigma(s)}}{Ad_\sigma(Iw^u)} \arrow[r, "\sim", "\text{(partial Frobenius)}"'] \arrow[d] & \dfrac{LG_{w'} \times^{Iw^u} LG_{\sigma(s)} \times^{Iw^u} LG_{\sigma(s)}}{Ad_\sigma(Iw^u)} \arrow[d] \\
\dfrac{LG_w}{Ad_\sigma Iw^u} \arrow[r, phantom, "\simeq"] \arrow[d] & \dfrac{LG_{s} \times^{Iw^u} LG_{w'} \times^{Iw} LG_{\sigma(s)}}{Ad_\sigma(Iw^u)} \arrow[r, "\sim"] \arrow[d] & \dfrac{LG_{w'} \times^{Iw} LG_{\sigma(s)} \times^{Iw^u} LG_{\sigma(s)}}{Ad_\sigma(Iw)} \arrow[d] \\
\dfrac{LG_w}{Ad_\sigma Iw} \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\shortparallel"] & \dfrac{LG_{s} \times^{Iw} LG_{w'} \times^{Iw} LG_{\sigma(s)}}{Ad_\sigma(Iw)} \arrow[r, "\sim"] \arrow[dr] & \dfrac{LG_{w'} \times^{Iw} LG_{\sigma(s)} \times^{Iw} LG_{\sigma(s)}}{Ad_\sigma(Iw)} \arrow[d] \\
Sht^{loc}_w & & \mathrm{Isoc}G
\end{tikzcd}
$$

This shows that

$$\tilde{R}^*_{\dot w} = Ch^{mon}_{LG,\phi}\left(\tilde{\nabla}^{mon}_{\dot w^{-1}}\right) \simeq Ch^{mon}_{LG,\phi}\left(\tilde{\nabla}^{mon}_{\sigma(\dot s)} *^u \tilde{\nabla}^{mon}_{\sigma(\dot s)} *^u \tilde{\nabla}^{mon}_{\dot w'^{-1}}\right)$$

By Prop. 4.47(3) we have a fiber sequence

$$\tilde{\nabla}^{mon}_e \longrightarrow \tilde{\nabla}^{mon}_{\sigma(\dot s)} *^u \tilde{\nabla}^{mon}_{\sigma(\dot s)} \longrightarrow \nabla_{\sigma(\dot s)}\left(\tilde{Ch}_{\sigma(s)}\right)[1]$$

hence a fiber sequence

$$\tilde{\nabla}^{mon}_{\dot w'^{-1}} \longrightarrow \tilde{\nabla}^{mon}_{\sigma(\dot s)} *^u \tilde{\nabla}^{mon}_{\sigma(\dot s)} *^u \tilde{\nabla}^{mon}_{\dot w'^{-1}} \longrightarrow \underbrace{\tilde{\nabla}^{mon}_{\sigma(\dot s)}\left(\tilde{Ch}_{\sigma(s)}\right) *^u \tilde{\nabla}^{mon}_{\dot w'^{-1}}[1]}_{\substack{\text{is} \\ \tilde{\nabla}^{mon}_{\sigma(\dot s)\dot w'^{-1}}\left(\tilde{Ch}_{\sigma(s)}\right)[1]}}$$

Applying $Ch^{mon}_{LG,\phi}$ we deduce

$$\tilde{R}^*_{w'} \longrightarrow \tilde{R}^*_w \longrightarrow R^{mon,*}_{w'\sigma(s),\sigma(s)}[1].$$

The version with ! is obtained similarly.

Recall that we have a fiber sequence in $\mathrm{Shv}_{\mathrm{mon}}(\mathcal{J}_k)$

$$\widetilde{Ch}_s \longrightarrow \underset{(=Ch_{mon})}{\widetilde{Ch}} \longrightarrow \widetilde{Ch}$$

We deduce the fiber sequence

$$\tilde{R}^{*}_{w'\sigma(s)} \longrightarrow \tilde{R}^{*}_{w'\sigma(s)} \longrightarrow R^{mon,*}_{w'\sigma(s),\sigma(s)}[1]$$

and similarly for the !-versions.

$\widetilde{Ch} \to \widetilde{Ch}$

When "killing the monodromy", the map becomes 0, so we obtain a direct sum decomposition. $\square$.

<u>Description of $\tilde{R}^{*}_{w}$ and $\tilde{R}^{!}_{w}$ for w of minimal length in its σ-conjugacy class</u>

Recall that $B(\tilde{W}) = \tilde{W}/$ σ-conjugation

There is a notion of <u>σ-straight</u> σ-conjugacy class (those who contain σ-straight elements), and a bijection

$$B(\tilde{W})_{str} \xrightarrow{\sim} \underset{\text{Kottwitz set}}{B(G)}.$$

Recall that we write $w \xrightarrow[\sigma]{t} w'$ if $w' = tw\sigma(t)^{-1}$, $\ell(w') \leq \ell(w)$, $\ell(t) \leq 1$

<u>Thm</u> (3.2(2)) [He-Nie]

$C \subset \tilde{W}$ σ-conjugacy class. $C_{min} \subset C$ elts of min. length.

$\forall v \in C$, there exists a sequence $v = v_0 \xrightarrow[\sigma]{s_1} v_1 \xrightarrow[\sigma]{s_2} \cdots \xrightarrow[\sigma]{s_n} v_n = v'$ with each s_i simple reflection, and a facet $\check{f} \subset \check{\mathfrak{a}}$ such that $v_n \in C_{min}$, $v_n = uw$, w σ-straight and of minimal length in $W_{\check{f}}w$, $u \in W_{\check{f}}$, $w\sigma(W_{\check{f}})w^{-1} = W_{\check{f}}$.

(facet of $B(G,\check{F})$ determined by choice of pinning)

In particular, if $v \in C_{min}$ then we have (assumed from now on)

$$\ell(v) = \ell(v_1) = \cdots = \ell(v')$$

hence

$$\begin{array}{ccccc} Sht^{loc}_{v} & \simeq & Sht^{loc}_{v_1} & \simeq \cdots \simeq & Sht^{loc}_{v'} \\ & \searrow & \downarrow & \swarrow & \\ & & Isoc_G & & \end{array}$$

(because if $\ell(xy) = \ell(y\sigma(x)) = \ell(x) + \ell(y)$, then

$$Sht^{loc}_{xy} = LG_{xy}/I_W \simeq LG_x \times^{I_W} LG_y / I_W$$

$$\underset{\text{(partial Frobenius)}}{\simeq} LG_y \times^{I_W} LG_{\sigma(x)}/I_W \simeq LG_{y\sigma(x)}/I_W = Sht^{loc}_{y\sigma(x)} \;)$$

Here we have $v_{i+1} = s_{i+1} v_i \sigma(s_{i+1})$, $\ell(v_i) = \ell(v_{i+1})$

If $\ell(s_{i+1} v_i) = \ell(v_i) - 1$ then

$$Sht^{loc}_{v_{i+1}} = Sht^{loc}_{(s_{i+1}v_i)\sigma(s_{i+1})} \simeq Sht^{loc}_{s_{i+1}s_{i+1}v_i} = Sht^{loc}_{v_i}$$

If $\ell(s_{i+1} v_i) = \ell(v_i) + 1$ and $v_{i+1} \neq v_i$, by the exchange condition we have ~~$\ell(s_{i+1}v_{i+1}) = \ell(v_{i+1}) - 1$~~,

$\ell(v_i \sigma(s_{i+1})) = \ell(v_i) - 1$. So

$$Sht^{loc}_{v_{i+1}} = Sht^{loc}_{\underbrace{s_{i+1}}\,\underbrace{v_i\sigma(s_{i+1})}} \simeq Sht^{loc}_{v_i\sigma(s_{i+1})\sigma(s_{i+1})} = Sht^{loc}_{v_i}.$$

As a consequence $\tilde{R}^*_v \simeq \tilde{R}^*_{v'}$, $\tilde{R}^!_v \simeq \tilde{R}^!_{v'}$.

So one can concentrate on $v' = uw$. (Let $b \in B(G)$ be the elt corresp. to the class of w)

We therefore consider $w, \breve{F}, u$ as in the thm. We have an associated standard parahoric subgp $\breve{\mathcal{P}}_{\breve{F}}$ ($\mathcal{O}_{\breve{F}}$-model of $G_{\breve{F}}$).

To w is associated $\tilde{\nu}_w \in X_*(T)^{I_F}_{\mathbb{Q}}$ and the Levi subgp $\breve{M}_w = Z_{G_{\breve{F}}}(\tilde{\nu}_w)$

To $\underset{\breve{\mathfrak{a}}}{\overset{\breve{F}}{\cap}}$ is attached $\underset{\breve{\mathfrak{a}}_{\breve{M}_w}}{\overset{\breve{F}_{\breve{M}_w}}{\cap}}$ in $\mathcal{B}(\breve{M}_w, \breve{F})$

$\breve M_w$ has an F-structure determined by $\sigma_{\dot w} := \mathrm{Ad}_{\dot w}\sigma$

The corresponding F-gp is $G_{\dot w}$ (= G_b in other talks):

$$(3.27)\qquad G_{\dot w}(F) = \{ g \in G(\breve F) \mid g^{-1}\dot w \sigma(g) = \dot w \}$$
$$= \{ g \in \breve M_w(\breve F) \mid \sigma_{\dot w}(g) = g \}$$

$\breve{\mathcal P}_{\breve F_{\breve M_w}}$ is rational wrt $\sigma_{\dot w}$, so we get a parahoric subgp

$$(3.28)\qquad P_b = P_{\dot w, \breve F} = \left(\breve{\mathcal P}_{\breve F}(\mathcal O_{\breve F}) \cap \breve M_w(\breve F) \right)^{\sigma_{\dot w}} \subset G_{\dot w}(F)$$

with Levi quotient $L_b = L_{\breve F}(k)^{\sigma_{\dot w}}$

($L_{\breve F}$ = Levi quotient of $(\breve{\mathcal P}_{\breve F})_k$).

~~Lemma (4.67)~~ We have $L_{\breve F} \simeq L_{\breve F_{\breve M_w}}$, so these gps have a k_F-structure

The image of I_W in $L_{\breve F}$ defines a Borel subgp $B_{L_{\breve F}}$ which is rational, and such that $L_{\breve F}/B_{L_{\breve F}} \simeq L^+\breve{\mathcal P}_{\breve F}/I_W$

~~We set $LG_{W_{\breve F}w} = \bigsqcup_{w' \in W_{\breve F}w} LG_{w'} \subset LG$~~

~~affine scheme~~ ~~lfp locally closed embedding~~

Probably one should consider $\tilde\Delta^{mon}_{\dot w^{-1}} *^u \mathcal F$ here instead

$= \bigsqcup_{y \in W_{\breve F}} LG_y$ unip. radical of $B_{\breve F}$

Lemma (4.67)

For any $\mathcal F \in \mathrm{Shv}_{mon}(I_W \backslash LG_{W_{\breve F}} / I_W) = \mathrm{Shv}_{mon}(U_{\breve F} \backslash L_{\breve F} / U_{\breve F})$

we have $\mathrm{Ch}^{mon}_{LG,\phi}(\mathcal F *^u \tilde\Delta^{mon}_{\dot w}) \simeq (i_b)_! \left(\mathrm{cind}^{G_b(F)}_{P_b} V \right)[-\langle 2\rho, \nu_b \rangle]$

where $V \in \mathrm{Rep}(L_b)$ is obtained as the (non affine!) Deligne-Lusztig

induction of $\mathcal F$ along

$$U_{\breve F} \backslash L_{\breve F} / U_{\breve F} \longleftarrow L_{\breve F}/\mathrm{Ad}_{\sigma_{\dot w}} U_{L_{\breve F}} \longrightarrow L_{\breve F}/\mathrm{Ad}_{\sigma_{\dot w}} L_{\breve F} \overset{\text{Lang's thm}}{\simeq} pt/L_b$$

Similarly, $\mathrm{Ch}^{mon}_{LG,\phi}\left(\mathcal{F} *^{u} \tilde{\nabla}^{mon}_{\dot{w}}\right) \simeq (i_b)_*\left(\text{c-ind}^{G_b(F)}_{\mathcal{P}_b}(V)\right)[-\langle 2\rho, \nu_b\rangle]$

Pf: We have

$$\begin{tikzcd}
 & \mathbb{B}_{proet}\, G_b(F) \arrow[r, phantom, "\simeq"] & \mathrm{Isoc}_{G,b} \arrow[r] & \mathrm{Isoc}_G \\
\mathbb{B}_{proet}\, \mathcal{P}_b \arrow[ur] \arrow[r, phantom, "\simeq"] & LG_{W_F^\vee w}/\mathrm{Ad}_\sigma L^+\mathcal{P}_F^\vee \arrow[ur] \arrow[urr, "\text{induced by } Nt"'] & &
\end{tikzcd}$$

(cf. Prop. 3.20)

$LG_{W_F^\vee w} = \bigsqcup_{y \in W_F^\vee w} LG_y$

(action stabilizes $LG_{W_F^\vee w}$ because $w\sigma(W_F^\vee)w^{-1} = W_F^\vee$)

Then we consider

$$\begin{tikzcd}
U_F^\vee \backslash L_F^\vee / U_F^\vee \times S_k \arrow[r, "m"] \arrow[dr, phantom, "\square"] & U_F^\vee \backslash L_F^\vee / U_F^\vee & \frac{L_F^\vee}{\mathrm{Ad}_\sigma U_{L_F^\vee}} \arrow[l] \arrow[r] & L_F^\vee / L_F^\vee \\
I_w^u \backslash L^+\mathcal{P}_F^\vee \times^{I_w^u} LG_w / I_w^u \arrow[u] \arrow[r] \arrow[d, hook] & I_w^u \backslash LG_{W_F^\vee w} / I_w^u \arrow[u] \arrow[d] \arrow[dr, phantom, "\square"] & \frac{LG_{W_F^\vee w}}{\mathrm{Ad}_\sigma I_w^u} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & \frac{LG_{W_F^\vee w}}{\mathrm{Ad}_\sigma L^+\mathcal{P}_F^\vee} \arrow[u] \arrow[d] \\
I_w^u \backslash LG \times^{I_w^u} LG / I_w^u \arrow[r] & I_w^u \backslash LG / I_w^u & \frac{LG}{\mathrm{Ad}_\sigma I_w^u} \arrow[l] \arrow[ddr, "Nt^u"'] & \mathrm{Isoc}_{G,b} \arrow[dd, "i_b"] \\
 & & & \\
 & & & \mathrm{Isoc}_G
\end{tikzcd}$$

and apply base change. $\square$

Corollary (4.68)

$\tilde{R}^*_{\dot{v}} \simeq (i_b)_* \,\text{c-ind}^{G_b(F)}_{\mathcal{P}_b}\left(\tilde{R}^{b,*}_{\dot{u}}\right)[-\langle 2\rho, \nu_b\rangle]$

$\tilde{R}^!_{\dot{v}} \simeq (i_b)_! \,\text{c-ind}^{G_b(F)}_{\mathcal{P}_b}\left(\tilde{R}^{b,!}_{\dot{u}}\right)[-\langle 2\rho, \nu_b\rangle]$

where $\tilde{R}^{b,*}_{\dot{u}}, \tilde{R}^{b,!}_{\dot{u}} \in \mathrm{Rep}(\mathcal{P}_b)$ are Deligne-Lusztig inductions for the Levi quotient L_b of $\mathcal{P}_b$.

In particular, if w is σ-straight we have

$\tilde{R}^*_{\dot{w}} \simeq (i_b)_*\left(\text{c-ind}^{G_b(F)}_{I_b^u}\Lambda\right)[-\langle 2\rho, \nu_b\rangle]$, $\tilde{R}^!_{\dot{w}} \simeq (i_b)_!\left(\text{c-ind}^{G_b(F)}_{I_b^u}\Lambda\right)[-\langle 2\rho, \nu_b\rangle]$

$R^*_{\dot{w}} \simeq (i_b)_*\left(\text{c-ind}^{G_b(F)}_{I_b}\Lambda\right)[-\langle 2\rho, \nu_b\rangle]$ $R^!_{\dot{w}} \simeq (i_b)_!\left(\text{c-ind}^{G_b(F)}_{I_b}\Lambda\right)[-\langle 2\rho, \nu_b\rangle]$

Pf: Follows from the proposition using $\tilde{\Delta}^{mon}_{\dot v} \simeq \underbrace{\tilde{\Delta}^{mon,u}_{\dot u}}_{} * \tilde{\Delta}^{mon}_{\dot w}$

supported on $Iw^u \backslash L^+ \mathcal{P}^u_{F} / Iw^u$

Deligne-Lusztig theory for finite groups (§4.4)

We consider the restriction of the constructions above to $G \subset LG$. The relevant diagram is therefore

$$U\backslash G/U \xleftarrow{\delta^u} G/_{Ad_\sigma U} \xrightarrow{Nt^u} G/_{Ad_\sigma G} \simeq \mathbb{B}G(k_F)$$

(given by Lang's thm: $G \to G$, $g \mapsto g\sigma(g)^{-1}$ induces $G/_{G(k_F)} \xrightarrow{\sim} G$)

~~so we have~~ and we consider

$$Ch^{mon}_{G,\phi} := (Nt^u)_*(\delta^u)^! : Shv_{mon}(U\backslash G/U) \to Shv(\mathbb{B}G(k_F)) \simeq Rep(G(k_F))$$

Note: this is left complete, and compactly generated by reg. rep; cf Lemma 9.14

(unbounded derived ∞-category of the abelian cat. of $\Lambda G(k_F)$-modules)

Lemma (4.84)

Let $P \supset B$ be a rational parabolic subgp, with Levi quotient L. Set
- B_L = image of $B \cap L$ (Borel subgp of L)
- U_L = ——— U ——— (= unip. rad. of B_L)
- $W_P \subset W$ Weyl gp of L

For $\mathcal{F} \in Shv_{mon}(U_L\backslash L/U_L) \simeq Shv_{mon}(U\backslash P/U) \subset Shv_{mon}(U\backslash G/U)$

we have $Ind^{G(k_F)}_{P(k_F)} Ch^{mon}_{L,\phi}(\mathcal{F}) \simeq Ch^{mon}_{G,\phi}(\mathcal{F})$

__Pf:__ Use base-change for

$$\begin{tikzcd}
 & & \mathbb{B}P(k_F) \arrow[d, phantom, "\cong" rotate=90] & \\
U\backslash P/U & P/\mathrm{Ad}_\sigma U \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P/\mathrm{Ad}_\sigma(P) \arrow[r] \arrow[d] & G/\mathrm{Ad}_\sigma G \arrow[d, phantom, "="] \\
U_L\backslash L/U_L \arrow[from=u] & L/\mathrm{Ad}_\sigma U_L \arrow[l] \arrow[r] & L/\mathrm{Ad}_\sigma L \arrow[d, phantom, "\cong" rotate=90] & \mathbb{B}G(k_F) \\
 & & \mathbb{B}L(k_F) &
\end{tikzcd}$$

$\square$

__Thm__ (4.85)

$\mathrm{Rep}(G(k_F))$ is generated as Λ-linear presentable stable ∞-category by:
- the objects $(\widetilde{R}_w^* : w \in W)$
- the objects $(\widetilde{R}_w^! : w \in W)$

__Rmk:__ the objects $\widetilde{R}_w^*$, $\widetilde{R}_w^!$ are compact (as before for LG).

Due to:
- Deligne–Lusztig when $\Lambda = \overline{\mathbb{Q}}_\ell$
- Bonnafé–Rouquier in general.

Zhu gives a sketch of proof when $\#W$ is invertible in Λ (cf. Rmk 4.99).